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On Existence Result of a Class of Nonlinear Integral Equation

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Abstract. Combining the approaches of functionals associated with $h-$concave functions and fixed point techniques, we study the existence and uniqueness of a solution for a class of nonlinear integral equation:

$$x(t) = g_1(t) - g_2(t) + \mu \int_0^t V_1(t,s)h_1(s,x(s))\ ds + \Lambda \int_0^T V_2(t,s)h_2(s,x(s))ds,$$

where $C([0,T],\mathbb{R})$ denotes the space of all continuous functions on $[0,T]$ equipped with the uniform metric and $t \in [0,T]$, μ, Λ are real numbers, $g_1, g_2 \in C([0,T],\mathbb{R})$ and $V_1(t,s)$, $V_2(t,s)$, $h_1(t,s)$, $h_2(t,s)$ are continuous real-valued functions in $[0,T] \times \mathbb{R}$.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

A coincidence point theorem is one which ensures the existence of a coincidence point of a pair of mappings under suitable assumptions both on the space and the mappings. Mostly, these assumptions are sufficient conditions which include containment of the ranges of the mappings, continuity of at least one of the mappings or weaker notion of continuity conditions, a contractive condition and every substantial coincidence point theorem attempts to soften or obtain a necessary version of one or more of these conditions. Apart from ensuring the existence of a coincidence point or point of coincidence, it often becomes necessary to prove the uniqueness of a coincidence point or point of coincidence. Besides, from a computational point of view, an algorithm for calculating the value of the coincidence point is desirable. Often such algorithms involve iterates of some of the given mappings.

The questions about the existence, uniqueness and approximation of a coincidence point or point of coincidence provide three significant distinct features of the general coincidence point theorem. Banach contraction principle for single mapping and Machuca' s coincidence point theorem for a pair of mappings answer all the three questions of existence, uniqueness and constructive algorithm convincingly.

The origin of a coincidence point theorem can be traced back to Machuca [9], who in 1967, used a constructive technique similar to Banach contraction principle to establish the existence of a coincidence

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point. In 1968, Goebel [7] gave an improved version of Machuca's result and utilized the result to find a solution of a differential equation. In 1976 Jungck [2] used Machuca's constructive technique which is equivalent to Picard iterates to generalize the celebrated Banach contraction principle by exploiting the idea of commuting mappings. Since then a lot of research has been carried out in proving the existence of coincidence or common fixed points of pairs of mappings under contractive or noncontractive conditions ([1, 3–5, 11]).

The proofs of common fixed theorems for a pair of mappings satisfying various contractive conditions have almost the same pattern up to finding a coincidence point of the mappings. In the first step one uses the most crucial part of fixed point theorems consisting of constructive procedures yielding a Cauchy sequence. The second step is to use completeness of the space or some subspaces to find limit point of the Cauchy sequence. In step three different sufficient conditions are used to show that there exists a coincidence point for a pair of mappings. In step four coincidence point gives rise to the existence of a common fixed point under some coincidence preserving mappings, like weak compatibility. In the last step uniqueness is established. Systematic comparison and illustration of different noncommuting notions (up to 2013) can be found in Agarwal et al. [1].

2. Basic Tools

Let $M = C([0, T], \mathbb{R})$ denote the space of all continuous functions on $[0, T]$, which, as it is well-known, is a complete metric space when it is equipped with the uniform metric d

$$d(u, v) = \sup_{t\in[0,T]} \{|u(t) - v(t)|\}, \quad u, v \in M. \tag{1}$$

Now we discuss an application of fixed point techniques to the solution of the nonlinear integral equation [8]:

$$x(t) = g_1(t) - g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, x(s))\, ds + \Lambda \int_0^T V_2(t, s)h_2(s, x(s))ds, \tag{2}$$

where $t \in [0, T]$, μ, Λ are real numbers, $g_1, g_2 \in C([0, T], \mathbb{R})$ and $V_1(t, s)$, $V_2(t, s)$, $h_1(t, s)$, $h_2(t, s)$ are continuous real-valued functions in $[0, T] \times \mathbb{R}$.

Further, we use some functional associated with h-concave and quasilinear functions studied in [6]. Let C be a convex cone in the linear space X over $\mathbb{R}$ and let L be a real number $L \neq 0$. A functional $\psi : C \longrightarrow \mathbb{R}$ is called *L-superadditive* on C if

$$f(x + y) \geq L\left(f(x) + f(y)\right), \quad \text{for any } x, y \in C.$$

Let K be a real non-negative function, a functional ψ satisfying

$$\psi(tx) = K(t)\psi(x)$$

for any $t \geq 0$ and $x \in C$, is called *K-positive homogeneous*. Notice that $K(1) = 1$.

