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Some Remarks on Schwarz Lemma at the Boundary

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Abstract. In this paper, a boundary version of the Schwarz lemma is investigated. We consider a function f holomorphic in the unit disc D, $f(a) = b$, $|a| < 1$ and $\left|f(z) - \alpha\right| < \alpha$ for $z \in D$, where α is a positive real number and $\frac{1}{2} < \alpha \leq 1$. We obtain sharp lower bounds on the angular derivative $f'(c)$ at the point c, where $f(c) = 2\alpha$, $|c| = 1$.

1. Introduction

One of the most quoted and central results in all of complex function theory is the Schwarz lemma. There is hardly a result that has been as influential. It is difficult to overestimate the significance of this lemma which gave a great push to the development of geometric function theory, fixed point theory of holomorphic mappings, hyperbolic geometry, and many other fields of analysis. A general form of this lemma, which is very simple and commanly used, is as follow:

Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $\left|f(z)\right| < 1$ for $|z| < 1$. For any point z in the disc D, we have $\left|f(z)\right| \leq |z|$ and $\left|f'(0)\right| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ ([5], p.329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [1]).

Let f be a holomorphic function on D, $f(a) = b$, $|a| < 1$ and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number and $\frac{1}{2} < \alpha \leq 1$.

Consider the functions

$$\varphi(z) = f\left(\frac{z + a}{1 + \overline{a}z}\right),$$

$$h(z) = \frac{\varphi(z) - \alpha}{\alpha}$$

and

$$\phi(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)}.$$

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$\phi(z)$ is holomorphic function in the unit disc D, $\phi(0) = 0$ and $\left|\phi(z)\right| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain

$$\left|f'(a)\right| \leq \frac{\alpha^2 - |b-\alpha|^2}{\alpha\left(1-|a|^2\right)}. \tag{1.1}$$

The inequality in (1.1) is sharp with equality for the function

$$f(z) = \alpha b \frac{1 + \left(\frac{z-a}{1-\bar{a}z}\right)}{\alpha + (b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)},$$

where $-1 < a \leq 0$ with $\alpha < b < 2\alpha$.

Robert Osserman [14] has given the inequalities which are called the boundary Schwarz lemma. He has first showed that

$$\left|f'(c)\right| \geq \frac{2}{1+\left|f'(0)\right|} \tag{1.2}$$

and

$$\left|f'(c)\right| \geq 1 \tag{1.3}$$

under the assumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and c is a boundary point to which f extends continuously and $\left|f(c)\right| = 1$. In addition, the equality in (1.3) holds if and only if $f(z) = ze^{i\theta}$, θ real. Also, $c = 1$ in the inequality (1.2) equality occurs for the function $f(z) = z\frac{z+\epsilon}{1+\epsilon z}$, $0 < \epsilon < 1$.

Furthermore, if $f(z) = c_p z^p + c_{p+1}z^{p+1} + \ldots$, then

$$\left|f'(c)\right| \geq p + \frac{1-\left|c_p\right|}{1+\left|c_p\right|} \tag{1.4}$$

and

$$\left|f'(c)\right| \geq p. \tag{1.5}$$

If, in addition, the function f has an angular limit $f(c)$ at $c \in \partial D$, $\left|f(c)\right| = 1$, then by the Julia-Wolff lemma the angular derivative $f'(c)$ exists and $1 \leq \left|f'(c)\right| \leq \infty$ (see [17]).

Inequality (1.3) and its generalizations have important applications in geometric theory of functions (see, e.g., [5], [17]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [3], [4], [6], [7], [9], [10], [14], [15], [16] and references therein).

The inequality (1.4) is a particular case of a result due to Vladimir N. Dubinin in (see [3]), who strengthened the inequality $\left|f'(c)\right| \geq 1$ by involving zeros of the function f.

D. M. Burns and S. G. Krantz [8] and D. Chelst [2] were studied the uniqueness portion of the Schwarz lemma.

X. Tang, T. Liu and J. Lu [10] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Some other types of results which are related to the subject can be found in (see, e.g., [11], [12]). In addition, (see [13]) was posed on ResearchGate where is discussed concerning results in more general aspects.

Also, M. Jeong [6] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [7] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section we give estimate below $\left|f'(c)\right|$ according to first nonzero Taylor coefficient of $f(z)-b$ about zero, namely $z=a$. The sharpness of these estimates is also proved.

