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Pre-Hausdorff and Hausdorff Proximity Spaces

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Abstract. In this paper, an explicit characterization of the separation properties for T_0, T_1, $PreT_2$ (pre-Hausdorff) and T_2 (Hausdorff) is given in the topological category of proximity spaces. Moreover, specific relationships that arise among the various T_i, $i = 0, 1, 2$ and $PreT_2$ structures are examined in this category. Finally, we investigate the relationships among generalized separation properties for T_i, $i = 0, 1, 2$ and $PreT_2$ (in our sense), separation properties at a point p and separation properties for T_i, $i = 0, 1, 2$ in the usual sense in this category.

1. Introduction

The notion of proximity on a set X was introduced by Efremovich [17]. But the germ of the theory of proximity spaces, which have today become the basic concepts of the theory, was emerged by Frigyes Riesz [30] at the mathematical congress in Roma in 1908. This theory was axiomatized by Efremovich [17], [18] in 1934, but not published until 1951. He characterized the proximity relation "A is close to B" as a binary relation on subsets of a set X. In the interim, in 1941, a study was made by Wallace [35], [36] regarding "separation of sets". This study can be considered as the primitive version of the same concept. Similar but weaker axioms than Efremovich's were used.

Efremovich [18] defined the closure of a subset A of X to be the collection of all points of X "close" A. Thereby he showed that a topology (completely regular) can be introduced in a proximity space. He also showed that every completely regular space X can be turned into a proximity space by using Urysohn's function. Efremovich later used proximity neighborhoods to obtain an equivalent set of axioms for a proximity space.

Much of the early work in proximity spaces was done by Smirnov [33] and [34]. He showed which topological spaces admit a proximity relation compatible with the given topology [34]. Smirnov was also the first to discover relationship between proximities and uniformities.

All our preliminary information on proximity spaces as well as additional information can be found in [28].

Various generalizations of the usual separation properties of topology and for an arbitrary topological category over sets separation properties at a point p are given in [2]. Baran [2] defined separation properties first at a point p, i.e., locally (see [3], [5], [10], [14], [23] and [24]), then they are generalized to point free

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definitions by using the generic element, [22] p. 39, method of topos theory. These generalizations are, for example, two notions of T_0 denoted by $\overline{T}_0$ and T'_0, each equivalent to the classical T_0 notion for topological spaces, and two notions of $PreT_2$ denoted by $Pre\overline{T}_2$ and $PreT'_2$, each equivalent to the classical $PreT_2$ notion (for each distinct pair x and y in X, if the set $\{x, y\}$ is not indiscrete, then there exist disjoint neighborhoods of x and y) for topological spaces.

The main goal of this paper is

1. to give the characterization of the separation properties for T_0, T_1, $PreT_2$ and T_2 in the topological category of proximity spaces,
2. to examine how these generalizations are related, and
3. to show that the relationships among generalized separation properties for T_i, $i = 0, 1, 2$ and $PreT_2$ (in our sense), separation properties at a point p and separation properties for T_i, $i = 0, 1, 2$ in the usual sense in this category.

2. Preliminaries

The following are some basic definitions and notations which we will use throughout the paper.

Let $\mathcal{E}$ and $\mathcal{B}$ be any categories. The functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be topological or that $\mathcal{E}$ is a topological category over $\mathcal{B}$, if $\mathcal{U}$ is concrete (i.e., faithful and amnestic), has small fibers, and for which every $\mathcal{U}$-source has an initial lift or, equivalently, for which each $\mathcal{U}$-sink has a final lift [1].

Note that a topological functor $\mathcal{U} : \mathcal{E} \longrightarrow \mathcal{B}$ is said to be normalized, if constant objects, i.e., subterminals, have a unique structure [1, 5, 13, 27, 29].

Recall in [1] or [29], that an object $X \in \mathcal{E}$ (where $X \in \mathcal{E}$ stands for $X \in \mathrm{Ob}(\mathcal{E})$), a topological category, is discrete iff every map $\mathcal{U}(X) \to \mathcal{U}(Y)$ lifts to a map $X \to Y$ for each object $Y \in \mathcal{E}$ and an object $X \in \mathcal{E}$ is indiscrete iff every map $\mathcal{U}(Y) \to \mathcal{U}(X)$ lifts to a map $Y \to X$ for each object $Y \in \mathcal{E}$.

Let $\mathcal{E}$ be a topological category and $X \in \mathcal{E}$. A is called a subspace of X if the inclusion map $i : A \to X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

Definition 2.1. *[28] An Efremovich proximity (EF-proximity) space is a pair (X, δ), where X is a set and δ is a binary relation on the power set of X such that*

(P1) $A \delta B$ *iff* $B \delta A$;
(P2) $A \delta (B \cup C)$ *iff* $A \delta B$ *or* $A \delta C$;
(P3) $A \delta B$ *implies* $A, B \neq \emptyset$;
(P4) $A \cap B \neq \emptyset$ *implies* $A \delta B$;
(P5) $A \not\delta B$ *implies there is an* $E \subseteq X$ *such that* $A \not\delta E$ *and* $(X - E) \not\delta B$;

where $A \not\delta B$ *means it is not true that* $A \delta B$.

A function $f : (X, \delta) \to (Y, \delta')$ between two proximity spaces is called a *proximity mapping (or a p-map)* iff $f(A)\ \delta'\ f(B)$ whenever $A \delta B$. It can easily be shown that f is a p-map iff, for subsets C and D of Y, $f^{-1}(C) \not\delta f^{-1}(D)$ whenever $C \not\delta' D$.

We denote the category of proximity spaces and proximity mappings by **Prox**. Hunsaker and Sharma [21] showed that the functor $\mathcal{U}$: **Prox** $\longrightarrow$ **Set** is topological.

Definition 2.2. *[31] Let X be a nonempty set and $P(X)$ be the set of all subsets of X. A proximity-base on X is a binary relation $\mathfrak{B}$ on $P(X)$ satisfying the axioms (B1) through (B5) given below:*

(B1) $(\emptyset, X) \notin \mathfrak{B}$;
(B2) $A \cap B \neq \emptyset$ *implies* $(A, B) \in \mathfrak{B}$;
(B3) $(A, B) \in \mathfrak{B}$ *iff* $(B, A) \in \mathfrak{B}$;
(B4) *If* $(A, B) \in \mathfrak{B}$ *and* $A \subseteq A^*$, $B \subseteq B^*$ *then* $(A^*, B^*) \in \mathfrak{B}$;
(B5) *If* $(A, B) \notin \mathfrak{B}$ *then there exists a set* $E \subseteq X$ *such that* $(A, E) \notin \mathfrak{B}$ *and* $(X - E, B) \notin \mathfrak{B}$.