The existence of solutions of the nonlinear integral equation (2) will be analyzed by using some auxiliary operators T and I (see below), belonging to the $\psi - (\alpha, \beta, \gamma)$-contraction class. Then, the conclusion is obtained from the existence of point of coincidence for the pair (T, I). Our result holds true if in equation (2) we replace the kernel $h_i(s, x(s))$ for one of the form $h_i(t, x(s))$.

The following lemma was proved in [6].

Lemma 2.1. *Let $u, v \in C$ and $\psi : C \longrightarrow \mathbb{R}$ be a non-negative, L-superadditive and K-positive homogeneous functional on C. If $M \geq m > 0$ are such that $u - mv$ and $Mv - u \in C$. Then*

$$LK(m)\psi(v) \leq \psi(u) \leq \frac{1}{L}K(M)\psi(v).$$

Recently, Morales et al.[10] introduced the notion of $\psi - (\alpha, \beta, \gamma)$-contraction mappings and proved the following theorem.

Theorem 2.2. *Let T and I be self-mappings on a complete metric space (M, d) such that $T(M) \subset I(M)$ and for all $x, y \in M$*

$$\psi\left(d(Tx, Ty)\right) \leq \alpha(d(Ix, Iy))\psi(d(Ix, Iy)) + \beta(d(Ix, Iy))\psi(d(Tx, Ix)) + \gamma(d(Ix, Iy))\psi(d(Ty, Iy)) \tag{3}$$

where $\alpha, \beta, \gamma : \mathbb{R}_+ \longrightarrow [0, 1)$ are functions satisfying the conditions $\alpha(t)+\beta(t)+\gamma(t) < 1$, for all $t \in \mathbb{R}_+(:= [0, +\infty))$, and

$$\limsup_{s\to 0^+} \gamma(s) < 1$$
$$\limsup_{s\to t^+} \frac{\alpha(s) + \beta(s)}{1 - \gamma(s)} < 1, \quad \forall t > 0, \tag{4}$$

and a continuous function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies that

$$\psi(t_n) \to 0 \text{ implies that } t_n \to 0. \tag{5}$$

Then the pair (T, I) has a unique point of coincidence (POC).

3. Main Result

In this section we study the existence and uniqueness of a solution for a class of nonlinear integral equation.

Theorem 3.1. *Suppose the following assumptions are satisfied:*
(i) $\int_0^T \sup_{t\in[0,T]} |V_i(t, s)|ds = L_i < \infty$, $i \in \{1, 2\}$,
(ii) for each $s \in [0, T]$ and for all $x, y \in M$, there is $M_i \geq 0$ such that

$$|h_i(s, x(s)) - h_i(s, y(s))| \leq M_i|x(s) - y(s)|, \quad i \in \{1, 2\}.$$

(iii) $\Lambda \int_0^T V_2(t, s)h_2\left(s, \mu \int_0^s V_1(s, \kappa)h_1(\kappa, x(\kappa))d\kappa + g_1(s) - g_2(s)\right)ds = 0.$
Then, the integral equation (2) has a unique solution, $\phi \in \mathcal{X}$, satisfying

$$g_1(t) = -\Lambda \int_0^1 V_2(t, s)h_2(s, \phi(s))ds$$

provided that

$$|\mu|L_1M_1 + |\Lambda|M_2L_2 = 1. \tag{6}$$

Proof. We define the following operators, for each $x \in M$,

$$Tx(t) = -g_2(t) + \mu \int_0^t V_1(t, s)h_1(s, x(s))ds$$

and

$$Ix(t) = x(t) - g_1(t) - \Lambda \int_0^T V_2(t,s)h_2(s,x(s))\,ds,$$

where $t \in [0,T]$, μ, Λ are real numbers, $g_1, g_2 \in C([0,T], \mathbb{R})$ and $V_1(t,s)$, $V_2(t,s)$, $h_1(t,s)$, $h_2(t,s)$ are continuous real-valued functions in $[0,T] \times \mathbb{R}$ satisfying assumptions (i)–(iii) above.

Clearly, T and I are self-operators on M. We prove that $T(M) \subseteq I(M)$. Using assumption (*iii*), for $x(t) \in \mathcal{X}$ we have

$$\begin{aligned} I(Tx(t) + g_1(t)) &= Tx(t) + g_1(t) - g_1(t) - \Lambda \int_0^T V_2(t,s)h_2(s, Tx(s) + g_1(s))ds \\ &= Tx(t) - \Lambda \int_0^T V_2(t,s)h_2\left(s, \mu \int_0^s V_1(s,\kappa)h_1(\kappa, x(\kappa))d\kappa + g_1(s) - g_2(s)\right)ds \\ &= Tx(t). \end{aligned}$$

