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The Split Common Fixed Point Problem of two Infinite Families of Demicontractive Mappings and the Split Common Null Point Problem

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Abstract. In this paper we introduce a new algorithm based on the viscosity iteration method for solving the split common fixed point problem of two infinite families of k-demicontractive mappings. We shall also study the split common null point problem, and the split equilibrium problem for this class of mappings. As an application, we obtain strong convergence theorems for the split monotone variational inclusion problem and the split variational inequality problem. Our results improve and extend the recent results of Cui and Wang [9], Takahashi [21], Tang and Lui [22], Moudafi [15], Eslamian and Vahidi [17], and many others.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Given nonlinear operators $T : H_1 \longrightarrow H_1$ and $U : H_2 \longrightarrow H_2$, the split fixed point problem (SFPP) is to find a point

$$x \in Fix(T) \quad \text{such that} \quad Ax \in Fix(U) \tag{1}$$

where $Fix(T)$ and $Fix(U)$ stand for, respectively, the fixed point sets of T and U. The (SFPP) has the following extension:
Let $T_i : H_1 \longrightarrow H_1$, $(1 \le i \le m)$ be nonlinear operators on H_1, and let $U_j : H_2 \longrightarrow H_2$, $(1 \le j \le n)$ be nonlinear operators on H_2. Then the split common fixed point problem (SCFPP) is to find a point

$$x \in \bigcap_{i=1}^{m} Fix\,(T_i) \quad \text{such that} \quad Ax \in \bigcap_{j=1}^{n} Fix\left(U_j\right). \tag{2}$$

In particular, if $T_i = P_{C_i}$ and $U_j = P_{Q_j}$, then the SCFPP (2) reduces to the multiple-sets split feasibility problem (MSSFP): find

$$x \in \bigcap_{i=1}^{m} P_{C_i} \quad \text{such that} \quad Ax \in \bigcap_{j=1}^{n} P_{Q_j}.$$

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where $\{C_i\}_{i=1}^m$ and $\{Q_j\}_{j=1}^n$ are nonempty closed convex sets in H_1 and H_2, respectively.

In [23] Censor and Segal introduced the iterative scheme

$$x_{n+1} = U\left(I - \rho_n A^*\left(I - T\right)A\right)x_n$$

which solves the problem (1) for directed operators. This algorithm was then extended to the case of quasi-nonexpansive mappings [13], as well as to the case of demicontractive mappings [14]. Finally, Wang et al. [24] solved the problem for infinitely many directed operators.

Moudafi [13] then introduced the following relaxed algorithm with weak convergence for the split fixed point problem

$$u_n = x_n + \gamma\beta A^*\left(T - I\right)Ax_n$$

$$x_{n+1} = \left(1 - \alpha_n\right)U_n + \alpha_n U\left(u_n\right)$$

where $\alpha_n \in (\delta, 1-\delta)$ for a small enough $\delta > 0$, $\beta \in (0,1)$ and $\gamma \in \left(0, \frac{1}{\lambda\beta}\right)$ with λ being the spectral radius of the operator A^*A. In [17], Eslamian and Vahidi studied an algorithm for solving the split common fixed point problem for an infinite family of quasi-nonexpansive mappings. They established the following theorem.

Theorem 1.1. *[17] Let H_1 and H_2 be two real Hilbert spaces, and $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Let $S_i : H_1 \longrightarrow H_1$ and $T_i : H_2 \longrightarrow H_2$, $(i \in \mathbb{N})$, be two infinite families of quasi-nonexpansive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Suppose that $\Omega = \{x \in \cap_{i=1}^{\infty} Fix\left(S_i\right) : Ax \in \cap_{i=1}^{\infty} Fix\left(T_i\right)\} \neq \emptyset$. Let f be a k-contraction of H_1 into itself, and $\{x_n\}$ be a sequence generated by $x_0 \in H_1$ and by*

$$\begin{cases} y_n = x_n + \sum_{i=1}^{\infty} \beta_{n,i}\gamma\beta A^*\left(T_i - I\right)Ax_n, \\ u_n = \alpha_{n,0}y_n + \sum_{i=1}^{\infty} \alpha_{n,i}S_i y_n, \\ x_{n+1} = \nu_n f\left(u_n\right) + \left(1 - \nu_n\right)u_n, \end{cases}$$

where $\beta \in (0,1)$, and $\gamma \in \left(0, \frac{1}{\lambda\beta}\right)$ with λ being the spectral radius of the operator A^*A, and the sequences $\{\alpha_{n,i}\}$, $\{\beta_{n,i}\}$ and $\{\nu_n\}$ satisfy the following conditions:
(i) $\sum_{j=1}^{\infty} \alpha_{n,j} = 1$, and $\liminf_n \alpha_{n,0}\alpha_{n,i} > 0$, for all $i \in \mathbb{N}$,
(ii) $\sum_{j=1}^{\infty} \beta_{n,j} = 1$, and $\liminf_n \beta_{n,i} > 0$, for all $i \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \nu_n = 0$ and $\sum_{n=1}^{\infty} \nu_n = \infty$.
Then, the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality

$$\langle x^* - f\left(x^*\right), x - x^*\rangle \geq 0, \quad \forall x \in \Omega.$$

Let $T : H \longrightarrow 2^H$ be a multivalued mapping with graph $G(T) = \{(x,y) : y \in Tx\}$, domain $D(T) = \{x \in H : Tx \neq \emptyset\}$ and range $R(T) = \cup\{Tx : x \in D(T)\}$. The mapping T is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $(x,u), (y,v) \in G(T)$. We denote the set $\{x \in H : 0 \in Tx\}$ by $T^{-1}(0)$. A monotone operator $T \subset H \times H$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. If $T \subset H \times H$ is maximal monotone, then the solution set $T^{-1}(0)$ is closed and convex. An operator T on a Hilbert space H is maximal if and only if $R(I + rT) = H$ for $r > 0$ (see Barbu [1]). If $T \subset H \times H$ is a maximal monotone operator, then for each $r > 0$ and $x \in H$, there corresponds a unique element $x_r \in D(T)$ satisfying

$$x = x_r + rTx_r.$$

We define the resolvent of T by $J_r x = x_r$. In other words, $J_r = (I + rT)^{-1}$ for all $r > 0$. The resolvent J_r is a single-valued mapping from H into $D(T)$. It is easy to see that $T^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ denotes the set of fixed points of J_r. We can also define, for each $r > 0$, the Yosida approximation of T by $T_r = (I - J_r)/r$. We know that $(J_r x, T_r x) \in T$ for all $r > 0$ and $x \in H$.

Byrne et al. [4] considered the following problem: For given set-valued mappings $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, and bounded linear operators $T_j : H_1 \to H_2$, the split common null point problem is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^{m} A_i^{-1}0\right) \cap \left(\bigcap_{j=1}^{n} T_j^{-1}B_j^{-1}0\right)$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j, respectively.

Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem introduced by Blum and Oettli [3] for $F : C \times C \longrightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \qquad \forall y \in C. \tag{3}$$

The set of solutions of (3) is denoted by EP(F). Numerous problems in physics, optimization, and economics reduce to finding a solution of (3) (see [8], [23]). The split equilibrium problem was introduced by Moudafi in [15]; indeed he considered the following pair of equilibrium problems in different spaces. Let H_1 and H_2 be two real Hilbert spaces, and $F_1 : C \times C \longrightarrow \mathbb{R}$ and $F_2 : Q \times Q \longrightarrow \mathbb{R}$ be nonlinear bifunctions, and let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Consider the nonempty closed convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$; then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \qquad \forall x \in C$$

and such that

$$y^* = Ax^* \in Q, \quad F_2(y^*, y) \geq 0, \qquad \forall y \in Q.$$

In this paper, we present a new algorithm based on the viscosity iterative method for solving the split common fixed point problem for k-demicontractive mappings, as well as the split common null point problem and the split equilibrium problem. We also consider some particular cases such as quasi-nonexpansive operators and directed operators. As application, we obtain strong convergence theorems for split monotone variational inclusion and split variational inequality problems. Our results improve and extend some recent results due to Cui and Wang [9], Takahashi [21], Tang and Liu [22], Moudafi [15], as well as Eslamian and Vahidi [17].

2. Preliminaries

In this section, we collect some basic facts which are needed for the proofs of the main results of this paper.

Lemma 2.1. *[20] Let C be a nonempty closed convex subset of a real Hilbert space H and $x \in H$. Then $x_0 = P_C x$ if and only if for all $y \in C$, $\langle x_0 - y, x - x_0 \rangle \geq 0$.*

Let T be a maximal monotone operator on a real Hilbert space H. It is known that the resolvent J_r of T for $r > 0$ is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y \rangle, \quad \forall x, y \in H.$$

It is also known that the inequality

$$\|J_\lambda x - J_\mu x\| \leq \left(\frac{|\lambda - \mu|}{\lambda}\right) \|x - J_\lambda x\|$$

holds true for all $\lambda, \mu > 0$ and $x \in H$ (for details, see [18]). Moreover, we have the following lemma due to Takahashi et al. [19].

Lemma 2.2. *[19] Let H be a real Hilbert space and let T be a maximal monotone operator on H. Then we have*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s,t>0$ and $x \in H$.

Lemma 2.3. *[7] Let H be a real Hilbert space and let $B_1(0) = \{x \in E : \|x\| \leq 1\}$. For any given sequence $\{x_n\}_{n=1}^{\infty} \subset B_1(0)$ and for any given sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty}\alpha_n = 1$ and for any positive integers i, j with $i<j$,*

$$\|\sum_{n=1}^{\infty}\alpha_n x_n\|^2 \leq \sum_{n=1}^{\infty}\alpha_n\|x_n\|^2 - \alpha_i\alpha_j\|x_i - x_j\|^2.$$

Lemma 2.4. *[25] Let $\{\gamma_n\}$ be a sequence in $(0,1)$ and $\{\delta_n\}$ be a sequence in $\mathbb{R}$ satisfying*
(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$,
(ii) $\limsup_{n\to\infty}\gamma_n \leq 0$ or $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$.
If $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)\,a_n + \gamma_n\delta_n,$$

for each $n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[11] Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \leq s_{n_{i+1}}$ for all $i \geq 0$. For every $n \in \mathbb{N}$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ and $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.

We call a bounded linear operator B on a real Hilbert space H strongly positive if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \forall x \in H.$$

Lemma 2.6. *[12] Assume that A is a strongly positive self-adjoint bounded linear operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

For solving the equilibrium problem, we assume that C is a nonempty closed convex subset of a real Hilbert space H. Let us assume that $f : C \times C \longrightarrow \mathbb{R}$ is a bifunction satisfying the following conditions:
(A_1) $F(x,x) = 0$ for all $x \in C$,
(A_2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$,
(A_3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x,y),$$

(A_4) $F(x,0)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.7. *[3] Let C be nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - x, z - x\rangle \geq 0, \qquad \forall y \in C.$$

Lemma 2.8. *[8] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A_1) - (A_4)$, and let $r > 0$ and $x \in E$. Define a mapping $T_r : E \longrightarrow C$ as follows:*

$$T_r x = \{z \in C : F(z, x) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following hold:
(i) T_r is single-valued;
(ii) T_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle.$$

(iii) $F(T_r) = EP(F)$;
(iv) EP(F) is closed and convex.

Definition 2.9. *Let $T : C \longrightarrow C$ be a mapping, then $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, imply $x = Tx$.*

Definition 2.10. *Let $T : H \longrightarrow H$ be a mapping with $Fix(T) \neq \emptyset$. Then*
(i) $T : H \longrightarrow H$ is called directed if

$$\langle z - Tx, x - Tx\rangle \leq 0, \qquad \forall z \in Fix(T), \quad \forall x \in H.$$

(ii) $T : H \longrightarrow H$ is called quasi-nonexpansive if

$$\|Tx - z\| \leq \|x - z\|, \qquad \forall z \in Fix(T), \quad \forall x \in H.$$

(iii) $T : H \longrightarrow H$ is called k-demicontractive with $k \leq 1$, if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + k\|(I - T)x\|^2, \qquad \forall z \in Fix(T), \quad \forall x \in H$$

or equivalently

$$\langle x - z, Tx - x\rangle \leq \frac{k-1}{2}\|x - Tx\|^2, \qquad \forall z \in Fix(T), \quad \forall x \in H.$$

(iv) $T : H \longrightarrow H$ is called averaged if there exists a nonexpansive operator $N : H \longrightarrow H$ and a number $\lambda \in (0, 1)$ such that

$$T = (1 - \lambda) I + \lambda N.$$

A typical example of a directed operator is the orthogonal projection P_C from a Hilbert space H onto a nonempty closed convex subset $C \subset H$ defined by

$$P_C x := \underset{y \in C}{argmin} \|x - y\|^2, \quad x \in H.$$

It is well known that the projection P_C is characterized by

$$y = P_C x \quad \text{if and only if} \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad \forall z \in C.$$

3. Main Results

This section is devoted to the main results of this paper. We start by proving a split common fixed point, and common null point problem.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, and $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that $F_1 : H_1 \longrightarrow 2^{H_1}$ and $F_2 : H_2 \longrightarrow 2^{H_2}$ are two maximal monotone operators such that $F_1^{-1}0 \neq \emptyset$ and $F_2^{-1}0 \neq \emptyset$. Let J_r be the resolvent of F_1 for $r > 0$ and Q_μ be the resolvent of F_2 for $\mu > 0$. Let, for $i \in \mathbb{N}$, $T_i : H_2 \longrightarrow H_2$ be an infinite family of k-demicotractive mappings and $S_i : H_1 \longrightarrow H_1$ be an infinite family of l-demicotractive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Assume further that $\Omega = \{x \in \left(F_1^{-1}0\right) \cap \left(\bigcap_{i=1}^{\infty} Fix\,(S_i)\right) : Ax \in \left(F_2^{-1}0\right) \cap \left(\bigcap_{i=1}^{\infty} Fix\,(T_i)\right)\} \neq \emptyset$. Suppose that f is a b-contraction of H_1 into itself and that B is a strongly positive bounded linear operator on H_1 with coefficient $\bar{\gamma} \geq 0$ and $0 < \gamma < \frac{\bar{\gamma}}{k}$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_n = J_{r_n}\left(x_n + \lambda A^*\left(\left(\alpha_n Q_{\mu_n} + \beta_n I + \sum_{i=1}^{\infty} \gamma_{n,i} T_i\left(Q_{\mu_n}\right) - I\right) A x_n\right)\right), \\ w_n = \delta_{n,0} y_n + \sum_{i=1}^{\infty} \delta_{n,i} S_i y_n, \\ x_{n+1} = a_n \gamma f(w_n) + (1 - a_n B) w_n, \end{cases} \tag{4}$$