2.3 *Let* $\mathfrak{B}$ *be a proximity-base on a set* X *and let a binary relation* δ *on* $P(X)$ *be defined as follows:* $(A, B) \in \delta$ *if, given any finite covers* $\{A_i : 1 \leq i \leq n\}$ *and* $\{B_j : 1 \leq j \leq m\}$ *of* A *and* B *respectively, then there exists a pair* (i, j) *such that* $(A_i, B_j) \in \mathfrak{B}$. δ *is a proximity on* X *finer than the relation* $\mathfrak{B}$ *[21] or [31].*

2.4 *Let* X *be a non-empty set, for each* $i \in I$, (X_i, δ_i) *be a proximity space and* $f_i : X \to (X_i, \delta_i)$ *be a source in* **Prox**. *Define a binary relation* $\mathfrak{B}$ *on* $P(X)$ *as follows: for* $A, B \in P(X)$, $A \mathfrak{B} B$ *iff* $f_i(A)\ \delta_i\ f_i(B)$*, for all* $i \in I$. $\mathfrak{B}$ *is a proximity-base on* X *(Theorem 3.8, [31]). The initial proximity structure* δ *on* X *generated by the proximity base* $\mathfrak{B}$ *is given by for* $A, B \in P(X)$, $A\ \delta\ B$ *iff for any finite covers* $\{A_i : 1 \leq i \leq n\}$ *and* $\{B_j : 1 \leq j \leq m\}$ *of* A *and* B *respectively, then there exists a pair* (i, j) *such that* $(A_i, B_j) \in \mathfrak{B}$ *[31].*

2.5 *Let* (X, δ) *be a proximity space,* Y *a non-empty set and* f *a function from a proximity space* (X, δ) *onto a set* Y*. The quotient proximity* δ^* *on* Y *is defined as follows for every* $A, B \subset Y$: $A\ \delta^*\ B$ *iff, for each binary rational* s *in* $[0, 1]$*, there is some* $C_s \subset Y$ *such that* $C_0 = A$, $C_1 = B$ *and* $s < t$ *implies* $f^{-1}(C_s)\ \delta\ f^{-1}(C_t)$ *[20] or [38] p. 276.*

2.6 *We write* Δ *for the diagonal in* X^2*, where* $X \in$ **Prox**. *For* $X \in$ **Prox** *we define the wedge* $X^2 \vee_\Delta X^2$*, as the final structure, with respect to the map* $X^2 \coprod X^2 \longrightarrow X^2 \vee_\Delta X^2$*, that is the identification of the two copies of* X^2 *along the diagonal* Δ*. An epi sink* $\{i_1, i_2 : (X^2, \delta) \longrightarrow (X^2 \vee_\Delta X^2, \delta')\}$*, where* i_1, i_2 *are the canonical injections, in* **Prox** *is a final lift if and only if the following statement holds. For each pair* A, B *in the different component of* $X^2 \vee_\Delta X^2$, $A\ \delta'\ B$ *iff there exist sets* C, D *and* U *in* X^2 *such that* $C\ \delta\ U$ *and* $U\ \delta\ D$ *with* $i_k^{-1}(A) = C$ *and* $i_j^{-1}(B) = D$ *for* $k, j = 1, 2$ *and* $k \neq j$*. If* A *and* B *are in the same component of wedge, then* $A\ \delta'\ B$ *iff there exist sets* C, D *in* X^2 *such that* $C\ \delta\ D$ *and* $i_k^{-1}(A) = C$ *and* $i_k^{-1}(B) = D$ *for some* $k = 1, 2$*. Specially, if* $i_k(E) = A$ *and* $i_k(F) = B$*, then* $(i_k(E), i_k(F)) \in \delta'$ *iff* $(i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta$*. This is a special case of 2.5.*

2.7 *Let* X *be a non-empty set. The discrete proximity structure* δ *on* X *is defined as follows for* $A, B \subset X$: $A\ \delta\ B$ *iff* $A \cap B \neq \emptyset$ *[28] p. 9.*

2.8 *Let* X *be a non-empty set. The indiscrete proximity structure* δ *on* X *is defined as follows for* $A, B \subset X$: $A\ \delta\ B$ *iff* $A \neq \emptyset$ *and* $B \neq \emptyset$ *[19] p. 5.*

3. T_0, T_1, $PreT_2$ and T_2 Proximity Spaces

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal, i.e., the result of pushing out Δ along itself. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1((x, y)_2)$ if (x, y) is in the first (resp. second) component of $B^2 \vee_\Delta B^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$ [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$. Note that $\pi_1 S = \pi_{11} = \pi_1 A$, $\pi_2 S = \pi_{21} = \pi_2 A$, $\pi_3 A = \pi_{12}$, and $\pi_3 S = \pi_{22}$, where $\pi_k : B^3 \to B$ the k-th projection $k = 1,\ 2,\ 3$ and $\pi_{ij} = \pi_i + \pi_j : B^2 \vee_\Delta B^2 \to B$, for $i, j \in \{1, 2\}$ [2].

Definition 3.1. *(cf. [2] and [13]) Let* $\mathcal{U} : \mathcal{E} \to$ ***Set*** *be a topological functor,* X *an object in* $\mathcal{E}$ *with* $\mathcal{U}(X) = B$*.*

1. X is $\overline{T}_0$ iff the initial lift of the $\mathcal{U}$−source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$ [2].
2. X is T_0' iff the initial lift of the $\mathcal{U}$−source $\{id : B^2 \vee_\Delta B^2 \to \mathcal{U}(B^2 \vee_\Delta B^2)' = B^2 \vee_\Delta B^2$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $(B^2 \vee_\Delta B^2)'$ is the final lift of the $\mathcal{U}$−sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$, i_1 and i_2 are the canonical injections, and $\mathcal{D}(B^2)$ is the discrete structure on B^2 [2], [6].
3. X is T_1 iff the initial lift of the $\mathcal{U}$−source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete [2].
4. X is $Pre\overline{T}_2$ iff the initial lift of the $\mathcal{U}$-sources $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ agree.
5. X is $PreT_2'$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ agree.
6. X is $\overline{T}_2$ iff X is $\overline{T}_0$ and $Pre\overline{T}_2$.
7. X is T_2' iff X is T_0' and $PreT_2'$.

Remark 3.2. 1. *Note that for the category* ***Top*** *of topological spaces,* $\overline{T}_0$, T_0' *or* T_1 *or* $Pre\overline{T}_2$, $PreT_2'$ *or* $\overline{T}_2$, T_2' *reduce to usual* T_0 *or* T_1 *or* $PreT_2$ *(for each distinct pair x and y in X, if the set* $\{x, y\}$ *is not indiscrete, then there exist disjoint neighborhoods of x and y) or* T_2 *separation axioms, respectively [2, 26, 37].*

2. *For an arbitrary topological category, we have* $\overline{T}_0$ *implies* T_0' *([6], Theorem 3.2) but the reverse implication is generally not true (see [6] or 3.3 and 3.4, below).*
3. *Let* $\mathcal{U} : \mathcal{E} \to$ ***Set*** *be a topological functor, X an object in* $\mathcal{E}$ *and* $p \in \mathcal{U}(X)$ *be a retract of X, i.e., the initial lift* $h : 1 \to X$ *of the* $\mathcal{U}$*-source* $p : 1 \to \mathcal{U}(X)$ *is a retract, where* 1 *is the terminal object in* ***Set****. Then if X is* $\overline{T}_0$ *(resp.* T_1*), then X is* $\overline{T}_0$ *at p (resp.* T_1 *at p) but the reverse implication is not true, in general ([5], Theorem 2.6).*
4. *If* $\mathcal{U} : \mathcal{E} \to$ ***Set*** *be a normalized topological functor, then* $\overline{T}_0$ *(resp.* T_1*) implies* $\overline{T}_0$ *at p (resp.* T_1 *at p) ([5], Corollary 2.9).*

Theorem 3.3. *An Efremovich proximity space* (X, δ) *is* $\overline{T}_0$ *if and only if, for each distinct pair x and y in X,* $(\{x\}, \{y\}) \notin \delta$.

