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Some Polynomially Solvable Cases of the Inverse Ordered 1-Median Problem on Trees

Kien Trung Nguyen[a]

[a]*Mathematics Department, Teacher College, Cantho University, Cantho, Vietnam.*

Abstract. We consider the problem of modifying the edge lengths of a tree at minimum cost such that a prespecified vertex become an ordered 1-median of the perturbed tree. We call this problem the inverse ordered 1-median problem on trees. **Gassner** showed in 2012 that the inverse ordered 1-median problem on trees is, in general, *NP*-hard. We, however, address some situations, where the corresponding inverse 1-median problem is polynomially solvable. For the problem on paths with n vertices, we develop an $O(n^3)$ algorithm based on a greedy technique. Furthermore, we prove the *NP*-hardness of the inverse ordered 1-median problem on star graphs and propose a quadratic algorithm that solves the inverse ordered 1-median problem on unweighted stars.

1. Introduction

In a non-inverse location problem we want to find optimal locations of new facilities. For reference, readers may refer to Eiselt and Marianov [10], Hamacher [16]. While objective functions concerning classical location problem are often median or center functions, the decision maker sometimes chooses other objective functions as the k-centrum function [29] or the center-median function [15]. Therefore, it raises a need to study a universal approach of location theory, i.e., algorithms and methods can be applied to solve the location problem with a class of objective functions. To unify the classical location problem, Nickel and Puerto [19] introduced the so-called ordered median function that generalizes most of known objective functions.

Recently, the inverse location problem has become an interesting topic in operations research. Here, we want to modify the parameters at minimum total cost so that the prespecified facilities become optimal with respect to the new parameters. In what follows we review some previous results, which were classified according to the objective function.

For the inverse 1-median problems, Burkard et al. [7] were the first who proposed the inverse models for the 1-median problem on trees and the 1-median problem on the plane with Manhattan norm. They also solved these problems in $O(n \log n)$ time. Then Galavii [11] improved the complexity of the inverse 1-median problem on trees to linear time. Additionally, the inverse 1-median problem on trees under uncertain costs was investigated and solved by Nguyen and Chi [27]. Nguyen [26] generalized the inverse

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Email address: trungkien@ctu.edu.vn (Kien Trung Nguyen)

1-median problem on trees to the corresponding problem on block graphs. He applied the convexity of the cost function to develop an algorithm that solve the problem in $O(n \log n)$ time. Furthermore, Burkard et al. [8] developed an $O(n^2)$ algorithm that solves the inverse 1-median problem on a cycle based on the concavity of the corresponding linear programmming constraints. Burkard et al. [6] solved the inverse Fermat-Weber problem in $O(n \log n)$, where the input points are not colinear. Otherwise, this problem can be formulated as a convex program. For the inverse location problem with serveral facilities, Bonab et al. [4] showed that the inverse p-median problem on networks with variable edge lengths is NP-hard. However, the inverse 2-median problem on a tree can be solved in polynomial time. Additionally, the problem is solvable in linear time if the underlying tree is a star. Sepasian and Rahbarnia [28] investigated the inverse 1-median problem on trees with both vertex weight and edge length modification. They solved the problem in $O(n \log n)$ time. While most recent papers concern the inverse 1-median problem under linear cost functions, Guan and Zhang [14] solved the inverse 1-median problem on trees under Chebyshev norm and Hamming distance by a binary search algorithm in linear time.

For the inverse center problems, Cai et al. [9] was the first who showed the NP-hardness of the inverse 1-center problem on networks, whereas the non-inverse 1-center problem can be solved in polynomial time. Therefore, it is interesting to focus on some special cases of the inverse 1-center problem, which can be solved in polynomial time. Alizadeh and Burkard [1, 2] investigated the inverse 1-center problem on unweighted trees and solved it efficiently. Moreover, Nguyen [20, 25] solved the reverse 1-center problem on a weighted tree in quadratic time and applied the proposed method to solve the inverse 1-center problem on weighted trees in polynomial time. Nguyen and Chassein [21] showed the NP-hardness for the inverse 1-center problem on a simple generalization of tree graphs, the cactus graphs. Furthermore, Nguyen and Sepasian [24] solved the inverse 1-center problem on trees under Chebyshev norm and Hamming distance in $O(n \log n)$ time if there is no topology change during the modification. Otherwise, the problem is solvable in quadratic time.

Concerning the inverse location problem with ordered median objective function, Gassner [13] showed that the inverse convex ordered 1-median problem on trees with variable edge lengths is NP-hard in both cases, the inverse ordered 1-median problem on unweighted trees and the inverse k-centrum problem on weighted trees. Moreover, the inverse k-centrum problem on unweighted trees can be solved in $O(n^3k^2)$ time by a dynamic programming algorithm. Nguyen and Anh [23] investigated the inverse k-centrum problem on trees with variable vertex weights and showed that the problem in NP-hard. The inverse 1-center problem on trees, a special case of the inverse k-centrum problem with $k = 1$, is however solvable in quadratic time. Also, the inverse convex ordered 1-median problem on trees under Chebyshev norm and Hamming distance was solved in $O(n^2 \log n)$ time, based on a binary search algorithm and a special property of the objective function; see Nguyen and Chassein [22].

This paper considers the inverse ordered 1-median problem on trees with variable edge lengths. Especially, we focus on some special cases in which the problem can be solved in polynomial time. The paper is organized as follows. Section 3 solves the uniform-cost inverse convex ordered 1-median problem on a path in $O(n^3)$ time by a greedy algorithm. Then we solve the inverse k-centrum problem on paths in linear time. We prove in Section 4 that the inverse k-centrum problem on a star is NP-hard. However, if the underlying star is unweighted, we develop an $O(n^2)$ algorithm.