where $\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$ and A^* is the adjoint of A. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{r_n\}$ and $\{a_n\}$ satisfy the following conditions:
(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$ *and* $\delta_{n,0} + \sum_{i=1}^{\infty} \delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $0 < a \leq r_n < \infty$ *and* $0 < b \leq \mu_n < \infty \quad (a, b \in \mathbb{R})$,
(iv) $k < \alpha_n < 1$ *and* $l < \delta_{n,0} < 1$,
(v) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty}(\alpha_n - k)\gamma_{n,i} > 0$ *and* $\liminf_{n\to\infty}(\delta_{n,0} - l)\delta_{n,i} > 0$.
Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality

$$\langle (B - \gamma f)\, x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega.$$

Proof. Since $P_\Omega\,(I - B + \gamma f)$ is a contraction, by the Banach contraction principle there exists $q \in \Omega$ such that $q = P_\Omega\,(I - B + \gamma f)\,(q)$; this is equivalent to saying that

$$\langle (I - B + \gamma f)\, q - q, q - p \rangle \geq 0, \quad \forall p \in \Omega.$$

Since $\lim_{n\to\infty} a_n = 0$, we may assume that $a_n \in \left(0, \|A\|^{-1}\right)$ for all $n \geq 0$. By Lemma 2.6, we have $\|I - a_n B\| \leq 1 - a_n\bar{\gamma}$. Take $p \in \Omega$ and $u_n = Q_{\mu_n} A x_n$. From Lemma 2.8, we know that for any $n \geq 0$,

$$\|u_n - Ap\| = \|Q_{\mu_n} A x_n - Q_{\mu_n} Ap\| \leq \|Ax_n - Ap\|.$$

We show that $\{x_n\}$ is bounded. Take $z_n = \alpha_n u_n + \beta_n A x_n + \sum_{i=1}^{\infty} \gamma_{n,i} T_i\,(u_n)$. Since for each $i \in \mathbb{N}$, T_i is demicontractive, from Lemma 2.3 we have

$$\begin{aligned} \|z_n - Ap\|^2 &= \|\alpha_n u_n + \beta_n A x_n + \sum_{i=1}^{\infty} \gamma_{n,i} T_i\,(u_n) - Ap\|^2 \\ &\leq \alpha_n \|u_n - Ap\|^2 + \beta_n \|Ax_n - Ap\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i} \|T_i\,(u_n) - Ap\|^2 \\ &- \alpha_n\beta_n \|Ax_n - u_n\|^2 - \alpha_n \gamma_{n,i} \|u_n - T_i u_n\|^2 \\ &\leq \alpha_n \|u_n - Ap\|^2 + \beta_n \|Ax_n - Ap\|^2 + \sum_{i=1}^{\infty} \gamma_{n,i} \left(\|u_n - Ap\|^2 + k\|u_n - T_i\,(u_n)\,\|^2\right) \\ &- \alpha_n\beta_n \|Ax_n - u_n\|^2 - \alpha_n \gamma_{n,i} \|u_n - T_i u_n\|^2 \\ &\leq \|Ax_n - Ap\|^2 - \alpha_n\beta_n \|Ax_n - u_n\|^2 - (\alpha_n - k) \sum_{i=1}^{\infty} \gamma_{n,i} \|u_n - T_i u_n\|^2 \\ &\leq \|Ax_n - Ap\|^2. \end{aligned}$$

From Lemma 2.8, we also have

$$\begin{aligned}
\|y_n - p\|^2 &= \|J_{r_n}\left(x_n + \lambda A^*\left(z_n - Ax_n\right)\right) - p\|^2 \\
&\leq \|x_n + \lambda A^*\left(z_n - Ax_n\right) - p\|^2 \\
&= \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\langle x_n - p, \lambda A^*\left(z_n - Ax_n\right)\rangle \\
&= \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\lambda\langle Ax_n - Ap, z_n - Ax_n\rangle \\
&= \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\lambda\left(\langle z_n - Ap, z_n - Ax_n\rangle - \|z_n - Ax_n\|^2\right) \\
&\leq \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\lambda\left(\frac{1}{2}\left(\|z_n - Ap\|^2\right.\right. \\
&\left.\left.+\|z_n - Ax_n\|^2 - \|Ax_n - Ap\|^2\right) - \|z_n - Ax_n\|^2\right) \\
&\leq \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\lambda\left(\frac{1}{2}\left(\|Ax_n - Ap\|^2 - \alpha_n\beta_n\|Ax_n - u_n\|^2\right.\right. \\
&\left.\left.- (\alpha_n - k)\sum_{i=1}^{\infty}\gamma_{n,i}\|u_n - T_iu_n\|^2 + \|z_n - Ax_n\|^2 - \|Ax_n - Ap\|^2\right) - \|z_n - Ax_n\|^2\right) \\
&\leq \|x_n - p\|^2 + \lambda^2\|A\|^2\|z_n - Ax_n\|^2 + 2\lambda\left(\frac{1}{2}\left(\|z_n - Ax_n\|^2 - \alpha_n\beta_n\|Ax_n - u_n\|^2\right)\right. \\
&\left.- (\alpha_n - k)\sum_{i=1}^{\infty}\gamma_{n,i}\|u_n - T_iu_n\|^2\right) - \|z_n - Ax_n\|^2\Big) \\
&= \|x_n - p\|^2 + \lambda\left(\lambda\|A\|^2 - 1\right)\|z_n - Ax_n\|^2 - \lambda\alpha_n\beta_n\|Ax_n - u_n\|^2 \\
&- \lambda\left(\alpha_n - k\right)\sum_{i=1}^{\infty}\gamma_{n,i}\|u_n - T_iu_n\|^2.
\end{aligned}$$

Similarly, since for each $i \in \mathbb{N}$, S_i is demicontractive, it follows from Lemma 2.3 that

$$\begin{aligned}
\|w_n - p\|^2 &= \|\delta_{n,0}y_n + \sum_{i=1}^{\infty}\delta_{n,i}S_iy_n - p\|^2 \\
&\leq \delta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{\infty}\delta_{n,i}\|S_iy_n - p\|^2 - \delta_{n,0}\delta_{n,i}\|y_n - S_iy_n\|^2 \\
&\leq \delta_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{\infty}\delta_{n,i}\left(\|y_n - p\|^2 + l\|y_n - S_iy_n\|^2\right) - \delta_{n,0}\delta_{n,i}\|y_n - S_iy_n\|^2 \\
&= \|y_n - p\|^2 - (\delta_{n,0} - l)\sum_{i=1}^{\infty}\delta_{n,i}\|y_n - S_iy_n\|^2 \\
&\leq \|y_n - p\|^2 - (\delta_{n,0} - l)\sum_{i=1}^{\infty}\delta_{n,i}\|y_n - S_iy_n\|^2 \\
&\leq \|x_n - p\|^2 + \lambda\left(\lambda\|A\|^2 - 1\right)\|z_n - Ax_n\|^2 - \lambda\alpha_n\beta_n\|Ax_n - u_n\|^2 \\
&- \lambda\left(\alpha_n - k\right)\sum_{i=1}^{\infty}\gamma_{n,i}\|u_n - T_iu_n\|^2 - (\delta_{n,0} - l)\sum_{i=1}^{\infty}\delta_{n,i}\|y_n - S_iy_n\|^2.
\end{aligned}$$