Proof. Suppose that (X, δ) is $\overline{T}_0$, i.e., 2.4, 2.7 and Definition 3.1, for any sets U, V on $X^2 \vee_\Delta X^2$, $\pi_{11}U \,\delta\, \pi_{11}V$, $\pi_{21}U \,\delta\, \pi_{21}V$, $\pi_{12}U \,\delta\, \pi_{12}V$ and $\nabla U \,\delta_d^2\, \nabla V$ iff $U \cap V \neq \emptyset$ (δ_d^2 is the discrete proximity structure on X^2).

We shall show that the condition holds. Suppose for some $x, y \in X$, $(\{x\}, \{y\}) \in \delta$ with $x \neq y$. Then, by 2.4, 2.7 and Definition 3.1, for $(U, V) \in \delta'$ (δ' is an Efremovich proximity structure on $X^2 \vee_\Delta X^2$) with $U = \{(x, y)_1\}$ and $V = \{(x, y)_2\}$, $\pi_{11}U \,\delta\, \pi_{11}V = \pi_1 A\{(x, y)_1\} \,\delta\, \pi_1 A\{(x, y)_2\} = \pi_1\{(x, y, x)\} \,\delta\, \pi_1\{(x, x, y)\} = \{x\} \,\delta\, \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_{21}U \,\delta\, \pi_{21}V = \pi_2 A\{(x, y)_1\} \,\delta\, \pi_2 A\{(x, y)_2\} = \pi_2\{(x, y, x)\} \,\delta\, \pi_2\{(x, x, y)\} = \{y\} \,\delta\, \{x\}$, i.e., $(\{y\}, \{x\}) \in \delta$, $\pi_{12}U \,\delta\, \pi_{12}V = \pi_3 A\{(x, y)_1\} \,\delta\, \pi_3 A\{(x, y)_2\} = \pi_3\{(x, y, x)\} \,\delta\, \pi_3\{(x, x, y)\} = \{x\} \,\delta\, \{y\}$, i.e., $(\{x\}, \{y\}) \in \delta$, where $\pi_i : X^3 \to X$, $i = 1, 2, 3$, are the projection maps, and $\nabla\{(x, y)_1\} \,\delta_d^2\, \nabla\{(x, y)_2\} = \{(x, y)\} \,\delta_d^2\, \{(x, y)\}$, i.e., $(\{(x, y)\}, \{(x, y)\}) \in \delta_d^2$, ($\delta_d^2$ is the discrete proximity structure on X^2). But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is $\overline{T}_0$. Hence if $(\{x\}, \{y\}) \in \delta$, then $x = y$.

Conversely, suppose that for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$. We need to show that (X, δ) is $\overline{T}_0$, i.e., by 2.4, 2.7 and Definition 3.1, we must show that the proximity structure δ' on $X^2 \vee_\Delta X^2$ induced by $A : X^2 \vee_\Delta X^2 \to \mathcal{U}((X^3, \delta^3)) = X^3$ and $\nabla : X^2 \vee_\Delta X^2 \to \mathcal{U}((X^2, \delta_d^2)) = X^2$ is discrete, where δ^3 and δ_d^2 are the product proximity structure on X^3 and the discrete proximity structure on X^2, respectively. Let (U, V) be any set in δ', i.e., $\pi_i A(U) \,\delta\, \pi_i A(V)$ $(i = 1, 2, 3)$ and $\nabla U \,\delta_d^2\, \nabla V$.

Since δ_d^2 is the discrete proximity structure and $\nabla U \,\delta_d^2\, \nabla V$, then $\nabla U \cap \nabla V \neq \emptyset$. It follows that there exists $(x, y) \in \nabla U \cap \nabla V$. Hence, there exist $t \in U$ and $z \in V$ such that $\nabla t = (x, y) = \nabla z$. If $x = y$, then $t = (x, y)_i = z$, $(i = 1, 2)$ and $(x, y)_i \in U \cap V$.

If $x \neq y$, then $t = (x, y)_i$, $z = (x, y)_j$ $(i, j = 1, 2)$. We need to show that $U \cap V \neq \emptyset$, i.e., U and V are in the first or in the second or in both component of $X^2 \vee_\Delta X^2$.

If U subset of the first component of $X^2 \vee_\Delta X^2$ and V subset of the second component of $X^2 \vee_\Delta X^2$, then $\{(x, y)_1\} \subseteq U$ and $\{(x, y)_2\} \subseteq V$. It follows that $\pi_3 A\{(x, y)_1\} \,\delta\, \pi_3 A\{(x, y)_2\} = \pi_3\{(x, y, x)\} \,\delta\, \pi_3\{(x, x, y)\} = \{x\} \,\delta\, \{y\}$, i.e., $(\{x\}, \{y\}) \in \delta$. Since $(\{x\}, \{y\}) \notin \delta$ (by assumption), $(\{(x, y)_1\}, \{(x, y)_2\}) \notin \delta'$ by the condition $(P2)$ of 2.1.

The case U subset of the second component of $X^2 \vee_\Delta X^2$ and V subset of the first component of $X^2 \vee_\Delta X^2$ can be handled similarly. Hence U and V can not be in different component of $X^2 \vee_\Delta X^2$.

If U and V are in both component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1, (x, y)_2\}$ and $V \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the first component of $X^2 \vee_\Delta X^2$ and V subset of both component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1\}$ and $V \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of both component of $X^2 \vee_\Delta X^2$ and V subset of the second component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1, (x, y)_2\}$ and $V \supseteq \{(x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U and V are in the first component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1\}$ and $V \supseteq \{(x, y)_1\}$. Hence $U \cap V \neq \emptyset$. Similarly if U and V are in the second component of $X^2 \vee_\Delta X^2$, then $U \cap V \neq \emptyset$.

If $(\{(x, y)_i\}, \{(x, y)_i\}) \in \delta'$, $(i = 1, 2)$, then $\pi_1 A\{(x, y)_1\} \,\delta\, \pi_1 A\{(x, y)_1\} = \{x\} \,\delta\, \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 A\{(x, y)_1\} \,\delta\, \pi_2 A\{(x, y)_1\} = \{y\} \,\delta\, \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$, $\pi_3 A\{(x, y)_1\} \,\delta\, \pi_3 A\{(x, y)_1\} = \{x\} \,\delta\, \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$ and $\pi_1 A\{(x, y)_2\} \,\delta\, \pi_1 A\{(x, y)_2\} = \{x\} \,\delta\, \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 A\{(x, y)_2\} \,\delta\, \pi_2 A\{(x, y)_2\} = \{x\} \,\delta\, \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_3 A\{(x, y)_2\} \,\delta\, \pi_3 A\{(x, y)_2\} = \{y\} \,\delta\, \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$.