2. Problem Definition and Optimality Criterion

We now repeat the definition of ordered median problem on a network; see [19]. Given a network $G = (V, E)$, $|V| = n$, each vertex $v \in V$ is associated with a nonnegative weight w_v and each edge $e \in E$ has a nonnegative length ℓ_e. If all vertices in G have equal weights, say 1, we get an unweighted network. A point in G is either a vertex or lies on an edge of the network. We further denote by $A(G)$ the set of all points in G. The distance $d(a, b)$ between two points a and b is the length of the shortest path connecting them. Assume that the vertices in G are numbered as $v_1, v_2, \ldots, v_n$. For a vector of multipliers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the ordered 1-median function at $\rho \in A(G)$ is

$$f_\lambda(\rho) = \sum_{i=1}^{n} \lambda_i w_{(i)} d(\rho, v_{(i)}).$$

Here, the operator (.) is a permutation of $\{1, 2, \ldots, n\}$ so that the weighted distances to the point ρ are sorted nondecreasingly, i.e.,

$$w_{(1)}d(\rho, v_{(1)}) \leq w_{(2)}d(\rho, v_{(2)}) \leq \ldots \leq w_{(n)}d(\rho, v_{(n)}).$$

The permutation (.) is called a feasible permutation. We further denote the set of all feasible permutations by Π. Additionally, a point ρ^* is, by definition, an ordered 1-median of G if

$$f_\lambda(\rho^*) \leq f_\lambda(\rho)$$

for all ρ in $A(G)$.

If the multipliers are nondecreasing, i.e., $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n$, and the underlying graph is a tree, the ordered 1-median function is convex along each simple path of the tree. Also, if $\lambda = (0, \ldots, 0, 1, \ldots, 1)$ with k $1's$, the corresponding objective function is called the k-centrum function.

Given a tree network $T = (V, E)$ and a prespecified vertex v^*. Denote by $\mathcal{T}(v^*)$ the set of all subtrees induced by deleting v^* and its incident edges from T. We revisit the optimality criterion for a vertex to be a 1-median on a tree T as follows.

Theorem 2.1. *(Optimality criterion, see [13, 18])*
Given a tree $T = (V, E)$ and a vector of multipliers $\lambda \in \mathbb{R}^n_+$ such that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Then the prespecified vertex v^* is an ordered 1-median of T if and only if for each subtree $T^{sub} \in \mathcal{T}(v^*)$ there exists a feasible permutation σ^{sub} (w.r.t. T^{sub}) such that

$$\sum_{v_{\sigma^{sub}(i)} \in T^{sub}} \lambda_i w_{\sigma^{sub}(i)} \leq \sum_{v_{\sigma^{sub}(i)} \notin T^{sub}} \lambda_i w_{\sigma^{sub}(i)}.$$

In the rest of this section, we formally define the so-called inverse ordered 1-median problem. Given a tree $T = (V, E)$, a prespecified vertex v^*, and a vector of multipliers $\lambda \in \mathbb{R}^n_+$. The length of each edge e can be increased or reduced by p_e or q_e. Moreover, we assume that the modifications are limited within certain bounds, i.e., $0 \leq p_e \leq \bar{p}_e$ and $0 \leq q_e \leq \bar{q}_e$ for $e \in E$. It means the modified length of e is $\tilde{\ell}_e := \ell_e + p_e - q_e$ and it is assumed to be nonnegative. The cost to increase or decrease one unit length of e is c^+_e or c^-_e, respectively. The inverse ordered 1-median problem on T is stated as follows.

1. The prespecified vertex v^* becomes an ordered 1-median of the perturbed tree.
2. The cost function $\sum_{e \in E}(c^+_e p_e + c^-_e q_e)$ is minimized.
3. Modifications are feasible, i.e., $0 \leq p_e \leq \bar{p}_e$ and $0 \leq q_e \leq \bar{q}_e$.

The inverse location problem can be applied in network design, evacuation planning, etc. For real-life applications of the inverse combinatorial optimization problem, we refer to the survey of Heuberger [17].

3. The Inverse Convex Ordered 1-median Problem on Unweighted Paths

3.1. Uniform-cost Convex Case

We consider the uniform-cost inverse ordered 1-median problem on an unweighted path graph $P = (V, E)$, i.e., the cost to modify one unit length of each edge $e \in E$ is $c^+_e = c^-_e = 1$. For a given prespecified vertex v^*, we denote by L and R the left and right part of P induced by deleting v^*. Assume in this section that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfies $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, i.e., the ordered 1-median function on P is convex. By Theorem 2.1, we derive the following result on a path.

Corollary 3.1. *(Optimality criterion for path graphs)*
Given an unweighted path graph $P = (v_1, v_2, \ldots, v_n)$ and a prespecified vertex v^*. Then v^* is an ordered median of P if and only if there exist feasible permutations σ_L and σ_R such that:

$$\sum_{v_{\sigma_L(i)} \in L} \lambda_i \leq \sum_{v_{\sigma_L(i)} \notin L} \lambda_i \quad \textit{and} \quad \sum_{v_{\sigma_R(i)} \in R} \lambda_i \leq \sum_{v_{\sigma_R(i)} \notin R} \lambda_i.$$

Denote by Λ_L and Λ_R the minimum ordered weighted sum of all vertices on the left and right of v^*, respectively. Here, all weights are 1. We can write

$$\Lambda_L = \min_{\sigma\in\Pi} \sum_{v_{\sigma(i)}\in L} \lambda_i \quad \text{and} \quad \Lambda_R = \min_{\sigma\in\Pi} \sum_{v_{\sigma(i)}\in R} \lambda_i.$$

We can find Λ_L in $O(n \log n)$ time by sorting all vertices v_i with respect to $d(v^*, v_i)$ and breaking all ties such that all vertices in L are preferred. This means that, if two vertices $v \in L$ and $v' \in P\backslash L$ have the same distance to v^*, then v gets a smaller index as v' in the ordering. For Λ_R, we apply the similar approach.

Assume that $\sigma^* \in \Pi$ satisfies $\Lambda_L = \sum_{v_{\sigma^*(i)}\in L} \lambda_i$, we denote by $\Lambda'_L = \sum_{v_{\sigma^*(i)}\notin L} \lambda_i$. We can check that $\Lambda'_L = \sum_{i=1}^n \lambda_i - \Lambda_L$. Similarly, we also define $\Lambda'_R = \sum_{i=1}^n \lambda_i - \Lambda_R$. The Corollary 3.1 can be reformulated as follows.

Corollary 3.2. *(Optimality criterion)*
Given a path P and a prespecified point $v^* \in P$. Then, v^* is an ordered 1-median of P if and only if $\Lambda_L \leq \Lambda'_L$ and $\Lambda_R \leq \Lambda'_R$.