Hence $\|w_n - p\| \leq \|x_n - p\|$. From Lemma 2.6, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|a_n(\gamma f(w_n)-Bp)+(I-a_nB)(w_n-p)\|\\ &\le a_n\|(\gamma f(w_n)-Bp\|+\|I-a_nB\|\|(w_n-p))\|\\ &\le a_n(\gamma\|f(w_n)-f(p)\|+\|\gamma f(p)-Bp\|)+(I-a_n\bar{\gamma})\|(w_n-p)\|\\ &\le a_nb\gamma\|w_n-p\|+a_n\|\gamma f(p)-Bp\|+(I-a_n\bar{\gamma})|(x_n-p)\|\\ &\le (I-a_n(\bar{\gamma}-b\gamma))\|x_n-p\|+a_n\|\gamma f(p)-Bp\|.\end{aligned}$$

Using mathematical induction, we obtain that

$$\|x_n-p\|\le\max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-Bp}{\bar{\gamma}-\gamma k}\right\},\quad n\ge 0.$$

This argument shows that $\{x_n\}$ is bounded. Now, it is easy to see that $\{w_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{f(w_n)\}$ are bounded too. Since $x_{n+1}-w_n=a_n(\gamma f(w_n)-Bw_n)$ and $\lim_{n\to\infty}a_n=0$, we obtain

$$x_{n+1}-w_n\to 0,\quad n\to\infty.$$

Next, we want to show that for each natural number i,

$$\lim_{n\to\infty}\|T_iu_n-u_n\|=0,\quad \lim_{n\to\infty}\|S_iy_n-y_n\|=0.$$

To this end, we note that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|a_n\gamma f(w_n)+(1-a_nB)w_n-p\|^2\\ &\le \|I-a_nB\|\|w_n-p\|^2+a_n\|\gamma f(w_n)-Bp\|^2\\ &\le (1-a_n\bar{\gamma})[\|x_n-p\|^2+\lambda\left(\lambda\|A\|^2-1\right)\|z_n-Ax_n\|^2-\lambda\alpha_n\beta_n\|Ax_n-u_n\|^2\\ &\quad-\lambda(\alpha_n-k)\sum_{i=1}^{\infty}\gamma_{n,i}\|u_n-T_iu_n\|^2-(\delta_{n,0}-l)\sum_{i=1}^{\infty}\delta_{n,i}\|y_n-S_iy_n\|]+a_n\gamma\|f(w_n)-Bp\|^2.\end{aligned}$$

This implies that

$$(1-a_n\bar{\gamma})\lambda\left(\lambda\|A\|^2-1\right)\|z_n-Ax_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+a_n\gamma\|f(w_n)-Bp\|^2, \tag{5}$$

and

$$(1-a_n\bar{\gamma})(\delta_{n,0}-l)\sum_{i=1}^{\infty}\delta_{n,i}\|y_n-S_iy_n\|\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\nu_n\gamma\|f(u_n)-Bx^*\|^2. \tag{6}$$

We now consider two cases:
Case 1: Assume that $\{\|x_n-p\|\}$ is a monotone sequence. We may assume that $\{\|x_n-p\|\}_{n\ge n_0}$ is either nondecreasing or nonincreasing. Since $\{\|x_n-p\|\}$ is bounded, it is convergent. Since $\lim_{n\to\infty}a_n=0$ and $\{f(u_n)\}$ and $\{x_n\}$ are bounded, in view of inequalities (5) and (6) we conclude that

$$\lim_{n\to\infty}(1-a_n\bar{\gamma})\lambda\left(\lambda\|A\|^2-1\right)\|z_n-Ax_n\|^2=0,$$

and

$$\lim_{n\to\infty}(1-a_n\bar{\gamma})(\delta_{n,0}-l)\delta_{n,i}\|y_n-S_iy_n\|=0.$$

By assumptions that $\liminf_n\delta_{n,i}(\delta_{n,0}-l)>0$, $\lim_{n\to\infty}a_n=0$ and $\lambda\in\left(0,\frac{1}{\|A\|^2}\right)$ we have

$$\lim_{n\to\infty}\|z_n-Ax_n\|=0, \tag{7}$$

and

$$\lim_{n\to\infty} \|y_n - S_i y_n\| = 0. \tag{8}$$

Using a similar argument, we can prove that

$$\lim_{n\to\infty} \|u_n - T_i u_n\| = 0, \tag{9}$$

$$\lim_{n\to\infty} \|Ax_n - u_n\| = 0. \tag{10}$$

Note that, from (4), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|J_{r_n}(x_n + \lambda A^*(z_n - Ax_n)) - J_{r_n} p\|^2 \\ &\le \langle y_n - p, x_n + \lambda A^*(z_n - Ax_n) - p\rangle \\ &= \frac{1}{2}\{\|y_n - p\|^2 + \|x_n + \lambda A^*(z_n - Ax_n) - p\|^2 - \|y_n - p - (x_n + \lambda A^*(z_n - Ax_n) - p)\|^2\} \\ &\le \frac{1}{2}\{\|y_n - p\|^2 + \|x_n - p\|^2 - \|y_n - x_n\|^2 - \lambda^2\|A^*(z_n - Ax_n)\|^2 \\ &+ 2\lambda\|A^*\|\|(y_n - x_n)\|\|z_n - Ax_n\|\}.\end{aligned}$$

On the other hand,

$$\|x_{n+1} - p\|^2 \le (1 - a_n\bar{\gamma})\left(\|y_n - p\|^2 - (\delta_{n,0} - l)\sum_{i=1}^{\infty} \delta_{n,i}\|y_n - S_i y_n\|^2\right) + a_n\gamma\|f(w_n) - Bp\|^2.$$

So, we have

$$\begin{aligned}(1 - a_n\bar{\gamma})\|x_n - y_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - (1 - a_n\bar{\gamma})(\delta_{n,0} - l)\sum_{i=1}^{\infty}\left(\delta_{n,i}\|y_n - S_i y_n\|^2\right) \\ &+ a_n\gamma\|f(w_n) - Bp\|^2 - a_n\bar{\gamma}\|x_n - p\| + 2\lambda(1 - a_n\bar{\gamma})\|Ay_n - Ax_n\|\|z_n - Ax_n\|.\end{aligned}$$

Since $\|x_n - p\|$ is convergent, $\lim_{n\to\infty} a_n = 0$, $\{x_n\}$, $\{y_n\}$ and $\{f(w_n)\}$ are bounded, it follows from (7) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{11}$$

Next, we show that

$$\limsup_{n\to\infty}\langle Bq - f(q), x_n - q\rangle \ge 0.$$

To prove this inequality, we can choose a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}\langle Bq - f(q), x_{ni} - q\rangle = \limsup_{n\to\infty}\langle Bq - f(q), x_n - q\rangle \ge 0.$$