We must have $(U, V) \supseteq (\{(x, y)_i\}, \{(x, y)_i\})$, $(i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by 2.4, 2.7 and Definition 3.1, (X, δ) is $\overline{T}_0$. □

Theorem 3.4. *An Efremovich proximity space is T_0'.*

Proof. Let (X, δ) be any Efremovich proximity space. By 2.4, 2.6, 2.7 and Definition 3.1, we will show that for any $(i_k(E), i_k(F)) \in \delta'$ (δ' is an Efremovich proximity structure on $X^2 \vee_\Delta X^2$), if $i_k(E, F) = (i_k(E), i_k(F)) \in \delta'$ $(k = 1, 2)$ for some $(E, F) \in \delta^2$ ($E, F \subset X^2$ and δ^2 is the product proximity structure on X^2) and $(\nabla(i_k(E)), \nabla(i_k(F))) \in \delta_d^2$ (δ_d^2 is the discrete proximity structure on X^2), then we will show that $(i_k(E), i_k(F)) \supseteq (\{(x, y)_k\}, \{(x, y)_k\})$, $(k = 1, 2)$, i.e., $i_k(E) \cap i_k(F) \neq \emptyset$.

Since δ_d^2 is the discrete proximity structure and $\nabla(i_k(E))\ \delta_d^2\ \nabla(i_k(F))$, then $\nabla(i_k(E)) \cap \nabla(i_k(F)) \neq \emptyset$. It follows that there exists $(x, y) \in \nabla(i_k(E)) \cap \nabla(i_k(F))$. Hence, there exist $t \in i_k(E)$ and $z \in i_k(F)$ such that $\nabla t = (x, y) = \nabla z$. If $x = y$, then $t = (x, y)_k = z$, $(k = 1, 2)$.

If $x \neq y$, then $t = (x, y)_k$, $z = (x, y)_n$ $(k, n = 1, 2)$. We need to show that $i_k(E) \cap i_k(F) \neq \emptyset$, i.e., $i_k(E)$ and $i_k(F)$ are in the first or in the second or in both component of $X^2 \vee_\Delta X^2$.

If $i_k(E)$ subset of the first component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of the second component of $X^2 \vee_\Delta X^2$, then $\{(x, y)_1\} \subseteq i_k(E)$ and $\{(x, y)_2\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x, y)_1\}, \{(x, y)_2\}) \in \delta'$ for some $(E, F) \in \delta^2$ and $k = 1$ (resp. $k = 2$), then $(\{(x, y)_1\}, \{(x, y)_2\}) \in (i_1(E), i_1(F))$ which shows that $(x, y)_2$ (resp. $(x, y)_1$) must be in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction since $x \neq y$.

Similarly, if $i_k(E)$ subset of the second component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of the first component of $X^2 \vee_\Delta X^2$, then $\{(x, y)_2\} \subseteq i_k(E)$ and $\{(x, y)_1\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x, y)_2\}, \{(x, y)_1\}) \in \delta'$ for some $(E, F) \in \delta^2$ and $k = 1$ (resp. $k = 2$), then $(\{(x, y)_2\}, \{(x, y)_1\}) \in (i_1(E), i_1(F))$ which shows that $(x, y)_2$ (resp. $(x, y)_1$) must be in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction since $x \neq y$. Hence $i_k(E)$ and $i_k(F)$ can not be in different component of $X^2 \vee_\Delta X^2$.

If $i_k(E)$ and $i_k(F)$ are in both component of $X^2 \vee_\Delta X^2$, then $i_k(E) \supseteq \{(x, y)_1, (x, y)_2\}$ and $i_k(F) \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ subset of the first component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of both component of $X^2 \vee_\Delta X^2$, then $i_k(E) \supseteq \{(x, y)_1\}$ and $i_k(F) \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ subset of both component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of the second component of $X^2 \vee_\Delta X^2$, then $i_k(E) \supseteq \{(x, y)_1, (x, y)_2\}$ and $i_k(F) \supseteq \{(x, y)_2\}$. Hence $i_k(E) \cap i_k(F) \neq \emptyset$.

If $i_k(E)$ and $i_k(F)$ are in the first component of $X^2 \vee_\Delta X^2$, then $i_k(E) \supseteq \{(x, y)_1\}$ and $i_k(F) \supseteq \{(x, y)_1\}$. Hence $i_k(E) \cap i_k(F) \neq \emptyset$. Similarly if $i_k(E)$ and $i_k(F)$ are in the second component of $X^2 \vee_\Delta X^2$, then $i_k(E) \cap i_k(F) \neq \emptyset$.

We must have $(i_k(E), i_k(F)) \supseteq (\{(x, y)_i\}, \{(x, y)_i\})$, $(i = 1, 2)$, i.e., $i_k(E) \cap i_k(F) \neq \emptyset$ and consequently, by 2.4, 2.6, 2.7 and Definition 3.1, (X, δ) is T_0'. □

Theorem 3.5. *An Efremovich proximity space (X, δ) is T_1 if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

Proof. Suppose that (X, δ) is T_1, i.e., 2.4, 2.7 and Definition 3.1, for any sets U, V on $X^2 \vee_\Delta X^2$, $\pi_{11}U\ \delta\ \pi_{11}V$, $\pi_{21}U\ \delta\ \pi_{21}V$, $\pi_{22}U\ \delta\ \pi_{22}V$ and $\nabla U\ \delta_d^2\ \nabla V$ iff $U \cap V \neq \emptyset$ (δ_d^2 is the discrete proximity structure on X^2).

We shall show that the condition holds. Suppose for some $x, y \in X$, $(\{x\}, \{y\}) \in \delta$ with $x \neq y$. Then, by 2.4, 2.7 and Definition 3.1, for $(U, V) \in \delta'$ (δ' is an Efremovich proximity structure on $X^2 \vee_\Delta X^2$) with $U = \{(x, y)_1\}$ and $V = \{(x, y)_2\}$, $\pi_{11}U\ \delta\ \pi_{11}V = \pi_1 S\{(x, y)_1\}\ \delta\ \pi_1 S\{(x, y)_2\} = \pi_1\{(x, y, y)\}\ \delta\ \pi_1\{(x, x, y)\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_{21}U\ \delta\ \pi_{21}V = \pi_2 S\{(x, y)_1\}\ \delta\ \pi_2 S\{(x, y)_2\} = \pi_2\{(x, y, y)\}\ \delta\ \pi_2\{(x, x, y)\} = \{y\}\ \delta\ \{x\}$, i.e., $(\{y\}, \{x\}) \in \delta$, $\pi_{22}U\ \delta\ \pi_{22}V = \pi_3 S\{(x, y)_1\}\ \delta\ \pi_3 S\{(x, y)_2\} = \pi_3\{(x, y, y)\}\ \delta\ \pi_3\{(x, x, y)\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$, where $\pi_i : X^3 \to X$, $i = 1, 2, 3$, are the projection maps, and $\nabla\{(x, y)_1\}\ \delta_d^2\ \nabla\{(x, y)_2\} = \{(x, y)\}\ \delta_d^2\ \{(x, y)\}$, i.e., $(\{(x, y)\}, \{(x, y)\}) \in \delta_d^2$, ($\delta_d^2$ is the discrete proximity structure on X^2). But $U \cap V = \emptyset$. This is a contradiction to the fact that (X, δ) is T_1. Hence if $(\{x\}, \{y\}) \in \delta$, then $x = y$.