If v^* is not an ordered 1-median of the path, we get either $\Lambda_L > \Lambda'_L$ or $\Lambda_R > \Lambda'_R$. Without loss of generality, we can assume that $\Lambda_L > \Lambda'_L$. Then we have to decrease the gap $\mathcal{G} = \Lambda_L - \Lambda'_L > 0$ until the optimality criterion in Corollary 3.2 holds. Assume that the edge lengths are always positive thoughout the modification, i.e., the set of vertex in L and R does not change. We obtain the following property of the modification.

Proposition 3.3. *In an optimal solution of the problem, we increase the lengths of edges in R and decrease the lengths of edges in L.*

By Proposition 3.3, we set $p_e := 0$ ($q_e := 0$) for $e \in L$ ($e \in R$) without changing the optimal solution of the problem. One can rewrite the cost function as $\sum_{e\in E} x_e$ where $x_e := p_e$ if $e \in R$ and $x_e := q_e$ otherwise. Here, the upper bound is $\bar{x}_e := \bar{q}_e$ ($\bar{x}_e = \bar{p}_e$) if $e \in L$ ($e \in R$). Now, modifying the length of an edge e means to increase or decrease its length if $e \in R$ or $e \in L$, respectively. In what follows we consider the relation of the gap $\mathcal{G}$ and the optimality of v^*.

Proposition 3.4. *The minimum cost to make $\mathcal{G} \leq 0$ is also the optimal cost so that v^* becomes an ordered 1-median of P.*

Proof. Assume that the modification $(x^*_e)_{e\in E}$ with the corresponding cost C^* being minimum to make $\mathcal{G} \leq 0$. Then we consider the following cases.

- If v^* is not an ordered 1-median of the path, we get $\Lambda_R > \Lambda'_R$. Let σ' be a feasible permutation such that $\sigma' = arg \max_{\sigma\in\Pi} \sum_{v_{\sigma(i)}\in L} \lambda_i$, we get

$$\sum_{v_{\sigma'(i)}\in L} \lambda_i < \sum_{v_{\sigma'(i)}\notin L} \lambda_i.$$

  For an edge e with $x^*_e > 0$, we set $x'_e := x^*_e - \varepsilon$ for a sufficient small ε. The new modification costs C' with $C' < C^*$, and the gap $\mathcal{G} \leq 0$ as σ' is still a feasible permutation. This contradicts to the optimality of cost C^*.

- If v^* is an ordered 1-median of P, we get $\mathcal{G} \leq 0$. Assume that there exists another modification such that the corresponding cost C' is strictly less than C^* but v^* is still an ordered 1-median of the path. Then $\mathcal{G} \leq 0$ because of the optimality criterion. This is also a contradiction.

Shortly, C^* is also the optimal cost such that v^* becomes an ordered median of the path. □

By Proposition 3.4, we aim to decrease the gap $\mathcal{G}$ until it becomes nonpositive with minimum cost in order to obtain an optimal solution. We can check the feasibility of the problem by modifying all edges in L and R by their upper bounds and recompute $\mathcal{G}$ in $O(n \log n)$ time. Assume from now on that, the uniform-cost inverse convex ordered 1-median problem on P is feasible.

Let σ be a feasible permutation such that $\Lambda_L = \sum_{v_{\sigma(i)} \in L} \lambda_i$. A vertex v_i is closer to v^* than v_j if $\sigma^{-1}(i) < \sigma^{-1}(j)$. Furthermore, an edge $e = (v', v)$ with $d(v^*, v') < d(v^*, v)$ is, by definition, the edge corresponding to v. An edge e is closer to v^* than e' if e corresponds to a closer vertex to v^* than e'. We can easily observe that any modification of edge lengths can be switched to the edge near the root node v^* first. To do that, we do not only change the cost but also decrease the gap $\mathcal{G}$ as much as possible. Therefore, the idea of the solution approach is to modify the edge, which is possible to be modified and closest to v^* in each step. We stop if the gap $\mathcal{G}$ becomes nonpositive.

Observe that, the gap $\mathcal{G}$ can be reduced if there exist $v' \in R$ and $v'' \in L$ such that $d(v^*, v') < d(v^*, v'')$ but $\tilde{d}(v^*, v') = \tilde{d}(v^*, v'')$ after the modification. Consider an edge e closest to v^* with $\bar{x}_e > 0$. If $e \in L$, we represent the modified distance from each vertex $v \in L$, where e corresponds to a vertex closer than v, to v^* as $\tilde{d}(v, v^*) = d(v, v^*) - x_e$. To compute the minimum cost such that the gap can be reduced, we have to compute the minimum modification x_e such that there exist a pair of vertices in L and R changing their orders. It can be done in linear time by first searching a vertex $v' \in R$ with the largest order and smaller than the order of the vertex v, for each $v \in L$ mentioned previously. Then we compute each amount such that the orders of v and v' changes. Finally, we choose the minimizer of the smallest amount and $\bar{x}_e$. For the case $e \in R$, we can use the similar approach.

Now we apply the previous observation to develop Algorithm 1 that solves the problem.

Algorithm 1 Solves the uniform-cost inverse convex ordered 1-median problem on paths.

Input: An instance of the problem
Check the optimality criterion. If it satisfies, then v^* is an ordered 1-median of the path. Otherwise, we consider the gap $\mathcal{G} := \Lambda_L - \Lambda'_L > 0$.
Set $Val := 0$.
while $\mathcal{G} > 0$ **do**
  Take an edge e corresponding to the closest vertex to v^* with $\bar{x}_e > 0$.
  Find the smallest modification t_e of e such that the gap $\mathcal{G}$ can be reduced.
  Modify the length of e by $x_e := \min\{t_e, \bar{x}_e\}$.
  Update $Val := Val + x_e$.
  Update $\bar{x}_e := \bar{x}_e - x_e$.
end while
Output: An optimal solution of the problem and the optimal cost Val.

In each iteration, we find the minimum amount such that the gap $\mathcal{G}$ can be decreased in linear time. There are at most $O(n^2)$ times, in which the orders of vertices change. Therefore, the algorithm runs in $O(n^3)$ time. Moreover, the correctness of the algorithm holds as in each iteration the gap is reduced as much as possible for a given cost.

Theorem 3.5. *The uniform-cost inverse convex ordered 1-median problem on unweighted paths can be solved in $O(n^3)$ time, for the paths with n vertices.*

We illustrate the algorithm to solve the inverse convex ordered 1-median problem on paths in the following example.