Theorem 2.1. *Let f be a holomorphic function in the disc D satisfying $f(a)=b$, $|a|<1$ and $\left|f(z)-\alpha\right|<\alpha$ for $|z|<1$, where α is a positive real number and $\frac{1}{2}<\alpha\leq 1$. Assume that, for some $c\in\partial D$, f has an angular limit $f(c)$ at c, $f(c)=2\alpha$. Then*

$$\left|f'(c)\right|\geq \alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}. \tag{1.6}$$

The inequality (1.6) *is sharp, with equality for the function*

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\overline{a}z}\right)}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)},$$

where $-1<a\leq 0$ with $\alpha<b<2\alpha$.

Proof. Let

$$\phi(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}.$$

Then $\phi(z)$ is holomorphic function in the unit disc D and $\left|\phi(z)\right|<1$ for $|z|<1$ and $\phi(0)=0$.
Moreover, for $z_0=\frac{c-a}{1-\overline{a}c}\in\partial D$,

$$\varphi(z_0)=f\left(\frac{z_0+a}{1+\overline{a}z_0}\right).$$

Therefore, we take

$$h\left(z_0\right)=\frac{\varphi(z_0)-\alpha}{\alpha}=\frac{f\left(\frac{z_0+a}{1+\overline{a}z_0}\right)-\alpha}{\alpha}=\frac{f(c)-\alpha}{\alpha}=\frac{2\alpha-\alpha}{\alpha}=1$$

and

$$\left|\phi(z_0)\right|=\left|\frac{h(z_0)-h(0)}{1-\overline{h(0)}h(z_0)}\right|=\left|\frac{1-h(0)}{1-\overline{h(0)}}\right|=1.$$

From (1.3), we obtain

$$\begin{aligned}
1 \quad\leq\quad & \left|\phi'(z_0)\right|=\frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}\left|h'(z_0)\right|\leq\frac{1+|h(0)|}{1-|h(0)|}\left|h'(z_0)\right| \\
= \quad & \frac{1}{\alpha}\frac{1-|a|^2}{\left|1+\overline{a}z_0\right|^2}\frac{1+\frac{|b-\alpha|}{\alpha}}{1-\frac{|b-\alpha|}{\alpha}}\left|f'\left(\frac{z_0+a}{1+\overline{a}z_0}\right)\right| \\
\leq \quad & \frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'\left(\frac{z_0+a}{1+\overline{a}z_0}\right)\right|.
\end{aligned}$$

Thus, we obtain

$$\left|f'(c)\right|\geq\alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}.$$

Now, we shall show that the inequality (1.6) is sharp. Let

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\bar{a}z}\right)}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)}.$$

Then

$$f'(z)=\alpha b\frac{\frac{1-|a|^2}{(1-\bar{a}z)^2}\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)\right)-(b-\alpha)\frac{1-|a|^2}{(1-\bar{a}z)^2}\left(1+\left(\frac{z-a}{1-\bar{a}z}\right)\right)}{\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)\right)^2}.$$

Therefore, we take

$$f'(1)=\alpha b\frac{\frac{1-|a|^2}{(1-\bar{a})^2}\left(\alpha+(b-\alpha)\left(\frac{1-a}{1-\bar{a}}\right)\right)-(b-\alpha)\frac{1-|a|^2}{(1-\bar{a})^2}\left(1+\left(\frac{1-a}{1-\bar{a}}\right)\right)}{\left(\alpha+(b-\alpha)\left(\frac{1-a}{1-\bar{a}}\right)\right)^2}.$$

Since $-1<a\leq 0$ with $\alpha<b<2\alpha$, we obtain

$$f'(1)=\alpha b\frac{1+a}{1-a}\frac{2\alpha-b}{b}.$$

□

Theorem 2.2. *Let f be a holomorphic function in the disc D satisfying $f(a)=b$, $|a|<1$ and $\left|f(z)-\alpha\right|<\alpha$ for $|z|<1$, where α is a positive real number and $\frac{1}{2}<\alpha\leq 1$. Assume that, for some $c\in\partial D$, f has an angular limit $f(c)$ at c, $f(c)=2\alpha$. Then*

$$\left|f'(c)\right|\geq 2\alpha\frac{1-|a|}{1+|a|}\frac{(\alpha-|b-\alpha|)^2}{\alpha^2-|b-\alpha|^2+\alpha\left(1-|a|^2\right)\left|f'(a)\right|}. \tag{1.7}$$

The equality in (1.7) occurs for the function

$$f(z)=\alpha b\frac{1+2d\left(\frac{z-a}{1-\bar{a}z}\right)+\left(\frac{z-a}{1-\bar{a}z}\right)^2}{\alpha(1+d\left(\frac{z-a}{1-\bar{a}z}\right))+(b-\alpha)\left(\left(\frac{z-a}{1-\bar{a}z}\right)^2+d\left(\frac{z-a}{1-\bar{a}z}\right)\right)},$$

where $-1<a\leq 0$ with $\alpha<b<2\alpha$ and $d=\frac{\alpha\left(1-|a|^2\right)\left|f'(a)\right|}{\alpha^2-|b-\alpha|^2}$ is arbitrary number from $[0,1]$ (see (1.1)).