Conversely, suppose that for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$. We need to show that (X, δ) is T_1, i.e., by 2.4, 2.7 and Definition 3.1, we must show that the proximity structure δ' on $X^2 \vee_\Delta X^2$ induced by $S : X^2 \vee_\Delta X^2 \to \mathcal{U}((X^3, \delta^3)) = X^3$ and $\nabla : X^2 \vee_\Delta X^2 \to \mathcal{U}((X^2, \delta_d^2)) = X^2$ is discrete, where δ^3 and δ_d^2 are the product proximity structure on X^3 and the discrete proximity structure on X^2, respectively. Let (U, V) be any set in δ', i.e., $\pi_i S(U)\ \delta\ \pi_i S(V)$ $(i = 1, 2, 3)$ and $\nabla U\ \delta_d^2\ \nabla V$.

Since δ_d^2 is the discrete proximity structure and $\nabla U \ \delta_d^2 \ \nabla V$, then $\nabla U \cap \nabla V \neq \emptyset$. It follows that there exists $(x, y) \in \nabla U \cap \nabla V$. Hence, there exist $t \in U$ and $z \in V$ such that $\nabla t = (x, y) = \nabla z$. If $x = y$, then $t = (x, y)_i = z$, $(i = 1, 2)$ and $(x, y)_i \in U \cap V$.

If $x \neq y$, then $t = (x, y)_i$, $z = (x, y)_j$ $(i, j = 1, 2)$. We need to show that $U \cap V \neq \emptyset$, i.e., U and V are in the first or in the second or in both component of $X^2 \vee_\Delta X^2$.

If U subset of the first component of $X^2 \vee_\Delta X^2$ and V subset of the second component of $X^2 \vee_\Delta X^2$, then $\{(x, y)_1\} \in U$ and $\{(x, y)_2\} \in V$. It follows that $\pi_2 S\{(x, y)_1\} \ \delta \ \pi_2 S\{(x, y)_2\} = \pi_2\{(x, y, y)\} \ \delta \ \pi_2\{(x, x, y)\} = \{y\} \ \delta \ \{x\}$, i.e., $(\{y\}, \{x\}) \in \delta$. Since $(\{x\}, \{y\}) \notin \delta$ (by assumption), $(\{(x, y)_1\}, \{(x, y)_2\}) \notin \delta'$ by the condition $(P2)$ of 2.1.

The case U subset of the second component of $X^2 \vee_\Delta X^2$ and V subset of the first component of $X^2 \vee_\Delta X^2$ can be handled similarly. Hence U and V can not be in different component of $X^2 \vee_\Delta X^2$.

If U and V are in both component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1, (x, y)_2\}$ and $V \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of the first component of $X^2 \vee_\Delta X^2$ and V subset of both component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1\}$ and $V \supseteq \{(x, y)_1, (x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U subset of both component of $X^2 \vee_\Delta X^2$ and V subset of the second component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1, (x, y)_2\}$ and $V \supseteq \{(x, y)_2\}$. Hence $U \cap V \neq \emptyset$.

If U and V are in the first component of $X^2 \vee_\Delta X^2$, then $U \supseteq \{(x, y)_1\}$ and $V \supseteq \{(x, y)_1\}$. Hence $U \cap V \neq \emptyset$. Similarly if U and V are in the second component of $X^2 \vee_\Delta X^2$, then $U \cap V \neq \emptyset$.

If $(\{(x, y)_i\}, \{(x, y)_i\}) \in \delta'$, $(i = 1, 2)$, then $\pi_1 S\{(x, y)_1\} \ \delta \ \pi_1 S\{(x, y)_1\} = \{x\} \ \delta \ \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 S\{(x, y)_1\} \ \delta \ \pi_2 S\{(x, y)_1\} = \{y\} \ \delta \ \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$, $\pi_3 S\{(x, y)_1\} \ \delta \ \pi_3 S\{(x, y)_1\} = \{y\} \ \delta \ \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$ and $\pi_1 S\{(x, y)_2\} \ \delta \ \pi_1 S\{(x, y)_2\} = \{x\} \ \delta \ \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_2 S\{(x, y)_2\} \ \delta \ \pi_2 S\{(x, y)_2\} = \{x\} \ \delta \ \{x\}$, i.e., $(\{x\}, \{x\}) \in \delta$, $\pi_3 S\{(x, y)_2\} \ \delta \ \pi_3 S\{(x, y)_2\} = \{y\} \ \delta \ \{y\}$, i.e., $(\{y\}, \{y\}) \in \delta$.

We must have $(U, V) \supseteq (\{(x, y)_i\}, \{(x, y)_i\})$, $(i = 1, 2)$, i.e., $U \cap V \neq \emptyset$ and consequently, by 2.4, 2.7 and Definition 3.1, (X, δ) is T_1. □

Remark 3.6. *Let (X, δ) be an Efremovich proximity space, it follows there are Theorems 3.3, 3.5 that (X, δ) is $\overline{T}_0$ if and only if (X, δ) is T_1 if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

Theorem 3.7. *An Efremovich proximity space is $Pre\overline{T}_2$.*

Proof. Let (X, δ) be any Efremovich proximity space. We will show that (X, δ) is $Pre\overline{T}_2$, i.e., by 2.4 and Definition 3.1, for any pair U and V in the wedge, $\pi_1 A(U) \ \delta \ \pi_1 A(V)$, $\pi_2 A(U) \ \delta \ \pi_2 A(V)$ and $\pi_3 A(U) \ \delta \ \pi_3 A(V)$ iff $\pi_1 S(U) \ \delta \ \pi_1 S(V)$, $\pi_2 S(U) \ \delta \ \pi_2 S(V)$ and $\pi_3 S(U) \ \delta \ \pi_3 S(V)$, respectively.

We consider various possibilities for U and V; namely $U \supseteq \{(x, y)_1\}$, $\{(x, y)_2\}$ or $\{(x, x)\}$ and $V \supseteq \{(z, w)_1\}$, $\{(z, w)_2\}$ or $\{(z, z)\}$ for some $x, y, z, w \in X$. By the condition $(P2)$ of 2.1 it is sufficient to take "equality" instead of "superset" for the possibilities above.

If $U = \{(x, y)_1\}$ and $V = \{(z, w)_1\}$, then $\pi_1 A(U) \ \delta \ \pi_1 A(V) = \{x\} \ \delta \ \{z\} = \pi_1 S(U) \ \delta \ \pi_1 S(V)$, $\pi_2 A(U) \ \delta \ \pi_2 A(V) = \{y\} \ \delta \ \{w\} = \pi_2 S(U) \ \delta \ \pi_2 S(V)$, and $\pi_3 A(U) \ \delta \ \pi_3 A(V) = \{x\} \ \delta \ \{z\}$ iff $\pi_3 S(U) \ \delta \ \pi_3 S(V) = \{y\} \ \delta \ \{w\}$.

If $U = \{(x, y)_1\}$ and $V = \{(z, w)_2\}$, then $\pi_1 A(U) \ \delta \ \pi_1 A(V) = \{x\} \ \delta \ \{z\} = \pi_1 S(U) \ \delta \ \pi_1 S(V)$, $\pi_2 A(U) \ \delta \ \pi_2 A(V) = \{y\} \ \delta \ \{z\} = \pi_2 S(U) \ \delta \ \pi_2 S(V)$. Note that $\pi_3 A(U) \ \delta \ \pi_3 A(V) = \{x\} \ \delta \ \{w\}$ iff $\pi_3 S(U) \ \delta \ \pi_3 S(V) = \{y\} \ \delta \ \{w\}$, (since δ is an Efremovich proximity structure).