Since $\{x_{ni}\}$ is a bounded sequence in a reflexive Banach space, there exists a subsequence $\{x_{ni_j}\}$ of $\{x_{ni}\}$ which converges weakly to ν. Without less of generality, we may assume that $x_{ni} \rightharpoonup \nu$. Since $\lim_{n\to\infty}\|Ax_n - u_n\| = 0$, we have $u_n \rightharpoonup A\nu$. We show that $\nu \in \Omega$. Let us verify that $A\nu \in F_2^{-1}0$. Note that $u_n = Q_{\mu_n}Ax_n$ and for $(h, h^*) \in G(F_2)$, we have $\langle h - u_n, h^* - \frac{Ax_n - u_n}{\mu_n}\rangle \ge 0$. Because of $\left(u_n, \frac{Ax_n - u_n}{\mu_n}\right) \in G(F_2)$ and the fact that F_2 is a monotone operator, $\|Ax_n - u_n\| \to 0$ and the condition (*iii*) we get

$$\frac{\|Ax_n - u_n\|}{\mu_n} \to 0.$$

Recall that $u_n = Q_{\mu_n}Ax_n \rightharpoonup A\nu$. Thus $\langle h - A\nu, h^*\rangle \ge 0$. So, the maximality of A implies that $A\nu \in F_2^{-1}0$. Now, from $y_{n_i} \rightharpoonup \nu$, the fact that $\lim_{n\to\infty}\|y_n - S_i y_n\| = 0$, and the demiclosedness of $I - S_i$ at zero, we conclude that $\nu \in \bigcap_{i=1}^{\infty} F(S_i)$. Note also that A is a bounded operator and $x_n \rightharpoonup \nu$, therefore $Ax_n \rightharpoonup A\nu$. In view of the inequality (9) and a similar argument as above we conclude that $A\nu \in \bigcap_{i=1}^{\infty} F(T_i)$.

Now to prove $\nu \in F_1^{-1}0$, we take $k_n = x_n + \lambda A^*(z_n - Ax_n)$. From inequality (3.12), it is easy to see that

$$\lim_{n\to\infty} \|k_n - x_n\| = 0. \tag{12}$$

Combining (11) with (12), we conclude that

$$\lim_{n\to\infty} \|y_n - k_n\| \le \lim_{n\to\infty} (\|y_n - x_n\| + \|x_n - k_n\|) = 0.$$

Since $y_n = J_{r_n} k_n$, a similar argument as above reveals that $\nu \in F_1^{-1}0$. This implies that $\nu \in \Omega$. Since $q = P_\Omega (I - B + \gamma f)(q)$ and $\nu \in \Omega$, we have

$$\lim_{i\to\infty} \langle Bq - f(q), x_{ni} - q\rangle = \limsup_{n\to\infty} \langle Bq - f(q), x_n - q\rangle = \langle Bq - f(q), \nu - q\rangle \ge 0.$$

As $x_{n+1} - q = \nu_n(\gamma f(w_n) - Bq) + (1 - \nu_n B)(w_n - q)$, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|a_n \gamma f(w_n) + (1 - a_n B) w_n - q\|^2 \\
&\le \|(I - a_n B)(w_n - q)\|^2 + 2a_n \langle \gamma f(w_n) - Bq, x_{n+1} - q\rangle \\
&\le (1 - a_n \bar{\gamma})^2 \|x_n - q\|^2 + 2a_n \langle \gamma f(w_n) - \gamma f(q), x_{n+1} - q\rangle + 2a_n \langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\
&\le (1 - a_n \bar{\gamma})^2 \|x_n - q\|^2 + b\gamma a_n \left(\|w_n - q\|^2 + \|x_{n+1} - q\|^2\right) + 2a_n \langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\
&= \left((1 - a_n \bar{\gamma})^2 + b\gamma a_n\right) \|x_n - q\|^2 + b\gamma a_n \|x_{n+1} - q\|^2 + 2a_n \langle \gamma f(q) - Bq, x_{n+1} - q\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \left(1 - \frac{2(\bar{\gamma} - b\gamma) a_n}{1 - b\gamma a_n}\right) \|x_n - q\|^2 + \frac{\bar{\gamma}^2 a_n^2}{1 - ba_n\gamma} \|x_n - q\|^2 + \frac{2a_n}{1 - ba_n\gamma} \langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\
&\le 1 - \frac{2(\bar{\gamma} - b\gamma) a_n}{1 - b\gamma a_n} \|x_n - q\|^2 + \frac{2(\bar{\gamma} - b\gamma) a_n}{1 - b\gamma a_n} \{\frac{a_n L}{2(1-b)} + \frac{1}{1-b} \langle \gamma f(q) - Bq, x_{n+1} - q\rangle\} \\
&= (1 - \eta_n) \|x_n - q\| + \eta_n \delta_n,
\end{aligned}$$

where $L = \sup\{\|x_n - q\| : n \ge 0\}$, $\eta_n = \frac{2(\bar{\gamma} - b\gamma)a_n}{1 - b\gamma a_n}$, and $\delta_n = \frac{a_n L}{2(1-b)} + \frac{1}{1-b}\langle \gamma f(q) - Bq, x_{n+1} - q\rangle$. Now, it is easy to see that $\eta_n \to 0$, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Lemma 2.4 now implies that the sequence $\{x_n\}$ converges strongly to $q = P_\Omega(I - B + \gamma f)(q)$.

Case 2: Assume that the sequence $\{x_n - q\}$ is not monotone. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N}; k \le n : \|x_k - q\| < \|x_{n+1} - q\|\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau_n \to \infty$ as $n \to \infty$, and for all $n \ge n_0$,

$$\|x_{\tau(n)} - q\| \le \|x_{\tau(n)+1} - q\|.$$

From (5) we obtain that

$$\lim_{n\to\infty} \|z_{\tau(n)} - Ax_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|u_{\tau(n)} - T_i u_{\tau(n)}\| = 0,$$

and

$$\lim_{n\to\infty} \|Ax_{\tau(n)} - u_{\tau(n)}\| = 0.$$

By a similar argument and from inequality (6) we get that

$$\lim_{n\to\infty} \|y_{\tau(n)} - S_i y_{\tau(n)}\| = 0, \qquad (i \in \mathbb{N}).$$

Again, as in Case 1, we arrive at

$$\|x_{\tau(n)+1} - q\|^2 \le \left(1 - \eta_{\tau(n)}\right) \|x_{\tau(n)} - q\|^2 + \eta_{\tau(n)}\delta_{\tau(n)}$$

where $\eta_{\tau(n)} \to 0$, $\sum_{n=1}^{\infty} \eta_{\tau(n)} = \infty$ and $\limsup_{n\to\infty} \delta_{\tau(n)} \le 0$. Hence, by Lemma 2.4, we obtain $\lim_{n\to\infty} \|x_{\tau(n)} - q\| = 0$, and $\lim_{n\to\infty} \|x_{\tau(n)+1} - q\| = 0$. Now Lemma 2.5 implies that

$$0 \le \|x_n - q\| \le \max\{\|x_{\tau(n)} - q\|, \|x_n - q\|\} \le \|x_{\tau(n)+1} - q\|.$$

Therefore, $\{x_n\}$ converges strongly to $q = P_{\Omega}(I - B + \gamma f)(q)$. $\square$

To prove our second main theorem, we need to recall the following statement from [19].