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. From (1.2), for $z_0=\frac{c-a}{1-\bar{a}c}\in\partial D$, we obtain

$$\begin{aligned}\frac{2}{1+\left|\phi'(0)\right|} &\leq \left|\phi'(z_0)\right|=\frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}|h'(z_0)|\leq|h'(z_0)|\frac{1+|h(0)|}{1-|h(0)|}\\ &= \frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'\left(\frac{z_0+a}{1+\bar{a}z_0}\right)\right|.\end{aligned}$$

Since

$$\phi'(z)=\frac{\left(1-|h(0)|^2\right)}{\left(1-h(z)\overline{h(0)}\right)^2}h'(z),$$

$$\phi'(0) = \frac{\left(1 - |h(0)|^2\right)}{\left(1 - h(0)\overline{h(0)}\right)^2} h'(0) = \frac{\left(1 - |h(0)|^2\right)}{\left(1 - |h(0)|^2\right)^2} h'(0) = \frac{h'(0)}{1 - |h(0)|^2}$$

and

$$\left|\phi'(0)\right| = \frac{|h'(0)|}{1 - |h(0)|^2} = \frac{1}{\alpha} \frac{\left(1 - |a|^2\right)\left|f'(a)\right|}{1 - \left|\frac{b-\alpha}{\alpha}\right|^2} = \alpha \frac{\left(1 - |a|^2\right)\left|f'(a)\right|}{\alpha^2 - |b - \alpha|^2}$$

we obtain

$$\frac{2}{1 + \alpha \frac{(1-|a|^2)|f'(a)|}{\alpha^2 - |b-\alpha|^2}} \leq \frac{1}{\alpha} \frac{1 + |a|}{1 - |a|} \frac{\alpha + |b - \alpha|}{\alpha - |b - \alpha|} \left| f'\left(\frac{z_0 + a}{1 + \bar{a} z_0}\right)\right|$$

and

$$\left|f'(c)\right| \geq 2\alpha \frac{1 - |a|}{1 + |a|} \frac{(\alpha - |b - \alpha|)^2}{\alpha^2 - |b - \alpha|^2 + \alpha\left(1 - |a|^2\right)\left|f'(a)\right|}.$$

Now, we shall show that the inequality (1.7) is sharp. Let

$$f(z) = \alpha b \frac{1 + 2d\left(\frac{z-a}{1-\bar{a}z}\right) + \left(\frac{z-a}{1-\bar{a}z}\right)^2}{\alpha(1 + d\left(\frac{z-a}{1-\bar{a}z}\right)) + (b - \alpha)\left(\left(\frac{z-a}{1-\bar{a}z}\right)^2 + d\left(\frac{z-a}{1-\bar{a}z}\right)\right)}.$$

Thus, since $-1 < a \leq 0$ with $\alpha < b < 2\alpha$, we take

$$f'(1) = 2\alpha \frac{1 + a}{1 - a} \frac{2\alpha - b}{(1 + d)b}.$$

□

If $f(z) = b + c_p(z - a)^p + c_{p+1}(z - a)^{p+1} + ...,\ p \geq 1$ is a holomorphic function in D, $f(a) = b$, $|a| < 1$ and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a real positive real number and $\frac{1}{2} < \alpha \leq 1$, then

$$\left|c_p\right| \leq \frac{\alpha^2 - |b - \alpha|^2}{\alpha\left(1 - |a|^2\right)^p}. \tag{1.8}$$

Theorem 2.3. *Let $f(z) = b + c_p(z - a)^p + c_{p+1}(z - a)^{p+1} + ...,\ p \geq 1$ be a holomorphic function in the disc D satisfying $f(a) = b$, $|a| < 1$ and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number and $\frac{1}{2} < \alpha \leq 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f(c) = 2\alpha$. Then*

$$\left|f'(c)\right| \geq \alpha p \frac{1 - |a|}{1 + |a|} \frac{\alpha - |b - \alpha|}{\alpha + |b - \alpha|}. \tag{1.9}$$

The inequality (1.9) is sharp, with equality for the function

$$f(z) = \alpha b \frac{1 + \left(\frac{z-a}{1-\bar{a}z}\right)^p}{\alpha + (b - \alpha)\left(\frac{z-a}{1-\bar{a}z}\right)^p},$$

where $-1 < a \leq 0$ with $\alpha < b < 2\alpha$.