If $U = \{(x, y)_1\}$ and $V = \{(z, z)\}$, then $\pi_1 A(U) \ \delta \ \pi_1 A(V) = \{x\} \ \delta \ \{z\} = \pi_1 S(U) \ \delta \ \pi_1 S(V)$, $\pi_2 A(U) \ \delta \ \pi_2 A(V) = \{y\} \ \delta \ \{z\} = \pi_2 S(U) \ \delta \ \pi_2 S(V)$, clearly, $\pi_3 A(U) \ \delta \ \pi_3 A(V) = \{x\} \ \delta \ \{z\}$ iff $\pi_3 S(U) \ \delta \ \pi_3 S(V) = \{y\} \ \delta \ \{z\}$.

Similarly, if $U = \{(x, y)_2\}$ or $\{(x, x)\}$ and $V = \{(z, w)_1\}$, $\{(z, w)_2\}$ or $\{(z, z)\}$, then we have $\pi_1 A(U) \ \delta \ \pi_1 A(V)$, $\pi_2 A(U) \ \delta \ \pi_2 A(V)$ and $\pi_3 A(U) \ \delta \ \pi_3 A(V)$ iff $\pi_1 S(U) \ \delta \ \pi_1 S(V)$, $\pi_2 S(U) \ \delta \ \pi_2 S(V)$ and $\pi_3 S(U) \ \delta \ \pi_3 S(V)$, respectively.

Hence (X, δ) is $Pre\overline{T}_2$. □

Theorem 3.8. *An Efremovich proximity space (X, δ) is $PreT_2'$ if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

Proof. Suppose (X, δ) is $PreT_2'$, i.e., by 2.4, 2.6 and Definition 3.1, for any sets U, V on $X^2 \vee_\Delta X^2$, (a) $\pi_{11} U \ \delta \ \pi_{11} V$, $\pi_{21} U \ \delta \ \pi_{21} V$ and $\pi_{22} U \ \delta \ \pi_{22} V$ iff (b) there exists a pair (a, b), $(c, d) \in X^2$ such that $\{(a, b)\} \ \delta^2 \ \{(c, d)\}$ and $i_k\{(a, b)\} = U$ and $i_k\{(c, d)\} = V$ for some $k = 1$ or 2, where δ^2 is the product proximity structure on X^2.

For each pair U, V in the different component of $X^2 \vee_\Delta X^2$, $U\ \delta'\ V$ iff there exist sets C, D and A in X^2 such that $C\ \delta^2\ A$ and $A\ \delta^2\ D$ with $i_k^{-1}(U) = C$ and $i_j^{-1}(V) = D$ for $k, j = 1,2$ and $k \neq j$. If U and V are in the same component of wedge, then $U\ \delta'\ V$ iff there exist sets C, D in X^2 such that $C\ \delta^2\ D$ and $i_k^{-1}(U) = C$ and $i_k^{-1}(V) = D$ for some $k = 1,2$. Specially, if $i_k(E) = U$ and $i_k(F) = V$, then $(i_k(E), i_k(F)) \in \delta'$ iff $(i_k^{-1}(i_k(E)), i_k^{-1}(i_k(F))) = (E, F) \in \delta^2$. This is a special case of 2.5. We shall show that the condition holds.

Suppose for some $x, y \in X$, $(\{x\}, \{y\}) \in \delta$ with $x \neq y$. Then, for $(U, V) \in \delta'$ (δ' is an Efremovich proximity structure on $X^2 \vee_\Delta X^2$) with $U \supseteq \{(x,y)_1\}$ and $V \supseteq \{(x,y)_2\}$, $\pi_{11}U\ \delta\ \pi_{11}V \supseteq \pi_1 S\{(x,y)_1\}\ \delta\ \pi_1 S\{(x,y)_2\} = \pi_1\{(x,y,y)\}\ \delta\ \pi_1\{(x,x,y)\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_{21}U\ \delta\ \pi_{21}V \supseteq \pi_2 S\{(x,y)_1\}\ \delta\ \pi_2 S\{(x,y)_2\} = \pi_2\{(x,y,y)\}\ \delta\ \pi_2\{(x,x,y)\} = \{y\}\ \delta\ \{x\}$, i.e., $(\{y\},\{x\}) \in \delta$, $\pi_{22}U\ \delta\ \pi_{22}V \supseteq \pi_3 S\{(x,y)_1\}\ \delta\ \pi_3 S\{(x,y)_2\} = \pi_3\{(x,y,y)\}\ \delta\ \pi_3\{(x,x,y)\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$, where $\pi_i : X^3 \to X$, $(i = 1,2,3)$, are the projection maps.

There exist sets E, F and A in X^2 such that $E\ \delta^2\ A$ and $A\ \delta^2\ F$ with $i_k^{-1}(U) = E$ and $i_j^{-1}(V) = F$ for $k, j = 1,2$ and $k \neq j$. $i_k(i_k^{-1}(U)) = i_k(E) \subseteq U$ and $i_j(i_j^{-1}(V)) = i_j(F) \subseteq V$.

If $i_k(E)$ subset of the first component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of the second component of $X^2 \vee_\Delta X^2$, then $\{(x,y)_1\} \subseteq i_k(E)$ and $\{(x,y)_2\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_1\}, \{(x,y)_2\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k = 1$ (resp. $k = 2$), then $(\{(x,y)_1\}, \{(x,y)_2\}) \supseteq (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction since $x \neq y$.

Similarly, if $i_k(E)$ subset of the second component of $X^2 \vee_\Delta X^2$ and $i_k(F)$ subset of the first component of $X^2 \vee_\Delta X^2$, then $\{(x,y)_2\} \subseteq i_k(E)$ and $\{(x,y)_1\} \subseteq i_k(F)$. But, if $(i_k(E), i_k(F)) \supseteq (\{(x,y)_2\}, \{(x,y)_1\}) \in \delta'$ for some $(E,F) \in \delta^2$ and $k = 1$ (resp. $k = 2$), then $(\{(x,y)_2\}, \{(x,y)_1\}) \in (i_1(E), i_1(F))$ which shows that $(x,y)_2$ (resp. $(x,y)_1$) must be in the first (resp. second) component of $X^2 \vee_\Delta X^2$, a contradiction since $x \neq y$. Hence $i_k(E)$ and $i_k(F)$ can not be in different component of $X^2 \vee_\Delta X^2$.

Conversely, suppose that for each $x \neq y$, $(\{x\}, \{y\}) \notin \delta$. We need to show that (X, δ) is $PreT_2'$, i.e., by 2.4, 2.6 and 3.1, (*a*) and (*b*) above are equivalent. We first show that (*a*) implies (*b*). Let (U, V) be any set in δ', i.e., $\pi_i S(U)\ \delta\ \pi_i S(V)$ $(i = 1,2,3)$.