Now, for all $x, y \in M$ and using (i)–(ii), we get

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq |\mu| \int_0^t |V_1(t,s)||h_1(s,x(s)) - h_1(s,y(s))|ds \\ &\leq |\mu| \int_0^t \sup_{t\in[0,T]} |V_1(t,s)||h_1(s,x(s)) - h_1(s,y(s))|ds \\ &\leq |\mu| \int_0^t \sup_{t\in[0,T]} |h_1(t,s)|M_1|x(s) - y(s)|ds \\ &\leq |\mu|L_1\|x-y\| \int_0^t \sup_{t\in[0,T]} |h_1(t,s)|ds \\ &\leq |\mu|M_1L_1\|x-y\|. \end{aligned}$$

This implies that

$$\|Tx - Ty\| = \sup_{t\in[0,T]} |Tx(t) - Ty(t)| \leq |\mu|M_1L_1\|x-y\|. \tag{7}$$

By a similar reasoning we get

$$\begin{aligned} &\left|\Lambda \int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda \int_0^T V_2(t,s)h_2(s,y(s))\,ds\right| \\ &\leq |\Lambda| \int_0^T |V_2(t,s)||h_2(s,x(s)) - h_2(s,y(s))|ds \\ &\leq |\Lambda| \int_0^T \sup_{t\in[0,T]} |V_2(t,s)||h_2(s,x(s)) - h_2(s,y(s))|ds \\ &\leq |\Lambda| \int_0^T \sup_{t\in[0,T]} |V_2(t,s)|M_2|x(s) - y(s)|ds \\ &\leq |\Lambda|M_2L_2\|x-y\|, \end{aligned}$$

which implies

$$\sup_{t\in[0,T]} \left|\Lambda \int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda \int_0^T V_2(t,s)h_2(s,y(s))\,ds\right| \leq |\Lambda|L_2M_2\|x-y\|.$$

Consequently, we note that

$$\begin{aligned} \|Ix - Iy\| &\geq \|x - y\| - \sup_{t\in[0,T]} \left| \Lambda \int_0^T V_2(t,s)h_2(s,x(s))\,ds - \Lambda \int_0^T V_2(t,s)h_2(s,y(s))\,ds \right| \\ &\geq (1 - |\Lambda| L_2 M_2)\|x - y\|, \end{aligned} \tag{8}$$

since condition (6) implies that $|\Lambda| L_2 M_2 < 1$, the above inequality gives

$$\|x - y\| \leq \frac{1}{1 - |\Lambda| L_2 M_2} \|Tx - Ty\|. \tag{9}$$

Finally, by (7), (9) and condition (6), we get

$$\|Tx - Ty\| \leq \|Ix - Iy\|.$$

Moreover, there exists $0 \leq m < 1$ depending of x and y, such that

$$m\|Ix - Iy\| \leq \|Tx - Ty\| \leq \|Ix - Iy\|. \tag{10}$$

Now, let ψ be a non-negative, continuous, 2-superadditive and K-positive homogeneous functional on the cone $\mathbb{R}_+$ satisfying (5). For $u = \|Tx - Ty\|$, $v = \|Ix - Iy\|$ and the inequality (10), the Lemma 2.1 allows us to conclude that,

$$\psi(\|Tx - Ty\|) \leq \frac{1}{2}\psi(\|Ix - Iy\|).$$

Let $\alpha, \beta, \gamma : \mathbb{R}_+ \longrightarrow [0, 1)$ satisfying (4) with $\frac{1}{2} \leq \alpha(t)$ for any $t \in \mathbb{R}_+$. Hence, we obtain

$$\begin{aligned} \psi(\|Tx - Ty\|) &\leq \frac{1}{2}\psi(\|Ix - Iy\|) \\ &\leq \alpha(\|Ix - Iy\|)\psi(\|Ix - Iy\|) + \beta(\|Ix - Iy\|)\psi(\|Tx - Ix\|) + +\gamma(\|Ix - Iy\|)\psi(\|Ty - Iy\|). \end{aligned}$$

Therefore, (T, I) is a $\psi - (\alpha, \beta, \gamma)$-contraction pair. In view of Theorem 2.2, the pair (T, I) has a unique POC (say y_0); i.e., $y_0 = Sx_*(t) = Tx_*(t)$. Thus,

$$-g_2(t) + \mu \int_0^t V_1(t,s)h_1(s,x_*(s))ds = x_*(t) - g_1(t) - \Lambda \int_0^T V_2(t,s)h_2(s,x_*(s))ds$$

or equivalently,

$$x_*(t) = g_1(t) - g_2(t) + \mu \int_0^t V_1(t,s)h_1(s,x_*(s))ds + \Lambda \int_0^T V_2(t,s)h_2(s,x_*(s))ds.$$

Therefore, $x_* \in M$ is a solution of the nonlinear integral equation (2). □

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