Proof. Using the inequality (1.5) for the function $\phi(z)$, for $z_0 = \frac{c-a}{1-\overline{a}c} \in \partial D$, we obtain

$$\begin{aligned} p &\leq \left|\phi'(z_0)\right| = \frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}|h'(z_0)| \leq \frac{1+|h(0)|}{1-|h(0)|}|h'(z_0)| \\ &\leq \frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(c)\right|. \end{aligned}$$

Thus, we take the inequality (1.9) with an obvious equality case. □

Theorem 2.4. *Under the same assumptions as in Theorem 2.3, we have*

$$\left|f'(c)\right| \geq \alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left[p+\frac{\alpha-|b-\alpha|-\alpha\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha-|b-\alpha|+\alpha\left(1-|a|^2\right)^p\left|c_p\right|}\right]. \tag{1.10}$$

Equality in (1.10) *is attained for the function*

$$f(z) = \alpha b\frac{1+k\left(\frac{z-a}{1-\overline{a}z}\right)+\left(\frac{z-a}{1-\overline{a}z}\right)^p\left(k+\frac{z-a}{1-\overline{a}z}\right)}{\alpha+\alpha k\left(\frac{z-a}{1-\overline{a}z}\right)+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\left(k+\frac{z-a}{1-\overline{a}z}\right)},$$

where $-1 < a \leq 0$ *with* $\alpha < b < 2\alpha$ *and* $k = \frac{\alpha\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}$ *is an arbitrary number on* $[0,1]$ (*see*, (1.8)).

Proof. Using the inequality (1.4) for the function $\phi(z)$, for $z_0 = \frac{c-a}{1-\overline{a}c} \in \partial D$, we obtain

$$\begin{aligned} p+\frac{1-\left|a_p\right|}{1+\left|a_p\right|} &\leq \left|\phi'(z_0)\right| = \frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}|h'(z_0)| \leq \frac{1+|h(0)|}{1-|h(0)|}|h'(z_0)| \\ &\leq \frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\overline{a}z_0})\right|. \end{aligned}$$

Since

$$\left|a_p\right| = \frac{\left|\phi^p(0)\right|}{p} = \frac{\alpha\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2},$$

Therefore, we take the inequality (1.10).

The equality in (1.10) is obtained for the function

$$f(z) = \alpha b\frac{1+k\left(\frac{z-a}{1-\overline{a}z}\right)+\left(\frac{z-a}{1-\overline{a}z}\right)^p\left(k+\frac{z-a}{1-\overline{a}z}\right)}{\alpha+\alpha k\left(\frac{z-a}{1-\overline{a}z}\right)+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\left(k+\frac{z-a}{1-\overline{a}z}\right)},$$

as show simple calculations. □

Theorem 2.5. *Let* $f(z) = b+c_p(z-a)^p+c_{p+1}(z-a)^{p+1}+...,\ p \geq 1$ *be a holomorphic function in the disc D satisfying* $f(a) = b$, $|a| < 1$ *and* $\left|f(z)-\alpha\right| < \alpha$ *for* $|z| < 1$, *where* α *is a positive real number and* $\frac{1}{2} < \alpha \leq 1$. *Assume that, for some* $c \in \partial D$, f *has an angular limit* $f(c)$ *at* c, $f(c) = 2\alpha$.. *Let* $z_1, z_2, .., z_n$ *be zeros of the function* $f(z)-b$ *in D that are different from* $z = a$. *Then we have the inequality*

$$\left|f'(c)\right| \geq \alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{\left|\frac{c-a}{1-\overline{a}c}-b_k\right|^2}+\frac{\left(\alpha^2-|b-\alpha|^2\right)\prod_{k=1}^{n}|z_k|-\alpha\left(1-|a|^2\right)^p\left|c_p\right|}{\left(\alpha^2-|b-\alpha|^2\right)\prod_{k=1}^{n}|z_k|-\alpha\left(1-|a|^2\right)^p\left|c_p\right|}\right). \tag{1.11}$$