Example 3.6. *Given an unweighted path P as in Figure 1. The vector of multipliers is $\lambda = (0, 1, 2, 3, 5)$ and v_1 is the prespecified vertex. On each edge, a pair $(\ell_e, \bar{x}_e)$ is given, where ℓ_e is the length and $\bar{x}_e$ is the upper bound of modification of the edge e.*

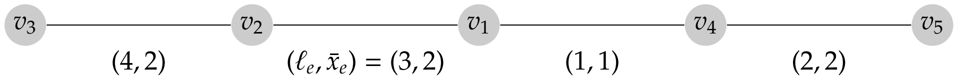

Figure 1: An instance of inverse convex ordered 1-median problem.

The left part is $L = \{v_1, v_2\}$ *with* $\Lambda_L = 8 > \Lambda'_L = 3$ *corresponds to the feasible permutation* $\sigma = (1, 4, 5, 2, 3)$. *As the optimality criterion does not hold, we solve the problem in these following iterations.*
Iteration 1. Take the edge (v_1, v_4) *and increase its length by* 1. *The orders of vertex does not change.*
Iteration 2. Take (v_1, v_2) *and reduce its length by* 1. *Then* $\sigma = (1, 2, 4, 5, 3)$ *is the feasible permutation corresponding to* Λ_L. *The gap* $\mathcal{G} = 1$ *is still positive.*
Iteration 3. The length of (v_1, v_2) *can be further reduced by* 1 *and the orders of vertex does not change.*
Iteration 4. We increase the length of (v_4, v_5) *by* 1 *and get the corresponding permutation* $\sigma = (1, 2, 4, 3, 5)$. *As the gap* $\mathcal{G} = \Lambda_L - \Lambda'_L = -3$ *is nonpositive, we stop and yield the path with optimal solution as in Figure* 2. *The optimal cost is* 4.

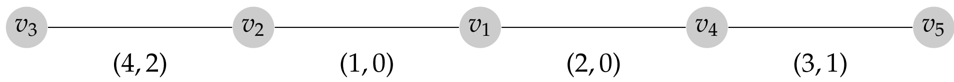

Figure 2: The path corresponding to the optimal solution.

3.2. k-centrum case

We consider the inverse k-centrum problem on an unweighted path, i.e., vector of multipliers is

$$\lambda = (0, 0, \ldots, 0, 0, \underbrace{1, 1, \ldots, 1, 1}_{\text{k 1's}}),$$

with $k < n$. We further investigate in this section the general cost coefficients. Thus, the cost function can be written as

$$\sum_{e \in E} (c_e^+ p_e + c_e^- q_e).$$

Recall that L (R) is the left (right) part of P. Rename the vertices in L by $u_1, u_2, \ldots, u_l$ and vertices in R by $v_1, v_2, \ldots, v_m$ such that the indices are sorted according to the distances of corresponding vertices to v^* as given in Figure 3.

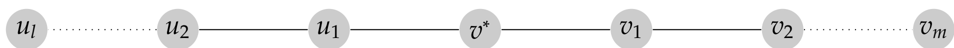

Figure 3: The path P with the renamed vertices.

Let $k' = \lfloor \frac{k+1}{2} \rfloor$ and assume that $m, l \geq k' - 1$. We can derive necessary and sufficient conditions for a vertex to be the k-centrum of P as follows.

Theorem 3.7. *(Optimality criterion, Nickel and Puerto [19]) Given an unweighted path P. A prespecified vertex* v^* *is the k-centrum of the path P,* $k < n$, *if and only if* v^* *is the midpoint of* $u_{l-k'+1}$ *and* $v_{m-k'+1}$.

By Theorem 3.7, the vertices $u_l, u_{l-1}, \ldots, u_{l-k'+2}$ and $v_m, u_{m-1}, \ldots, u_{m-k'+2}$ play no role in the optimality of v^*. Thus, we can delete these corresponding vertices from P.

Remark 3.8. *The inverse k-centrum problem on path graph is infeasible if* $l < k' - 1$ *or* $m < k' - 1$.

Therefore, we assume that $m, l \geq k' - 1$. Then we delete all vertices $u_l, u_{l-1}, \ldots, u_{l-k'+2}$ and $v_m, u_{m-1}, \ldots, u_{m-k'+2}$ and their incident edges from P to get the new path P'. The deletion costs $O(k)$ time. Rename the vertices in P' as $V = \{a_1, a_2, \ldots, a_s\}$, where there are at most $O(n-k)$ vertices in P'. We get the following result.

Corollary 3.9. *The prespecified vertex v^* is the k-centrum of the path P if and only if v^* is the midpoint of the path P'.*

Then we trivially have the relation.

Lemma 3.10. *The inverse k-centrum problem on P is equivalent to the inverse 1-center problem on P'.*

By Lemma 3.10, we focus on dealing with the inverse 1-center problem on P' in order to solve (*InvkP*). If v^* is not the k-centrum of path P, then v^* is not the midpoint of path P'. We modify the edge lengths of P' such that

$$\tilde{d}(v^*, a_1) = \tilde{d}(v^*, a_s).$$

Here, $\tilde{d}(u, v)$ is the distance between two vertices u, v in P' with respect to the modified edge lengths. By deleting v^* and its incident edges from P' to get the left part L' and right part R'. Moreover, denote by $\hat{L}'$, $\hat{R}'$ the subpath of P' induced by $L' \cup \{v^*\}$, $R' \cup \{v^*\}$, respectively. Assume that $d(v^*, a_1) > d(v^*, a_s)$, the modification of edge lengths in P' satisfies the conditions as follows.