Proposition 3.2. *[19] Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy the conditions $(A_1) - (A_4)$. Let A_f be a set-valued mapping on H defined by*

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \ge \langle y - x, z\rangle \quad \forall y \in C\}, & x \in C, \\ \emptyset \quad x \in H \setminus C. & \end{cases}$$

Then $EP(f) = A_f^{-1}0$ and A_f is a maximal monotone operator with $\mathrm{dom}\left(A_f\right) \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of f coincides with the resolvent of A_f, i.e.,

$$T_r x = \left(I + rA_f\right)^{-1} x.$$

Using Theorem 3.1, we obtain the following strong convergence theorem for finding solutions of equilibrium problems in Hilbert spaces.

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces, and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed subsets. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that $f_1 : C \times C \longrightarrow \mathbb{R}$ and $f_2 : Q \times Q \longrightarrow \mathbb{R}$ are two bifunctions satisfying $(A_1) - (A_4)$ and that f_2 is upper semicontinuous. Let, for $(i \in \mathbb{N})$, $T_i : H_2 \longrightarrow H_2$ be an infinite family of k-demicotractive mappings and $S_i : H_1 \longrightarrow H_1$ be an infinite family of l-demicotractive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Assume further that $\Omega = \{x \in EP(f_1) \cap (\bigcap_{i=1}^{\infty} Fix(S_i)) : Ax \in EP(f_2) \cap (\bigcap_{i=1}^{\infty} Fix(T_i))\} \neq \emptyset$. Suppose that g is a b-contraction on H_1 and that B is a strongly positive bounded linear operator on H_1 with coefficient $\bar{\gamma} \ge 0$ and $0 < \gamma < \frac{\bar{\gamma}}{k}$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_n = T_{r_n}^{f_1}\left(x_n + \lambda A^*\left(\left(\alpha_n T_{r_n}^{f_2} + \beta_n I + \sum_{i=1}^{\infty} \gamma_{n,i} T_i\left(T_{r_n}^{f_2}\right) - I\right) A x_n\right)\right), \\ w_n = \delta_{n,0} y_n + \sum_{i=1}^{\infty} \delta_{n,i} S_i y_n, \\ x_{n+1} = a_n \gamma g(w_n) + (1 - a_n B) w_n, \end{cases}$$

where $\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$ and A^* is the adjoint of A. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{r_n\}$ and $\{a_n\}$ satisfy the following conditions:
(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$ and $\delta_{n,0} + \sum_{i=1}^{\infty} \delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0, 1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$,
(iv) $k < \alpha_n < 1$ and $l < \delta_{n,0} < 1$,
(v) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} (\alpha_n - k)\gamma_{n,i} > 0$ and $\liminf_{n\to\infty} (\delta_{n,0} - l)\delta_{n,i} > 0$.
Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality

$$\langle (B - \gamma g) x^*, x - x^*\rangle \ge 0 \quad \forall x \in \Omega.$$

Proof. For the bifunctions $f_1 : C \times C \to \mathbb{R}$ and $f_2 : Q \times Q \to \mathbb{R}$, we define A_{f_1} and A_{f_2} as in the Proposition 3.2. We take $F_1 = A_{f_1}$ and $F_2 = A_{f_2}$ in Theorem 3.1, it then follows from Proposition 3.2 that $J_{r_n} = \left(I + r_n A_{f_1}\right)^{-1}$ and $Q_{\mu_n} = \left(I + \mu_n A_{f_2}\right)^{-1}$ for all $r_n > 0$ and $\mu_n > 0$. Thus the desired result follows from Theorem 3.1. □

Example 3.4. *Let $H_1 = H_2 = \mathbb{R}$ be equipped with the usual inner product and norm. Let $C = [0, +\infty)$ and $Q = (-\infty, 0]$. Let $f_1 : C \times C \longrightarrow \mathbb{R}$ and $f_2 : Q \times Q \longrightarrow \mathbb{R}$ be defined by*

$$f_1(x, y) = 0, \quad x, y \in C,$$

and

$$f_2(u, v) = u(v - u), \quad u, v \in Q.$$

It is easy to see that the bifunctions f_1 and f_2 satisfy the conditions (A_1)–(A_4), moreover the bifunction f_2 is upper-semicontinuous. From Lemma 2.8, we conclude that $T^{f_1}_{r_n} x = x$. Indeed, for any $x, y \in C$ and $r > 0$ we have

$$f_1(x, y) + \frac{1}{r}\langle x - z, y - x\rangle \geq 0,$$

from which it follows that

$$(x - z)(y - x) \geq 0, \quad \forall y \in C.$$

This implies that for $y \geq x$ we have $x \geq z$, and for $y \leq x$ we have $x \leq z$. Therefore, $z = x$, and so

$$T^{f_1}_{r_n} x = x.$$

Similarly, we can prove that

$$T^{f_2}_{r_n} u = (r_n + 1) u.$$

We now consider, for $x \in \mathbb{R}$, the mappings $g(x) = \frac{1}{8}x$, $A(x) = -\frac{1}{2}x$, and $Bx = 2x$. For $i \in \mathbb{N}$, define the mappings $T_i : H_2 \longrightarrow H_2$ and $S_i : H_1 \longrightarrow H_1$ by

$$T_i(x) = \begin{cases} \dfrac{x}{2i}\sin\dfrac{1}{x} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

and

$$S_i(x) = \frac{1}{1+i}x.$$

Then $\bigcap_{i=1}^{\infty} Fix(T_i) = \{0\}$, $\bigcap_{i=i}^{\infty} Fix(S_i) = \{0\}$,

$$\begin{aligned} |T_i x - 0|^2 = \frac{x^2}{4i^2}\sin^2\frac{1}{x} &\leq x^2 - \left(x - \frac{x}{2i}\sin\frac{1}{x}\right)^2 \\ &= |x - 0|^2 - |x - Tx|^2, \end{aligned}$$

and

$$|S_i x - 0| = |\frac{1}{i+1}x| \leq |x|.$$

So, each T_i is a -1-demicontractive mapping, and each S_i is a 0-demicontractive mapping. Note that the mapping g is contraction with constant $k = \frac{1}{4}$, A is a bounded linear operator on $\mathbb{R}$ with adjoint operator A^* and $\|A\| = \|A^*\| = \frac{1}{2}$, and B is a strongly positive bounded linear self-adjoint operator with constant $\bar{\gamma} = 1$ on $\mathbb{R}$. On the other hand, we *can take* $\gamma = 2$ *which satisfies* $0 < \gamma < \frac{\bar{\gamma}}{k} < \gamma + \frac{1}{k}$. *We can now define, for* $n \in \mathbb{N}$, $\alpha_n = \frac{1}{3}$, $\beta_n = \frac{1}{3}$, $\gamma_{n,i} = \frac{1}{4^i}$, $\delta_{n,0} = \frac{1}{2}$ *and* $\delta_{n,i} = \frac{1}{3^i}$. *It is easy to see that* $EP(f_1) = [0, \infty)$ *and* $EP(f_2) = \{0\}$. *Furthermore, we have*

$$\Omega = \{x \in EP(f_1) \cap \left(\bigcap_{i=1}^{\infty} Fix(S_i)\right) : Ax \in EP(f_2) \cap \left(\bigcap_{i=1}^{\infty} Fix(Ti)\right)\} = \{0\}.$$