In addition, the equality in (1.11) *occurs for the function*

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\bar{a}z}\right)^{p}\prod\limits_{k=1}^{n}\frac{z-a-z_k(1-\bar{a}z)}{1-\bar{a}z-(z-a)\overline{z_k}}}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)^{p}\prod\limits_{k=1}^{n}\frac{z-a-z_k(1-\bar{a}z)}{1-\bar{a}z-(z-a)\overline{z_k}}},$$

where $-1<a\leq 0$ *with* $\alpha<b<2\alpha$ *and* $z_1,z_2,...,z_n$ *are positive real numbers.*

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1 and $z_1,z_2,..,z_n$ be zeros of the function $f(z)-b$ in D that are different from $z=a$.

$$B(z)=z^{p}\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}$$

is a holomorphic function in D and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have

$$\left|\phi(z)\right|\leq|B(z)|.$$

The auxiliary function

$$\Upsilon(z)=\frac{\phi(z)}{B(z)}$$

is a holomorphic in D, and $|\Upsilon(z)|<1$ for $|z|<1$. In particular, we have

$$|\Upsilon(0)|=\frac{\alpha\left(1-|a|^2\right)^{p}\left|c_p\right|}{\left(\alpha^2-|b-\alpha|^2\right)\prod\limits_{k=1}^{n}|z_k|}.$$

Moreover, it can be seen that, for $z_0=\frac{c-a}{1-\bar{a}c}\in\partial D$

$$\frac{z_0\phi'(z_0)}{\phi(z_0)}=\left|\phi'(z_0)\right|\geq|B'(z_0)|=\frac{z_0B'(z_0)}{B(z_0)}.$$

Besides, by applying some simple calculations, we take

$$|B'(z_0)|=\frac{z_0B'(z_0)}{B(z_0)}=p+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|z_0-z_k|^2}.$$

The composite function

$$k(z)=\frac{\Upsilon(z)-\Upsilon(0)}{1-\overline{\Upsilon(0)}\Upsilon(z)}$$

satisfies the assumption of the Schwarz lemma on the boundary, whence we obtain

$$\begin{aligned}1\quad&\leq\quad|k'(z_0)|=\frac{1-|\Upsilon(0)|^2}{\left|1-\overline{\Upsilon(0)}\Upsilon(z)\right|^2}|\Upsilon'(z_0)|\leq\frac{1+|\Upsilon(0)|}{1-|\Upsilon(0)|}\left|\frac{z_0\phi'(z_0)}{\phi(z_0)}-\frac{z_0B'(z_0)}{B(z_0)}\right|\\&=\quad\frac{1+|\Upsilon(0)|}{1-|\Upsilon(0)|}\left\{\left|\phi'(z_0)\right|-|B'(z_0)|\right\}\end{aligned}$$

and

$$1 \leq \frac{1 + \frac{\alpha\left(1-|a|^2\right)^p |c_p|}{\left(\alpha^2-|b-\alpha|^2\right)\prod_{k=1}^{n}|z_k|}}{1 - \frac{\alpha\left(1-|a|^2\right)^p |c_p|}{\left(\alpha^2-|b-\alpha|^2\right)\prod_{k=1}^{n}|z_k|}} \left\{ \frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\bar{a}z_0})\right| - \left(p + \sum_{k=1}^{n}\frac{1-|z_k|^2}{|z_0-z_k|^2}\right)\right\}.$$

Therefore, we take the inequality (1.11).

The equality in (1.11) is obtained for the function

$$f(z) = \alpha b \frac{1 + \left(\frac{z-a}{1-\bar{a}z}\right)^p \prod_{k=1}^{n} \frac{z-a-z_k(1-\bar{a}z)}{1-\bar{a}z-(z-a)\overline{z_k}}}{\alpha + (b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)^p \prod_{k=1}^{n} \frac{z-a-z_k(1-\bar{a}z)}{1-\bar{a}z-(z-a)\overline{z_k}}},$$

as show simple calculations. □

The inequality (1.10) can be strengthened as below by taking into account c_{p+1} which is second coefficient in the expansion of the function $f(z)$.