Proposition 3.11. *In the optimal solution of the inverse 1-center problem on P', it is sufficient to reduce the lengths of edges in $\hat{L}'$ and increase the lengths of edges in $\hat{R}'$.*

By Proposition 3.11, we can set $p_e := 0$ for $e \in \hat{L}'$ and $q_e := 0$ for $e \in \hat{R}'$. Let $b := d(v^*, a_1) - d(v^*, a_s)$, the inverse 1-center problem on P' can be formulated as

$$\begin{aligned}
\min \quad & \sum_{e \in \hat{L}'} (c_e^- q_e + \sum_{e \in \hat{R}'} c_e^+ p_e) \\
\text{s.t.} \quad & \sum_{e \in \hat{L}'} q_e + \sum_{e \in \hat{R}'} p_e = b, \\
& 0 \leq p_e \leq \bar{p}_e, \qquad \forall e \in \hat{R}', \\
& 0 \leq q_e \leq \bar{q}_e, \qquad \forall e \in \hat{L}'.
\end{aligned} \tag{1}$$

Let us number the edges of P' as $e_1, e_2, \ldots, e_{s-1}$. Then we represent variables and cost coefficients of (1) as $x_i := p_{e_i}$, $\bar{x}_i = \bar{p}_{e_i}$, $c_i = c_{e_i}^+$ if $e_i \in \hat{R}'$ and $x_i := q_{e_i}$, $\bar{x}_i = \bar{q}_{e_i}$, $c_i = c_{e_i}^-$ if $e_i \in \hat{L}'$. Problem (1) can be reformulated as:

$$\begin{aligned}
\min \quad & \sum_{i=1}^{s-1} c_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{s-1} x_i = b, \\
& 0 \leq x_i \leq \bar{x}_i, \quad \forall i = 1, \ldots, s-1.
\end{aligned} \tag{2}$$

Problem (2) is a continuous knapsack problem and it can be solved in $O(s) = O(n-k)$ time by using the algorithm of Balas and Zemmel [3]. In short, as $1 < k \leq n, O(\max(k; n-k)) = O(n)$ we get the following result.

Theorem 3.12. *The inverse k-centrum problem on an unweighted path (with general positive cost coefficients) can be solved in linear time.*

4. The Inverse Ordered Median Problem on Star Graph

In this section we consider a star graph $S = (V, E)$, where v_0 is the central vertex and v_i for $i = 1, \ldots, n$ are the leaf vertices. If $e = (v_0, v)$, we denote by ℓ_v the length of edge e. Additionally, if the vertices are numbered as $v_1, v_2, \ldots, v_n$, we denote the length of edge (v_0, v_i) by ℓ_i for $i = 1, \ldots, n$, respectively. In the inverse ordered 1-median problem on a star graph, we modify the edge lengths at minimum cost such that the central vertex v_0 becomes an ordered 1-median. Star graphs are special cases of trees, where there are $|V| - 1$ leaves. Therefore, we can derive an optimality criterion for the convex case as in the following theorem.

Corollary 4.1. *(Optimality criterion, S. Nickel [19])*
Assume that the vector of multipliers λ satisfies $0 \le \lambda_0 \le \lambda_1 \le \ldots \le \lambda_n$. Then v_0 is an ordered 1-median of S if and only if for each $v_i \neq v_0$ there exists a feasible permutation σ so that

$$\lambda_{\sigma^{-1}(i)} w_i \le \sum_{\sigma(j)\neq i} \lambda_j w_{\sigma(j)}.$$

We can easily observe that, if v_0 is not an ordered 1-median of S, there exists exactly one vertex v_{i_0} which contradicts the optimality criterion, i.e., we get:

$$\lambda_{\sigma^{-1}(i_0)} w_{i_0} > \sum_{\sigma(j)\neq i_0} \lambda_j w_{\sigma(j)}.$$

Gassner [13] showed that the inverse k-centrum problem on trees is NP-hard. We further strengthen this result by proving that, NP-hardness also holds for the inverse k-centrum problem on stars.

Theorem 4.2. *The inverse k-centrum problem on general weighted star graphs is NP-hard.*

Proof. We consider an instance (I) of the $k'^{\le}$-partition problem as follows: Given a set $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{N}$ with $1 \le a_1 \le a_2 \le \ldots \le a_n$ and $\sum_{i=1}^{n} a_i = 2B$. We can assume that $B > 1$. Does there exists $S' \subset S$ such that $|S'| \le k'$ and $\sum_{a_i \in S'} a_i = B$? The $k'^{\le}$-partition problem is NP-hard, see [12].

The decision version of inverse k-centrum problem on a star graph (Inv) is 'given an instance of the inverse k-centrum problem on a star graph, does there exist a feasible solution with objective value is less than or equal to C'?

Given an instance (I), we construct an instance of (Inv) as follows.

- Let $S = (V, E)$, where $V = V_1 \cup V_2 \cup \{v_0, X\}$. Here, $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{v_1, \ldots, v_{k'}\}$. The set of edge $E = E_1 \cup E_2 \cup \{(v_0, X)\}$ with $E_1 = \{(v_0, x_j)\}_{j=1,\ldots,n}$ and $E_2 = \{(v_0, v_i)\}_{i=1,\ldots,k'}$.
- The weights of vertices are given as in the following. Let $m := 1 + \frac{a_n^2}{B-1}$, we set $w_{v_0} := 0$, $w_X := B + km$, $w_{v_i} := m$ for $i = 1, \ldots, k'$, and $w_{x_j} := a_j + m$ for $j = 1, \ldots, n$.
- We additionally choose $\ell(v_0, X) = \ell(v_0, v_i) = B$ for $i = 1, \ldots, k'$ and $\ell(v_0, x_j) = \frac{(B-a_j)m - a_j^2}{a_j + m} > 0$ for $j = 1, \ldots, n$.
- Only the lengths of (v_0, x_j), for $j = 1, \ldots, n$, can be increased by an upper bound a_j. Other edge lengths are fixed.
- The cost to modify one unit edge length is 1.
- Consider the inverse k-centrum problem on S with $k := k' + 1$ and the cost $C := B$.

Observe that

$$w_{x_j} \ell(v_0, x_j) = Bm - a_i m - a_i^2 < Bm = w_{v_i} \ell(v_0, v_i).$$

for all $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k'$. Therefore, the ordered weighted sum of the subgraph induced by X is $B + k'm$ and that of $S \backslash X$ is $k'm$. The optimality criterion does not hold for v_0. Furthermore, if the length of (v_0, x_j) increases by its upper bound $\bar{p}_{(v_0, x_j)} := a_j$, for some $j \in \{1, \ldots, n\}$, then $w_{x_i} \tilde{\ell}(v_0, x_i) = w_{v_i} \ell(v_0, v_i)$ for all $i = 1, \ldots, k'$. We aim to increase the length of $\ell(v_0, x_j)$ for $j = 1, \ldots, n$ to obtain the optimality criterion. In what follows we prove that the answer to (I) is 'yes' if and only if the answer to (Inv) is 'yes'.