Now, all the assumptions in Theorem 3.1 are satisfied. Let us consider the following numerical algorithm:

$$z_n = T_{r_n}^{f_2} A x_n,$$

$$y_n = x_n + \frac{1}{8}A^*\left(\frac{1}{3}z_n - \frac{x_n}{6} + \sum_{i=1}^{\infty} \frac{z_n}{2i4^i} \sin\frac{1}{z_n} + \frac{1}{2}x_n\right),$$

$$w_n = \left(\frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{(i+1)\,3^i}\right) y_n,$$

$$x_{n+1} = \frac{1}{4}a_n w_n + (1 - 2a_n)\, w_n,$$

where $a_n = \frac{4}{n+8}$. *If* $r_n = 1$, *then*

$$y_n = x_n - \frac{1}{8}x_n \sin\frac{1}{x_n}\sum_{i=1}^{\infty}\frac{1}{i4^i},$$

$$w_n = \left(\frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{(i+1)\,3^i}\right) y_n,$$

%

$$x_{n+1} = \frac{n+1}{n+8} w_n.$$

By Theorem 3.3, the sequence $\{x_n\}$ *converges to a solution of the variational inequality stated in the theorem.*

The following statements are now easy consequences of our main result.

Theorem 3.5. *Let* H_1 *and* H_2 *be two real Hilbert spaces and* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be two nonempty closed subsets. Let* $A : H_1 \longrightarrow H_2$ *be a bounded linear operator. Assume that* $F_1 : C \times C \longrightarrow \mathbb{R}$ *and* $F_2 : Q \times Q \longrightarrow \mathbb{R}$ *are two bifunctions satisfying* $(A_1) - (A_4)$, *and* F_2 *is upper-semicontinuous. Let, for* $(i \in \mathbb{N})$, $T_i : H_2 \longrightarrow H_2$ *and* $S_i : H_1 \longrightarrow H_1$ *be two infinite families of quasi-nonexpansive mappings such that* $S_i - I$ *and* $T_i - I$ *are demiclosed at* 0. *Assume that* $\Omega = \{x \in EP(F_1) \cap (\bigcap_{i=1}^{\infty} Fix(S_i)) : Ax \in EP(F_2) \cap (\bigcap_{i=1}^{\infty} Fix(T_i))\} \neq \emptyset$. *Suppose that* f *is a b-contraction on* H_1 *and that* B *is a strongly positive bounded linear operator on* H_1 *with coefficient* $\bar{\gamma} \geq 0$, *and* $0 < \gamma < \frac{\bar{\gamma}}{k}$. *Let* $\{x_n\}$ *be the sequence generated by* $x_0 \in C$ *and*

$$\begin{cases} y_n = T_{r_n}^{F_1}\left(x_n + \lambda A^*\left(\left(\alpha_n T_{r_n}^{F_2} + \beta_n I + \sum_{i=1}^{\infty}\gamma_{n,i}T_i\left(T_{r_n}^{F_2}\right) - I\right)Ax_n\right)\right), \\ w_n = \delta_{n,0}y_n + \sum_{i=1}^{\infty}\delta_{n,i}S_i y_n, \\ x_{n+1} = a_n\gamma f(w_n) + (1 - a_n B)\, w_n. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{r_n\}$, *and* $\{a_n\}$ *satisfy the following conditions:*
(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty}\gamma_{n,i} = 1$ *and* $\delta_{n,0} + \sum_{i=1}^{\infty}\delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$,
(iv) $k < \alpha_n < 1$ *and* $l < \delta_{n,0} < 1$,
(v) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}(\alpha_n - k)\gamma_{n,i} > 0$ *and* $\liminf_{n\to\infty}(\delta_{n,0} - l)\delta_{n,i} > 0$.
Then the sequence $\{x_n\}$ *converges strongly to* $x^* \in \Omega$ *which solves the variational inequality*

$$\langle (B - \gamma f)\, x^*, x - x^* \rangle \geq 0, \qquad \forall x \in \Omega.$$

Proof. Since every quasi-nonexpansive operator is clearly 0-demicontractive, the result follows. □

Theorem 3.6. *Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be two nonempty closed subsets. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Assume that $F_1 : C\times C \longrightarrow \mathbb{R}$ and $F_2 : Q\times Q \longrightarrow \mathbb{R}$ are two bifunctions satisfying the conditions $(A_1) - (A_4)$, and that F_2 is upper-semicontinuous. Let, for $(i \in \mathbb{N})$, $T_i : H_2 \longrightarrow H_2$ and $S_i : H_1 \longrightarrow H_1$ be two infinite families of directed mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Assume that $\Omega = \{x \in EP(F_1) \cap (\bigcap_{i=1}^{\infty} Fix(S_i)) : Ax \in EP(F_2) \cap (\bigcap_{i=1}^{\infty} Fix(T_i))\} \neq \emptyset$. Suppose f is a b-contraction on H_1 and that B is a strongly positive bounded linear operator on H_1 with coefficient $\bar{\gamma} \geq 0$, and $0 < \gamma < \frac{\bar{\gamma}}{k}$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_n = T_{r_n}^{F_1}\left(x_n + \lambda A^*\left(\left(\alpha_n T_{r_n}^{F_2} + \beta_n I + \sum_{i=1}^{\infty}\gamma_{n,i}T_i\left(T_{r_n}^{F_2}\right) - I\right)Ax_n\right)\right), \\ w_n = \delta_{n,0}y_n + \sum_{i=1}^{\infty}\delta_{n,i}S_iy_n, \\ x_{n+1} = a_n\gamma f(w_n) + (1 - a_nB)w_n, \end{cases}$$

where $\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$ and A^* is the adjoint of A. Assume the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$, $\{r_n\}$, and $\{a_n\}$ satisfy the following conditions:
(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty}\gamma_{n,i} = 1$ *and* $\delta_{n,0} + \sum_{i=1}^{\infty}\delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$,
(iv) $k < \alpha_n < 1$ *and* $l < \delta_{n,0} < 1$,
(v) $\liminf_{n\to\infty}\alpha_n\beta_n > 0$, $\liminf_{n\to\infty}(\alpha_n - k)\gamma_{n,i} > 0$ *and* $\liminf_{n\to\infty}(\delta_{n,0} - l)\delta_{n,i} > 0$.
Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \qquad \forall x \in \Omega.$$

Proof. A simple calculation shows that every directed operator is −1-demicontractive, thus the result follows. □