Theorem 2.6. *Let $f(z) = b + c_p(z-a)^p + c_{p+1}(z-a)^{p+1} + ..., c_p > 0, p \geq 1$ be a holomorphic function in the disc D satisfying $f(a) = b$, $|a| < 1$ and $\left|f(z) - \alpha\right| < \alpha$ for $|z| < 1$, where α is a positive real number and $\frac{1}{2} < \alpha \leq 1$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f(c) = 2\alpha$. Then*

$$\left|f'(c)\right| \geq \alpha \frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\Bigg(p+ \tag{1.12}$$

$$\frac{2\left(\alpha^2-|b-\alpha|^2-\alpha^2\left(1-|a|^2\right)^p\left|c_p\right|\right)^2}{\left(\alpha^2-|b-\alpha|^2\right)^2-\alpha^2\left(1-|a|^2\right)^{2p}\left|c_p\right|^2+\alpha\left(\alpha^2-|b-\alpha|^2\right)\left(1-|a|^2\right)^p\left|\left(1-|a|^2\right)c_{p+1}-\bar{a}pc_p\right|}\Bigg).$$

The inequality (1.12) *is sharp, with equality for the function*

$$f(z) = \alpha b \frac{1 + \left(\frac{z-a}{1-\bar{a}z}\right)^p}{\alpha + (b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)^p},$$

where $-1 < a \leq 0$ with $\alpha < b < 2\alpha$.

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1 and $B_0(z) = z^p$. By the maximum principle for each $z \in D$, we have

$$\left|\phi(z)\right| \leq |B_0(z)|.$$

Therefore,

$$p(z) = \frac{\phi(z)}{B_0(z)}$$

is holomorphic function in D and $\left|p(z)\right| < 1$ for $|z| < 1$.

In particular, we have

$$\left|p(0)\right| = \alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2} \leq 1. \tag{1.13}$$

and

$$\left|p'(0)\right| = \alpha \frac{\left(1-|a|^2\right)^p \left|\left(1-|a|^2\right)c_{p+1} - \bar{a}pc_p\right|}{\alpha^2 - |b-\alpha|^2}.$$

Moreover, since the expression $\frac{z_0\phi'(z_0)}{\phi(z_0)}$ is real number greater than or equal to 1 ([1]), we take that, for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \left|\frac{z_0\phi'(z_0)}{\phi(z_0)}\right| \geq \left|\phi'(z_0)\right|.$$

Since $\left|\phi(z)\right| \leq |B_0(z)|$, we take

$$\frac{1-\left|\phi(z)\right|}{1-|z|} \geq \frac{1-|B_0(z)|}{1-|z|}.$$

Taking angular limit in the last inequality yields

$$\left|\phi'(z_0)\right| \geq \left|B_0'(z_0)\right|.$$

Therefore, we obtain

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} \geq \left|\phi'(z_0)\right| \geq \left|B_0'(z_0)\right| = \frac{z_0 B_0'(z_0)}{B_0(z_0)}.$$

The function

$$\Phi(z) = \frac{p(z)-p(0)}{1-p(z)\overline{p(0)}}$$

is holomorphic in the unit disc D, $|\Phi(z)| < 1$ for $|z| < 1$, $\Phi(0) = 0$ and $|\Phi(z_0)| = 1$ for $z_0 = \frac{c-a}{1-\bar{a}c} \in \partial D$. From (1.2), we obtain

$$\begin{aligned}
\frac{2}{1+|\Phi'(0)|} &\leq |\Phi'(z_0)| \leq \frac{1+\left|p(0)\right|}{1-\left|p(0)\right|}\left|\frac{\phi'(z_0)}{B_0(z_0)} - \frac{\phi(z_0)B_0'(z_0)}{B_0^2(z_0)}\right| \\
&= \frac{1+\left|p(0)\right|}{1-\left|p(0)\right|}\left\{\left|\phi'(z_0)\right| - \left|B_0'(z_0)\right|\right\} \\
&\leq \frac{1+\left|p(0)\right|}{1-\left|p(0)\right|}\left\{\frac{1+|h(0)|}{1-|h(0)|}\frac{1}{\alpha}\frac{1-|a|^2}{\left|1+\bar{a}z_0\right|^2}\left|f'(\frac{z_0+a}{1+\bar{a}z_0})\right| - p\right\} \\
&\leq \frac{\alpha^2-|b-\alpha|^2+\alpha\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2-\alpha\left(1-|a|^2\right)^p\left|c_p\right|}\left\{\frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\bar{a}z_0})\right| - p\right\}.
\end{aligned}$$