If the answer to (I) is 'yes', there exists a subset S' such that $|S'| \leq k'$ and $\sum_{a_j \in S'} a_j = B$. We set $\tilde{\ell}(v_0, x_j) := \ell(v_0, x_j) + a_j$ for $a_j \in S'$, then $w_{x_j}\tilde{\ell}(v_0, x_j) = w_{v_i}\ell(v_0, v_i)$ for all j with $a_j \in S'$ and $i = 1, \ldots, k'$. As a result, there exists a feasible permutation such that the ordered weighted sum of $S\backslash X$ is $\sum_{a_j \in S'} a_j + k'm = B + k'm$. The optimality criterion is satisfied and the objective value is B. The answer to problem (Inv) is also 'yes'.

Conversely, if there exists a feasible solution of (Inv) problem with objective value being at most B. Denote by (p^*, q^*) the mentioned solution. Assume that there exists some index $j \in \{1, \ldots, n\}$ such that $0 < p^*_{x_j} < a_j$, we get $w_{x_j}\tilde{\ell}(v_0, x_j) < w_{v_i}\ell(v_0, v_i)$ for all $i = 1, \ldots, k$. In other words, $w_{x_j}\tilde{\ell}(v_0, x_j)$ is not one of the $k' + 1$ largest weighted distances. It means we have to pay avoidable cost. Thus, we can assume $p^*_{x_j} = 0$ or $p^*_{x_j} = a_j$ for all $j = 1, \ldots, n$. Furthermore, if $p^*_{x_j} = a_j$, we can choose $w_{x_j}\tilde{\ell}(v_0, x_j)$ as one of the $k' + 1$ largest weighted distance to v_0. If $w_{x_j}\tilde{\ell}(v_0, x_j)$ is not chosen as one of the $k' + 1$ largest weighted distance to v_0, we set $p^*_{x_j} = 0$. Let $J := \{j \in \{1, \ldots, n\} | p^*_{x_j} = a_j\}$. We observe that $|J| \leq k'$ as there are at most k' modifying edges. By the optimality criterion $\sum_{j \in J} a_j + k'm \geq B + k'm$ or $\sum_{j \in J} a_j \geq B$. The objective value satisfies $\sum_{j \in J} a_j \leq B$. We finally get $\sum_{j \in J} a_j = B$. In orther words, the set $S' = \{a_i \in S | i \in J\}$ satisfies $|S'| \leq k'$ and $\sum_{a_j \in S'} a_j = B$. The answer to (I) is 'yes'. □

By Theorem 4.2, the inverse k-centrum problem is NP-hard even on weighted stars. We now consider the underlying problem on an unweighted star S. We further investigate the problem, where the vector of multipliers is arbitrary, i.e., $\lambda \in \mathbb{R}^{n+1}_+$. We get the following property.

Proposition 4.3. *Given an unweighted star S. If $\lambda_i \leq \sum_{j \neq i} \lambda_j$ for all $i = 0, \ldots, n$, the central vertex v_0 is an ordered 1-median of the star.*

Proof. Take $v_i \in V\backslash\{v_0\}$ and $x(t) \in [v_0, v_i]$. Denote by $\ell(v_0, v_i) = \ell_i$ and assume that $d(v_0, x(t)) = t$. Let σ be a feasible permutation w.r.t. the distances to $x(t)$ and suppose that $\sigma(k) = i$. Then we write the ordered 1-median function as

$$f_\lambda(x(t)) = \sum_{\sigma(j) \neq i} \lambda_j(\ell_{\sigma(j)} + t) + \lambda_k(\ell_i - t) = (\sum_{\sigma(j) \neq i} \lambda_j - \lambda_k)t + constant.$$

As the slope of $f_\lambda(x(t))$ is $\sum_{\sigma(j) \neq i} \lambda_j - \lambda_k \geq 0$, this function is increasing along the direction from v_0 to v_i. Therefore, v_0 is an ordered 1-median of the star. □

Let us consider the case, in which the condition of Proposition 4.3 does not hold. Assume that there exists $i_0 \in \{0, 1, \ldots, n\}$ such that $\lambda_{i_0} > \sum_{i \neq i_0} \lambda_i$ and the edge lengths are supposed to be sorted nondecreasingly, i.e., $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_n$. We consider the following theorem.

Lemma 4.4. *Given a star graph S with $\lambda \in \mathbb{R}^n_+$ and assume that there exists an index i_0 such that $\lambda_{i_0} > \sum_{i \neq i_0} \lambda_i$, then we get two cases:*

1. *If $i_0 = 0$ then v_0 is the ordered median of the star.*
2. *If $i_0 > 0$ then the midpoint of $(v_{i_0 - 1}, v_{i_0})$ is the optimal location of the ordered median problem on the star.*

Proof. See Appendix. □

From Lemma 4.4, we get the optimality criterion

Theorem 4.5. *(Optimality criterion)*
Given an unweighted star $S = \{v_0, v_1, \ldots, v_n\}$, where v_0 is the central vertex and the the edge lengths are sorted nondecreasingly, i.e., $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_n$. Furthermore, the vector $\lambda \in \mathbb{R}^n_+$ satisfies that, there exists i_0 with $\lambda_{i_0} > \sum_{i \neq i_0} \lambda_i$ and $i_0 > 0$. Then, the central vertex v_0 is an ordered 1-median of the star if and only if v_0 is the midpoint of $(v_{i_0 - 1}, v_{i_0})$, i.e., $\ell_{i_0 - 1} = \ell_{i_0}$, where we define $\ell_0 = 0$.

Assume that v_0 is not an ordered 1-median of S. If $\lambda_1 > \sum_{j\neq 1} \lambda_j$, we have to reduce an edge length at minimum cost so that it is zero. On the other hand, if there exists an index $i_0 > 1$ such that $\lambda_{i_0} > \sum_{j\neq i_0} \lambda_j$, we get the following lemma.

Lemma 4.6. *In an optimal solution of the problem we have to increase the length of at most one edge in* $\{e_1, e_2, \ldots, e_{i_0-1}\}$ *and decrease the length of one edge in* $\{e_{i_0}, e_{i_0+1}, \ldots, e_n\}$.