If $i_k(E) \subseteq U$ subset of the first component of $X^2 \vee_\Delta X^2$ and $i_k(F) \subseteq V$ subset of the second component of $X^2 \vee_\Delta X^2$, then $\{(x,y)_1\} \supseteq i_k(E)$ and $\{(x,y)_2\} \supseteq i_k(F)$. It follows that $\pi_2 S\{(x,y)_1\}\ \delta\ \pi_2 S\{(x,y)_2\} = \pi_2\{(x,y,y)\}\ \delta\ \pi_2\{(x,x,y)\} = \{y\}\ \delta\ \{x\}$, i.e., $(\{y\},\{x\}) \in \delta$. Since $(\{x\},\{y\}) \notin \delta$ (by assumption), $(\{(x,y)_1\}, \{(x,y)_2\}) \notin \delta'$ by the condition ($P2$) of 2.1.

The case $i_k(E) \subseteq U$ subset of the second component of $X^2 \vee_\Delta X^2$ and $i_k(F) \subseteq V$ subset of the first component of $X^2 \vee_\Delta X^2$ can be handled similarly. Hence $i_k(E)$ and $i_k(F)$ can not be in different component of $X^2 \vee_\Delta X^2$.

If $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in both component of $X^2 \vee_\Delta X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1, (x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1, (x,y)_2\}$.

If $i_k(E) \subseteq U$ subset of the first component of $X^2 \vee_\Delta X^2$ and $i_k(F) \subseteq V$ subset of both component of $X^2 \vee_\Delta X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1, (x,y)_2\}$.

If $i_k(E) \subseteq U$ subset of both component of $X^2 \vee_\Delta X^2$ and $i_k(F) \subseteq V$ subset of the second component of $X^2 \vee_\Delta X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1, (x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_2\}$.

If $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the first component of $X^2 \vee_\Delta X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_1\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_1\}$. Similarly if $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the second component of $X^2 \vee_\Delta X^2$, then $U \supseteq i_k(E) \supseteq \{(x,y)_2\}$ and $V \supseteq i_k(F) \supseteq \{(x,y)_2\}$.

If $(\{(x,y)_i\}, \{(x,y)_i\}) \in \delta'$, $(i = 1,2)$, then $\pi_1 S\{(x,y)_1\}\ \delta\ \pi_1 S\{(x,y)_1\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_2 S\{(x,y)_1\}\ \delta\ \pi_2 S\{(x,y)_1\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$, $\pi_3 S\{(x,y)_1\}\ \delta\ \pi_3 S\{(x,y)_1\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$ and $\pi_1 S\{(x,y)_2\}\ \delta\ \pi_1 S\{(x,y)_2\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_2 S\{(x,y)_2\}\ \delta\ \pi_2 S\{(x,y)_2\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_3 S\{(x,y)_2\}\ \delta\ \pi_3 S\{(x,y)_2\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$.

It follows that $(i_k(E), i_k(F)) \supseteq (\{(x,y)_i\}, \{(x,y)_i\})$, $(i = 1,2)$, i.e., $i_k(E) \subseteq U$ and $i_k(F) \subseteq V$ are in the first or in the second or in both component of $X^2 \vee_\Delta X^2$. So there exists a pair (a,b), $(c,d) \in X^2$ such that $\{(a,b)\}\ \delta^2\ \{(c,d)\}$ and $i_k\{(a,b)\} = U$ and $i_k\{(c,d)\} = V$ for some $k = 1$ or 2. This shows that (*a*) implies (*b*).

We now show that (*b*) implies (*a*). Suppose (*b*) holds. We need to show that for any sets U, V on $X^2 \vee_\Delta X^2$, $\pi_{11}U\ \delta\ \pi_{11}V$, $\pi_{21}U\ \delta\ \pi_{21}V$ and $\pi_{22}U\ \delta\ \pi_{22}V$. There exists a pair (a,b), $(c,d) \in X^2$ such that $\{(a,b)\}\ \delta^2\ \{(c,d)\}$ and $i_k\{(a,b)\} = U$ and $i_k\{(c,d)\} = V$ for some $k = 1$ or 2. By using the similar argument as above, we must have $(i_k\{(a,b)\}, i_k\{(c,d)\}) \supseteq (\{(x,y)_i\}, \{(x,y)_i\})$, $(i = 1,2)$. For $i = 1$ if $(\{(x,y)_1\}, \{(x,y)_1\}) \in \delta'$, then $\pi_1 S\{(x,y)_1\}\ \delta\ \pi_1 S\{(x,y)_1\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_2 S\{(x,y)_1\}\ \delta\ \pi_2 S\{(x,y)_1\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$, $\pi_3 S\{(x,y)_1\}\ \delta\ \pi_3 S\{(x,y)_1\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$. For $i = 2$ if $(\{(x,y)_2\},\{(x,y)_2\}) \in \delta'$, then $\pi_1 S\{(x,y)_2\}\ \delta\ \pi_1 S\{(x,y)_2\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_2 S\{(x,y)_2\}\ \delta\ \pi_2 S\{(x,y)_2\} = \{x\}\ \delta\ \{x\}$, i.e., $(\{x\},\{x\}) \in \delta$, $\pi_3 S\{(x,y)_2\}\ \delta\ \pi_3 S\{(x,y)_2\} = \{y\}\ \delta\ \{y\}$, i.e., $(\{y\},\{y\}) \in \delta$. Hence $\pi_i S(U)\ \delta\ \pi_i S(V)$ $(i = 1,2,3)$. This shows that (b) implies (a).

Hence (X,δ) is $PreT_2'$. □

Remark 3.9. *If an Efremovich proximity space (X,δ) is $PreT_2'$, then it is $Pre\overline{T}_2$. However, the converse is not true generally. For example, let $X = \{1,2\}$ and $\delta = \{(X,X),(\{1\},\{1\}),(\{2\},\{2\}),(X,\{1\}),(\{1\},X),(X,\{2\}),(\{2\},X),(\{1\},\{2\}),(\{2\},\{1\})\}$. Then (X,δ) is $Pre\overline{T}_2$, but it is not $PreT_2'$ since $(\{1\},\{2\}) \in \delta$ but $1 \neq 2$.*

Theorem 3.10. *An Efremovich proximity space (X,δ) is $\overline{T}_2$ iff, for each distinct pair x and y in X, $(\{x\},\{y\}) \notin \delta$.*

Proof. It follows from Definition 3.1 and Theorems 3.3, 3.7. □

Theorem 3.11. *An Efremovich proximity space (X,δ) is T_2' iff, for each distinct pair x and y in X, $(\{x\},\{y\}) \notin \delta$.*

Proof. It follows from Definition 3.1 and Theorems 3.4, 3.8. □

4. T_0 and T_1 Proximity Spaces at a Point p and Relationships

Let B be set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([2] p. 334), i.e., two disjoint copies of B identified at p, or in other words, the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in **Set**, the category of sets). More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1(x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principal p-axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x,p)$ and $A_p(x_2) = (p,x)$. The skewed p-axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x,x)$ and $S_p(x_2) = (p,x)$. The fold map at p, $\nabla_p : B \vee_p B \to B$ is given by $\nabla_p(x_i) = x$ for $i = 1,2$ [2, 4].

Note that the maps A_p, S_p and ∇_p are the unique maps arising from the above pushout diagram for which $A_p i_1 = (id, f)$, $S_p i_1 = (id, id) : B \to B^2$, $A_p i_2 = S_p i_2 = (f, id) : B \to B^2$, and $\nabla_p i_j = id$, $j = 1,2$, respectively, where, $id : B \to B$ is the identity map and $f : B \to B$ is the constant map at p [2].