Theorem 3.7. *[16]. Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. Let, for $(i \in \mathbb{N})$, $T_i : H_2 \longrightarrow H_2$ be an infinite family of k-demicotractive mappings and $S_i : H_1 \longrightarrow H_1$ be an infinite family of l-demicotractive mappings such that $S_i - I$ and $T_i - I$ are demiclosed at 0. Assume that $\Omega = \{x \in EP(F_1) \cap (\bigcap_{i=1}^{\infty} Fix(S_i)) : Ax \in EP(F_2) \cap (\bigcap_{i=1}^{\infty} Fix(T_i))\} \neq \emptyset$. Let f be a b-contraction on H_1, and let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_n = x_n + \lambda A^*\left(\sum_{i=1}^{\infty}\gamma_{n,i}T_i - I\right)Ax_n, \\ w_n = \delta_{n,0}y_n + \sum_{i=1}^{\infty}\delta_{n,i}S_iy_n, \\ x_{n+1} = a_n\gamma f(w_n) + (1 - a_n)w_n, \end{cases}$$

where $\lambda \in \left(0, \frac{1}{\|A\|^2}\right)$ and A^* is the adjoint of A. Suppose the sequences $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$ and $\{a_n\}$ satisfy the following conditions:
(i) $\sum_{i=1}^{\infty}\gamma_{n,i} = 1$ *and* $\delta_{n,0} + \sum_{i=1}^{\infty}\delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $l < \delta_{n,0} < 1$,
(iv) $\liminf ?n \to \infty\gamma_{n,i} > 0$ *and* $\liminf ?n \to \infty(\delta_{n,0} - l)\delta_{n,i} > 0$.
Then $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solves the variational inequality

$$\langle (I - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \Omega.$$

Proof. Putting $F_1(x,y) = 0$ for all $x, y \in C$, $F_2(x,y) = 0$ for all $x, y \in Q$ and $r_n = 1$ in Theorem 3.1, we have $T_{r_n}^{F_1} = T_{r_n}^{F_2} = I$. Now, by taking $B = I$ and $\gamma = 1$ in Theorem 3.3, we obtain the desired result. □

4. The Split Monotone Variational Inclusion Problem

Following this line of ideas, Moudafi [15] introduced the Split Monotone Variational Inclusion Problem (SMVIP). We first review the basic definitions of the literature and then will provide an application of our theorem to approximate the solution of SMVIP.

Let H_1 and H_2 be two real Hilbert spaces. Let two mappings $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and two set-valued mappings $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be given. The SMVIP is formulated as follows:

$$\text{find a point} \quad x^* \in C \quad \text{such that} \quad 0 \in f(x^*) + B_1(x^*)$$

and such that the point

$$y^* = A(x^*) \in H_2 \quad \text{solves} \quad 0 \in g(y^*) + B_2(y^*).$$

Note that if C and Q are nonempty closed convex subsets of H_1 and H_2, (resp.), and $B_1 = N_C\, B_2 = N_Q$ where N_C and N_Q are normal cones to C and Q, (resp.); then the split monotone variational inclusion problem reduces to the split variational inequality problem (SVIP) which is formulated as follows:

$$\text{find a point} \quad x^* \in C \quad \text{such that} \langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in C$$

and such that the point

$$y^* = Ax^* \in Q \quad \text{solves} \quad \langle g(y^*), y - y^* \rangle \geq 0 \quad \text{for all} \quad y \in Q.$$

SVIP is quite useful in the study of the split minimization between two spaces, because the image of a solution point of one minimization problem, under a given bounded linear operator, is a solution point of another minimization problem.

Let $h : H \to H$ be an operator and let $C \subset H$. The operator h is called inverse strongly monotone with constant $\beta > 0$ if

$$\langle h(x) - h(y), x - y \rangle \geq \beta \| h(x) - h(y) \|^2, \quad \forall x, y \in H.$$

Remark 4.1. *If $h : H \to H$ is an α-inverse strongly monotone operator on H and if $B : H \to 2^H$ is a maximal monotone operator, Then $J_\lambda^B (I - \lambda h)$ is averaged for each $\lambda \in (0, 2\alpha)$.*

Proposition 4.2. *[2] Let $T : H \to H$ be a nonexpansive mapping. Then for all $\lambda \in (0, 1]$ and $(x, y) \in H \times H$, the averaged operator T_λ satisfies*

$$\|T_\lambda x - T_\lambda y\|^2 \leq \|x - y\|^2 - \frac{1-\lambda}{\lambda} \| (I - T_\lambda) x - (I - T_\lambda) y\|^2.$$

Theorem 4.3. *Let H_1 and H_2 be two real Hilbert spaces, and $T : H_1 \to H_2$ be a bounded linear operator. Let for $i \in \mathbb{N}$, $A_i : H_1 \to 2^{H_1}$ and $B_i : H_2 \to 2^{H_2}$ be maximal monotone mappings such that $\cap_{i=1}^\infty A_i^{-1} 0 \neq \emptyset$ and $\cap_{i=1}^\infty B_i^{-1} 0 \neq \emptyset$ and that for each i, $h_i : H_1 \to H_1$ is an α_i-inverse strongly monotone operator, and $g_i : H_2 \to H_2$ is a β_i-inverse strongly monotone operator. Assume that $\rho = inf_{i\in\mathbb{N}} \alpha_i \beta_i > 0$ and that $\tau \in (0, 2\rho)$. Suppose that the SMVI*

$$\begin{cases} x^* \in \cap_{i=1}^\infty A_i^{-1} 0 & 0 \in f(x^*) + A_i(x^*) \qquad \forall i \in \mathbb{N}, \\ y^* = Tx^* \in \cap_{i=1}^\infty B_i 0 & 0 \in g(y^*) + B_i(y^*) \qquad \forall i \in \mathbb{N}, \end{cases}$$

has a nonempty solution set Ω. Suppose further that f is a k-contraction on H, and $\{x_n\}$ is the sequence generated by $x_0 \in H$, and

$$\begin{cases} y_n = x_n + \lambda T^* \left(\sum_{i=1}^\infty \gamma_{n,i} J_r^{A_i} (I - \tau h_i) - I \right) T x_n, \\ w_n = \delta_{n,0} y_n + \sum_{i=1}^\infty \delta_{n,i} J_\mu^{B_i} (I - \tau g_i) y_n, \\ x_{n+1} = a_n \gamma f(w_n) + (1 - a_n) w_n, \end{cases}$$

where $\lambda \in \left(0, \frac{1}{\|T\|^2}\right)$ *and* T^* *is the adjoint of* T. *Suppose the sequences* $\{\delta_{n,i}\}$, $\{\gamma_{n,i}\}$ *and* $\{a_n\}$ *satisfy the following conditions:*
(i) $\sum_{i=1}^{\infty} \gamma_{n,i} = 1$ *and* $\delta_{n,0} + \sum_{i=1}^{\infty} \delta_{n,i} = 1$,
(ii) $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(iii) $l < \delta_{n,0} < 1$,
(iv) $\liminf_{n\to\infty} \gamma_{n,i} > 0$ *and* $\liminf_{n\to\infty} (\delta_{n,0} - l)\, \delta_{n,i} > 0$.
Then $\{x_n\}$ *converges strongly to* $x^* \in \Omega$ *which solves the variational inequality*

$$\langle (I - f)\, x^*, x - x^* \geq 0, \quad \forall x \in \Omega.$$

Proof. Since $J_r^{A_i}(I - \tau h_i)$ and $J_r^{B_i}(I - \tau g_i)$ are τ-averaged, from Proposition 4.2 we conclude that $J_r^{A_i}(I - \tau h_i)$ and $J_r^{B_i}(I - \tau g_i)$ are $-\frac{1-\tau}{\tau}$-demicontractive mappings. Thus, the result follows from Theorem 3.7. □

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