Since

$$\Phi'(z) = \frac{1-\left|p(0)\right|^2}{\left(1-\overline{p(0)}p(z)\right)^2}p'(z)$$

and

$$|\Phi'(0)| = \frac{\left|p'(0)\right|}{1-\left|p(0)\right|^2},$$

we get

$$\frac{2}{1+\alpha\frac{\left(\alpha^2-|b-\alpha|^2\right)\left(1-|a|^2\right)^p\left|\left(1-|a|^2\right)c_{p+1}-\bar{a}pc_p\right|}{\left(\alpha^2-|b-\alpha|^2\right)^2-\alpha^2\left(1-|a|^2\right)^{2p}\left|c_p\right|^2}} \leq \frac{\alpha^2-|b-\alpha|^2+\alpha\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2-\alpha\left(1-|a|^2\right)^p\left|c_p\right|} \times\left\{\frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'\left(\frac{z_0+a}{1+\bar{a}z_0}\right)\right|-p\right\}.$$

Thus, we obtain (1.12) with an obvious equality case. □

If $f(z)-b$ has no zeros different from $z=a$ in Theorem 2.6, the inequality (1.12) can be further strengthened. This is given by the following Theorem.

Theorem 2.7. *Let $f(z)=b+c_p(z-a)^p+c_{p+1}(z-a)^{p+1}+...,\ c_p>0,\ p\geq 1$ be a holomorphic function in the disc D satisfying $f(a)=b$, $|a|<1$ and $\left|f(z)-\alpha\right|<\alpha$ for $|z|<1$, where α is a positive real number and $\frac{1}{2}<\alpha\leq 1$ and $f(z)-b$ has no zeros in D except $z=a$. Assume that, for some $c\in\partial D$, f has an angular limit $f(c)$ at c, $f(c)=2\alpha$. Then*

$$\left|f'(c)\right|\geq\alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left[p-\frac{2\ln^2\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)}{2\left|c_p\right|\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)-\left|\left(1-|a|^2\right)c_{p+1}-\bar{a}pc_p\right|}\right]. \tag{1.14}$$

The inequality (1.14) *is sharp, with equality for the function*

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\bar{a}z}\right)^p}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\bar{a}z}\right)^p},$$

where $-1<a\leq 0$ with $\alpha<b<2\alpha$.

Proof. Let $p(z)$ be as in the proof of Theorem 2.6. Having in the mind inequality (1.13), we denote by $\ln p(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln p(0)=\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)<0.$$

The composite function

$$\Theta(z)=\frac{\ln p(z)-\ln p(0)}{\ln p(z)+\ln p(0)}$$

is holomorphic in the unit disc D, $|\Theta(z)|<1$ for $|z|<1$, $\Theta(0)=0$ and $|\Theta(z_0)|=1$ for $z_0=\frac{c-a}{1-\bar{a}c}\in\partial D$. From (1.2), we obtain

$$\begin{aligned}\frac{2}{1+|\Theta'(0)|} &\leq |\Theta'(z_0)|=\frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|\frac{p'(z_0)}{p(z_0)}\right|=\frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|p'(z_0)\right| \\ &= \frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|\frac{\phi'(z_0)}{B_0(z_0)}-\frac{\phi(z_0)B_0'(z_0)}{B_0^2(z_0)}\right| \\ &= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(z_0)}\left\{\left|\phi'(z_0)\right|-\left|B_0'(z_0)\right|\right\}.\end{aligned}$$

Replacing $\arg^2 p(z_0)$ by zero, we take

$$\begin{aligned}\frac{2}{1+|\Theta'(0)|} &\leq \frac{-2}{\ln\left(\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}\right)}\left(\frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}|h'(z_0)|-p\right)\leq\frac{1+|h(0)|}{1-|h(0)|}|h'(z_0)| \\ &= \frac{-2}{\ln\left(\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}\right)}\left(\frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\overline{a}z_0})\right|-p\right).\end{aligned}$$

Since

$$\Theta'(z)=2\ln p(0)\frac{p'(z)}{p(z)\left(\ln p(z)+\ln p(0)\right)^2}$$

and

$$\Theta'(0)=\frac{p'(0)}{2p(0)\ln p(0)},$$

we get

$$\frac{2}{1-\frac{\left|\left(1-|a|^2\right)c_{p+1}-\overline{a}pc_p\right|}{2\left|c_p\right|\ln\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}}}\leq\frac{-2}{\ln\left(\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}\right)}\left(\frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\overline{a}z_0})\right|-p\right)$$

$$\left|f'(c)\right|\geq\alpha\frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left[p-\frac{2\ln^2\left(\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}\right)}{2\left|c_p\right|\ln\alpha\frac{\left(1-|a|^2\right)^p|c_p|}{\alpha^2-|b-\alpha|^2}-\left|\left(1-|a|^2\right)c_{p+1}-\overline{a}pc_p\right|}\right].$$