Remark 4.1. *We define p_1, p_2 by $1 + p$, $p + 1 : B \vee_p B \to B$, respectively where $1 : B \to B$ is the identity map, $f : B \to B$ is constant map at p (i.e., having value p). Note that $\pi_1 A_p = p_1 = \pi_1 S_p$, $\pi_2 A_p = p_2$, $\pi_2 S_p = \nabla_p$, where $\pi_i : B^2 \to B$ is the i-th projection, $i = 1,2$. When showing A_p and S_p are initial it is sufficient to show that (p_1 and p_2) and (p_1 and ∇_p) are initial lifts, respectively [2, 4].*

Definition 4.2. *[2] Let $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set** be a topological functor, X an object in $\mathcal{E}$, and p be a point in $\mathcal{U}(X) = B$.*

1. *X is $\overline{T}_0$ at p iff the initial lift of the $\mathcal{U}$-source $\{A_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.*
2. *X is T_0' at p iff the initial lift of the $\mathcal{U}$ -source $\{id : B \vee_p B \to \mathcal{U}(X \vee_p X) = B \vee_p B$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete, where $X \vee_p X$ is the wedge in $\mathcal{E}$ i.e., the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X) = B \to B \vee_p B\}$ where i_1, i_2 denote the canonical injections.*
3. *X is T_1 at p iff the initial lift of the $\mathcal{U}$-source $\{S_p : B \vee_p B \to \mathcal{U}(X^2) = B^2$ and $\nabla_p : B \vee_p B \to \mathcal{U}\mathcal{D}(B) = B\}$ is discrete.*

Remark 4.3. *Note that for the category **Top** of topological spaces, $\overline{T}_0$ at p, T_0' at p, or T_1 at p reduce to usual T_0 at p or T_1 at p, respectively, where a topological space X is called T_0 at p (resp. T_1 at p) if for each $x \neq p$, there exists a neighborhood of x not containing p or (resp. and) there exists a neighborhood of p not containing x [7].*

Theorem 4.4. *Let (X,δ) be an Efremovich proximity space and $p \in X$. (X,δ) is T_1 at p iff, for each $x \neq p$, $(\{x\},\{p\}) \notin \delta$ [25].*

Theorem 4.5. *Let (X, δ) be an Efremovich proximity space and $p \in X$. (X, δ) is $\overline{T}_0$ at p iff, for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$ [25].*

Remark 4.6. *Let (X, δ) be an Efremovich proximity space and $p \in X$. It follows from 4.4, 4.5 that (X, δ) is $\overline{T}_0$ at p if and only if (X, δ) is T_1 at p if and only if, for each $x \neq p$, $(\{x\}, \{p\}) \notin \delta$ [25].*

Theorem 4.7. *An Efremovich proximity space is T_0' at p [25].*

Remark 4.8. *If an Efremovich proximity space (X, δ) is $\overline{T}_0$ at $p \in X$ or T_1 at $p \in X$, then it is T_0' at p. However, the converse is not true generally. For example, let $X = \{a, b\}$ and $\delta = \{(X, X), (\{a\}, \{a\}), (\{b\}, \{b\}), (X, \{a\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}$. Then (X, δ) is T_0' at $p = a$ but it is not $\overline{T}_0$ at $p = a$ or T_1 at p since $(\{a\}, \{b\}) \in \delta$ but $a \neq b$ [25].*

Definition 4.9. *[28, 32] An Efremovich proximity space (X, δ) is said to be a*

- $\mathbf{T_0}$-***space*** *if $x \neq y$ for $x, y \in X$ implies that $x \not\delta y$.*
- $\mathbf{T_1}$-***space*** *if $x \neq y$ for $x, y \in X$ implies that $x \not\delta y$.*
- $\mathbf{T_2}$-***space (Hausdorff)*** *if $x \,\delta\, y$ for $x, y \in X$ implies that $x = y$.*

We give explicit relationships among the generalized separation properties T_0 and T_1, the separation properties at a point p and separation properties T_i, $i = 0, 1, 2$ in Definition 4.9 in the topological category of proximity spaces.

Remark 4.10. *Let (X, δ) be an Efremovich proximity space.*

(i) *By Theorems 3.3, 3.5, 3.8, 3.10, 3.11, and Definition 4.9, (X, δ) is $\overline{T}_0$ if and only if (X, δ) is $\mathbf{T_0}$ if and only if (X, δ) is T_1 if and only if (X, δ) is $\mathbf{T_1}$ if and only if (X, δ) is $PreT_2'$ if and only if (X, δ) is $\overline{T}_2$ if and only if (X, δ) is T_2' if and only if (X, δ) is $\mathbf{T_2}$ if and only if, for each distinct pair x and y in X, $(\{x\}, \{y\}) \notin \delta$.*

(ii) *By Theorems 3.3, 3.4, 3.5, 3.7, 3.8, 3.10, 3.11, and Definition 4.9, (X, δ) is T_0' or $Pre\overline{T}_2$ if (X, δ) is $\overline{T}_0$ or $\mathbf{T_0}$ or T_1 or $\mathbf{T_1}$ or $PreT_2'$ or $\overline{T}_2$ or T_2' or $\mathbf{T_2}$. But the converse implication is not true, in general. For example, let $X = \{x, y\}$ and $\delta = \{(X, X), (\{x\}, \{x\}), (\{y\}, \{y\}), (X, \{x\}), (\{x\}, X), (X, \{y\}), (\{y\}, X), (\{x\}, \{y\}), (\{y\}, \{x\})\}$. Then (X, δ) is T_0' and $Pre\overline{T}_2$ but it is not $\overline{T}_0$ or $\mathbf{T_0}$ or T_1 or $\mathbf{T_1}$ or $PreT_2'$ or $\overline{T}_2$ or T_2' or $\mathbf{T_2}$, since $(\{x\}, \{y\}) \in \delta$ but $x \neq y$.*

(iii) *By Theorems 3.4, 3.7 and 4.7, (X, δ) is $Pre\overline{T}_2$ if and only if (X, δ) is T_0' if and only if (X, δ) is T_0' at p for all points p in X.*

We can infer the following result.

Remark 4.11. *Let (X, δ) be an Efremovich proximity space. By Theorems 3.3, 3.5, 3.8, 3.10, 3.11, 4.4, 4.5, Definition 4.9, and Remark 4.10 then the following are equivalent:*

(i) *(X, δ) is $\overline{T}_0$.*
(ii) *(X, δ) is $\mathbf{T_0}$.*
(iii) *(X, δ) is T_1.*
(iv) *(X, δ) is $\mathbf{T_1}$.*
(v) *(X, δ) is $PreT_2'$.*
(vi) *(X, δ) is $\overline{T}_2$.*
(vii) *(X, δ) is T_2'.*
(viii) *(X, δ) is $\mathbf{T_2}$.*
(ix) *For any distinct pair of points x and y in X, $(\{x\}, \{y\}) \notin \delta$.*
(x) *(X, δ) is T_1 at p for all points p in X.*
(xi) *(X, δ) is $\overline{T}_0$ at p for all points p in X.*

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