Proof. Assume that (v_0, u) and (v_0, v) is the $(i_0 - 1)^{th}$ and i_0^{th} largest edges of the star graph and their lengths are equal in order to obtain the optimality criterion. Hence, other edges, which are different from (v_0, u) and (v_0, v), can not be modified as we have to pay avoidable costs. Moreover, as the edge lengths are sorted nondecreasingly according to the indices, the lemma holds. □

We introduce the concept of a candidate pair as follows.

Definition 4.7. *A pair* $u, v \in V$, *where* $\ell_u \leq \ell_{i_0-1} \leq \ell_{i_0} \leq \ell_v$, *is called a candidate pair if* $\ell_u + \bar{p}_u \geq \ell_{i_0-1}$, $\ell_v - \bar{q}_v \leq \ell_{i_0}$ *and* $\ell_u + \bar{p}_u \geq \ell_v - \bar{q}_v$.

Remark 4.8. *For a candidate pair, we are sure to get a feasible solution, i.e., their modified lengths are equal and they are the* $(i_0 - 1)^{th}$ *and* i_0^{th} *largest edges. Therefore, we consider only the candidate pairs.*

Denote by C the set of all candidate pairs. For each candidate pair $\{u, v\} \in C$, we solve a problem $(P_{\{u,v\}})$.

$$
\begin{aligned}
\min \quad & (c_u p_u + c_v q_v) \\
\text{s.t.} \quad & \ell_u + p_u \geq \ell_{i_0-1} \\
& \ell_v - q_v \leq \ell_{i_0} \\
& \ell_u + p_u = \ell_v - q_v \\
& 0 \leq p_u \leq \bar{p}_u \\
& 0 \leq q_v \leq \bar{q}_v
\end{aligned}
\tag{3}
$$

We first presolve $(P_{\{u,v\}})$ by increasing the length of (v_0, u) to ℓ_{i_0-1} and decrease the length of (v_0, v) to ℓ_{i_0}. In other words, we choose $p'_u := \ell_{i_0-1} - \ell_u$ and $q'_v := \ell_v - \ell_{i_0}$. Then we update the upper bound $\bar{p}_u := \bar{p}_u - p'_u$ and $\bar{q}_v := \bar{q}_v - q'_v$. We get a new problem $(P'_{\{u,v\}})$ as follows.

$$
\begin{aligned}
\min \quad & (c_u p_u + c_v q_v) \\
\text{s.t.} \quad & \ell_u + p_u = \ell_v - q_v \\
& 0 \leq p_u \leq \bar{p}_u \\
& 0 \leq q_v \leq \bar{q}_v
\end{aligned}
\tag{4}
$$

1. If $c_u \leq c_v$, the optimal solution of (4) is $p^*_u := \min\{\bar{p}_u, \ell_v - \ell_u\}$ and $q^*_v := \ell_v - \ell_u - p^*_u$.
2. If $c_v < c_u$, the optimal solution of (4) is $q^*_v := \min\{\bar{q}_v, \ell_v - \ell_u\}$ and $p^*_u := \ell_v - \ell_u - q^*_v$.

The optimal objective value of $(P_{\{u,v\}})$ is equal to the optimal objective value of $(P'_{\{u,v\}})$ adding an amount $(c_u p'_u + c_v p'_v)$. Therefore, each problem $(P_{\{u,v\}})$ is solvable in $O(1)$ time.

In Algorithm 2 follows we assume that there exists $i_0 > 1$ such that $\lambda_{i_0} > \sum_{i\neq i_0} \lambda_i$. Otherwise, the solution approach is trivial.

Algorithm 2 Solves the inverse ordered median problem on an unweighted star graph.

Input: An instance of the inverse ordered 1-median problem on a star $S = (V, E)$ with sorted edge lengths.

for any pair of vertices $\{u, v\}$ in V such that it satisfies $\ell_u \leq \ell_{i_0-1} \leq \ell_{i_0} \leq \ell_v$ **do**
  if $\{u, v\}$ is not a candidate pair **then**
    The subproblem $P_{\{u,v\}}$ is infeasible.
  else
    Solve the subproblem $P_{\{u,v\}}$ and get the optimal objective value $Val_{\{u,v\}}$
  end if
end for
Compare all $Val_{\{u,v\}}$ and get the minimum value. The corresponding solution is the optimal solution of the problem.
Output: The optimal solution of the problem.

We analyze the complexity of Algorithm 2. We first sort all the lengths of the star graph in $O(n \log n)$ time. For each pair $\{u, v\}$, we can check if it is a candidate pair in $O(1)$ time and solve the corresponding problem $P_{\{u,v\}}$ in $O(1)$ time. As there are at most $O(n^2)$ pairs, all subproblems can be solved in $O(n^2)$ time. We then choose the smallest objective value in $O(n^2)$ time. Finally, we get the result on the solvability of the inverse ordered 1-median problem on a star graph as follows.

Theorem 4.9. *The inverse convex ordered 1-median problem on an unweighted star graph can be solved in $O(n^2)$ time, where $n + 1$ is the number of vertices.*

5. Conclusions and Outlook

We investigated in this paper some polynomially solvable cases of the inverse ordered 1-median problem on trees. Precisely, the inverse convex ordered 1-median problem on paths and the inverse ordered 1-median problem on unweighted stars can be solved in polynomial time. We further improved the result in [13] by proving that, the inverse k-centrum problem on weighted stars is NP-hard.

For future research, the results in this paper play an important role in the following.

- As the topology structure of the caterpillar trees is somehow related to that of paths and stars, the solution approaches in this paper form the basis for considering the inverse ordered 1-median problem on caterpillars.

- It is worthwhile to investigate the inverse ordered 1-median problem in the real line, or say $\mathbb{R}^1$, as the line is closely related to path graphs. Then, we can also extend the corresponding results in the line to the inverse ordered 1-median problem on the plane.

- Another promising topic is to find an exact solution approach for the inverse convex ordered 1-median problem on trees; e.g., through a mixed integer program.

- The inverse ordered 1-median problem on unweighted paths was investigated with non-decreasing multipliers, it is therefore interesting to study the problem with general multipliers.

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Appendix

Proof of Lemma 4.4.

Let $f_\lambda^{[v_i,v_j]}(.)$ denote the ordered 1-median objective function, which is defined on the edge (v_i, v_j). We recall that $\ell_0 = 0$. We consider some cases as follows.