Now, we shall show that the inequality (1.14) is sharp. Let

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\overline{a}z}\right)^p}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p}.$$

Then

$$\begin{aligned}f'(z) &= \alpha b\frac{p\left(\frac{z-a}{1-\overline{a}z}\right)^{p-1}\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)}{\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)^2} \\ &\quad -\frac{(b-\alpha)p\left(\frac{z-a}{1-\overline{a}z}\right)^{p-1}\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)\left(1+\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)}{\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)^2}.\end{aligned}$$

Thus, since $-1<a\leq 0$ with $\alpha<b<2\alpha$, we take

$$f'(1)=\alpha p\frac{2\alpha-b}{b}\frac{1+a}{1-a}.$$

Since $\left|c_p\right|=\frac{b(2\alpha-b)}{\alpha(1-a^2)^p}$, (1.14) is satisfied with equality. □

We note that the inequality (1.2) has been used in the proofs of Theorem 2.6 and Theorem 2.7. So, there are both c_p and c_{p+1} in the right side of the inequalities. But, if we use (1.3) instead of (1.2), we obtain weaker but more simpler inequality (not including c_{p+1}). It is formulated in the following Theorem.

Theorem 2.8. *Under the hypotheses of Theorem 2.7, we have*

$$\left|f'(c)\right| \geq \alpha \frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left[p-\frac{1}{2}\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)\right]. \tag{1.15}$$

The equality in (1.15) *occurs for the function*

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\overline{a}z}\right)^p}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p},$$

where $-1<a\leq 0$ *with* $\alpha<b<2\alpha$*.*

Proof. From Theorem 2.7, using the inequality (1.3) for the function $\Theta(z)$, we obtain

$$\begin{aligned}
1 &\leq |\Theta(z_0)| = \frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|\frac{p'(z_0)}{p(z_0)}\right| = \frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|p'(z_0)\right| \\
&= \frac{\left|2\ln p(0)\right|}{\left|\ln p(z_0)+\ln p(0)\right|^2}\left|\frac{\phi'(z_0)}{B_0(z_0)}-\frac{\phi(z_0)B_0'(z_0)}{B_0^2(z_0)}\right| \\
&= \frac{-2\ln p(0)}{\ln^2 p(0)+\arg^2 p(z_0)}\left\{\left|\phi'(z_0)\right|-\left|B_0'(z_0)\right|\right\}.
\end{aligned}$$

Replacing $\arg^2 p(z_0)$ by zero, we take

$$\begin{aligned}
1 &\leq \frac{-2}{\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)}\left(\frac{\left(1-|h(0)|^2\right)}{\left|1-h(z_0)\overline{h(0)}\right|^2}|h'(z_0)|-p\right)\leq\frac{1+|h(0)|}{1-|h(0)|}|h'(z_0)| \\
&= \frac{-2}{\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)}\left(\frac{1}{\alpha}\frac{1+|a|}{1-|a|}\frac{\alpha+|b-\alpha|}{\alpha-|b-\alpha|}\left|f'(\frac{z_0+a}{1+\overline{a}z_0})\right|-p\right)
\end{aligned}$$

and

$$\left|f'(c)\right| \geq \alpha \frac{1-|a|}{1+|a|}\frac{\alpha-|b-\alpha|}{\alpha+|b-\alpha|}\left[p-\frac{1}{2}\ln\left(\alpha\frac{\left(1-|a|^2\right)^p\left|c_p\right|}{\alpha^2-|b-\alpha|^2}\right)\right].$$

Now, we shall show that the inequality (1.15) is sharp. Let

$$f(z)=\alpha b\frac{1+\left(\frac{z-a}{1-\overline{a}z}\right)^p}{\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p}.$$

Then

$$\begin{aligned}
f'(z) &= \alpha b\frac{p\left(\frac{z-a}{1-\overline{a}z}\right)^{p-1}\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)}{\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)^2} \\
&\quad -\frac{(b-\alpha)p\left(\frac{z-a}{1-\overline{a}z}\right)^{p-1}\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)\left(1+\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)}{\left(\alpha+(b-\alpha)\left(\frac{z-a}{1-\overline{a}z}\right)^p\right)^2}.
\end{aligned}$$

Thus, since $-1 < a \leq 0$ with $\alpha < b < 2\alpha$, we take

$$f'(1) = \alpha p \frac{2\alpha - b}{b} \frac{1+a}{1-a}.$$

Since $\left|c_p\right| = \frac{b(2\alpha - b)}{\alpha(1-a^2)^p}$, (1.15) is satisfied with equality. □

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