1. If $\lambda_0 > \sum_{i\neq 0} \lambda_i$, we consider a point $x(t) \in (v_0, v_i)$ such that $d(v_0, x(t)) = t$. The distance from $x(t)$ to v_j is $\ell_j + t$ for $j \neq i$ and to v_i is $\ell_i - t$. For $0 \leq t \leq \frac{\ell_i}{2}$, the function $f_\lambda(x(t))$ is a piecewise linear increasing function as the slope of $f_\lambda(x(t))$ is positive. For $\frac{\ell_i}{2} \leq t \leq \ell_i$, i.e., $\ell_i - t \leq t$, the function $f_\lambda(x(t))$ is decreasing. Therefore, the minimum value of $f_\lambda(x(t))$ is either $f_\lambda^{[v_0,v_i]}(x(\ell_i)) = f_\lambda(v_i)$ for $i = 1, \ldots, n$, or $f_\lambda(v_0)$. Moreover, $f_\lambda^{[v_0,v_i]}(\ell_i) = f(v_i) \geq f(v_0)$ as $d(v_{(j)}, v_0) < d(v_{(j)}, v_i)$ for $j = 1, 2, \ldots, n$. and $d(v_{(0)}, v_0) = d(v_{(0)}, v_i) = 0$. In conclusion, v_0 is an ordered 1-median of the star, see Figure 4.

2. If there exists $i_0 > 0$ such that $\lambda_{i_0} > \sum_{i\neq i_0} \lambda_i$, we take a point $x(t)$ on an edge of the star S. We consider the following situations.

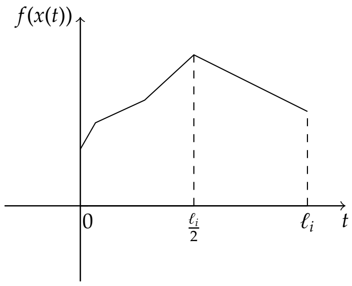

Figure 4: The ordered 1-median function satisfies $f(v_i) > f(v_0)$.

a. If $x(t) \in (v_0, v_j)$ for some $j < i_0$, the distance from $x(t)$ to v_i is $\ell_i + t$ for $i \neq j$ and to v_j is $\ell_j - t$. The slope of $f_\lambda(x(t))$ is always nonnegative. Therefore, $f_\lambda(x(t))$ is a piecewise linear increasing function in (v_0, v_j) and obtain the smallest value at v_0; see Figure 5.

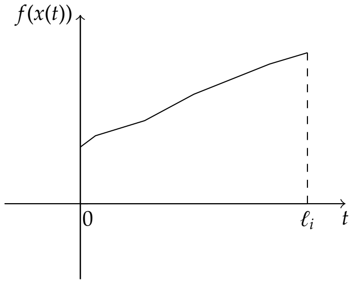

Figure 5: The ordered median function is increasing in (v_0, v_j).

b. If $x(t) \in (v_0, v_{i_0})$, the distance from $x(t)$ to v_i is $\ell_i + t$ for $i \neq i_0$ and to v_{i_0} is $\ell_{i_0} - t$. Then, $f_\lambda(x(t))$ decreases in the interval $0 \leq t \leq \frac{\ell_{i_0} - \ell_{i_0 - 1}}{2}$ since its slope is negative. However, when $t \geq \frac{\ell_{i_0} - \ell_{i_0 - 1}}{2}$ the slope of $f_\lambda(x(t))$ is nonnegative. Thus, the minimal objective value of $f_\lambda(x(t))$ in the edge (v_0, v_{i_0}) is

$$A := f^{[v_0, v_{i_0}]}(\frac{\ell_{i_0} - \ell_{i_0-1}}{2}).$$

Moreover, $f_\lambda^{[v_0, v_{i_0}]}(\frac{\ell_{i_0} - \ell_{i_0-1}}{2}) < f_\lambda(v_0)$ as the function is strictly decreasing when $0 \leq t \leq \frac{\ell_{i_0} - \ell_{i_0-1}}{2}$, see Figure 6.

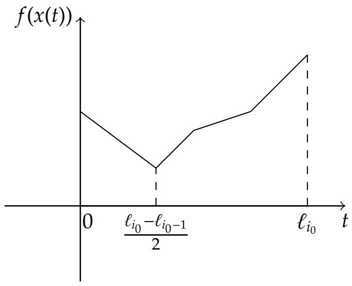

Figure 6: The ordered median function gets its minimal value at $t = \frac{\ell_{i_0} - \ell_{i_0-1}}{2}$.

c. If $x(t) \in (v_0, v_j)$ for some $j > i_0$, the distance from $x(t)$ to v_i is $\ell_i + t$ for $i \neq j$ and to v_j is $\ell_j - t$. The function $f_\lambda(x(t))$ increases on $[0, \frac{\ell_j - \ell_{i_0}}{2}] \cup [\frac{\ell_j - \ell_{i_0-1}}{2}, \ell_j]$ and decreases on $[\frac{\ell_j - \ell_{i_0}}{2}, \frac{\ell_j - \ell_{i_0-1}}{2}]$. Therefore, the two possible minimal values of $f_\lambda(x(t))$ in the edge (v_0, v_j) are $f_\lambda(v_0)$ and $f_\lambda^{[v_0,v_j]}(\frac{\ell_j-\ell_{i_0-1}}{2})$; see Figure 7. We get

$$B := f_\lambda^{[v_0,v_j]}(\frac{\ell_j - \ell_{i_0-1}}{2}) < f_\lambda(v_0).$$

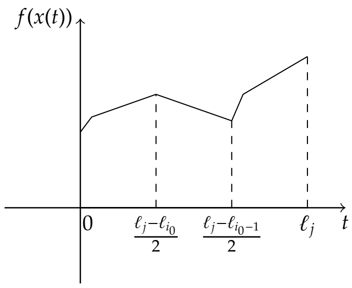

Figure 7: The two candidate minimal value of $f(x(t))$ are obtained at $t = 0$ and $t = \frac{\ell_j-\ell_{i_0-1}}{2}$.

By elementary calculation, we get $B - A > 0$. Therefore, it implies $f_\lambda^{[v_0,v_j]}(\frac{\ell_j-\ell_{i_0-1}}{2}) > f_\lambda^{[v_0,v_{i_0}]}(\frac{\ell_{i_0}-\ell_{i_0-1}}{2})$. In other words, the midpoint of (v_{i_0-1}, v_{i_0}) is an ordered 1-median of the star graph.