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A Categorical Approach to Soft S-Acts

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Abstract. The purpose of this paper is to study certain categorical properties of the categories **SoftAct** of all soft S-acts and soft homomorphisms, and **WSoftAct** of all soft S-acts and weak soft homomorphisms. We investigate the interrelations of some particular morphisms, limits and colimits in **SoftAct** and **WSoftAct** with their corresponding notions in the categories **Act-**S and **Set**. It is proved that **SoftAct** has non-empty soft coproducts and soft coequalizers and then soft pushouts. Moreover, **WSoftAct** has arbitrary w-soft products and non-empty w-soft coproducts. Some results concerning soft equalizers and w-soft pullbacks are also presented.

1. Introduction

The theory of soft sets was initiated by Molodtsov [15] as a new mathematical tool for dealing with uncertainties which is free from the difficulties affecting classical methods. A soft set is a parameterized family of subsets of an initial universe set. The absence of any restrictions on the approximate description in soft set theory makes this theory very convenient and easily applicable. Soft set theory has potential applications in many different fields including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. The study on connections between soft sets and algebraic structures has been of interest for some authors. As the first step in this direction, Aktaş and Çağman in [4] studied soft groups. Sun et al. [16] defined soft modules and investigated their basic properties (see also [17]). Soft rings were introduced by Akar et al. [2], and Atagün and Sezgin [7] studied soft substructures of rings and modules. Soft semigroups were touched in [6] and then extensively studied in [10] by means of soft relations. Soft sets were applied to soft ordered semigroups by Jun et al. [11] and a lattice structure on soft sets was considered in [1]. Recently on this area some new papers appeared, such as [9, 13].

Acts over a semigroup S, namely S-acts, appeared and were used in a variety of applications like graph theory, combinatorial problems, algebraic automata theory, mathematical linguistics, theory of machines and theoretical computer science. Over the past three decades, an extensive theory of the properties of S-acts has been developed. A comprehensive survey of this area was published in 2000 by Kilp et al. in [12]. In [5], Ali et al. introduced the concept of soft S-act to characterize general fuzzy soft S-acts. Some authors have investigated some categorical properties including limits and colimits in the category **SoftSet** (see, for example, [19, 20]). Inspired by these studies, in this paper we investigate some limits and colimits such as products, coproducts, equalizers, coequalizers, pullbacks and pushouts in the categories **SoftAct**

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and **WSoftAct**, and show that the mentioned categorical constructions in these two categories are mostly relevant to their counterparts in **Act-**S and **Set**. Let P be a category-theoretic notion in **SoftAct**. We use the term "soft corresponding notion" for P. For example, by a soft monomorphism we mean a monomorphism in **SoftAct**, a soft pullback refers to a pullback in **SoftAct**, etc. Analogously, we use "w-soft corresponding notion" for a category-theoretic notion in **WSoftAct**.

Here we first consider some particular morphisms in **SoftAct** and **WSoftAct**. In particular, we characterize soft monomorphisms, soft epimorphisms and soft isomorphisms in terms of monomorphisms, epimorphisms and isomorphisms in **Act-**S and **Set**. Next we investigate products and coproducts in **SoftAct** and **WSoftAct** and show that **WSoftAct** has arbitrary w-soft products and non-empty w-soft coproducts. As for **SoftAct**, soft products are not necessarily constructed via products in **Act-**S and **Set** whereas, in analogy to the category **WSoftAct**, one can construct non-empty soft coproducts via their correspondings in **Act-**S and **Set**. Afterwards, it is shown that **SoftAct** has soft coequalizers and find a necessary and sufficient condition under which soft equalizers exist. Finally, **SoftAct** has soft pushouts and a result dealing with w-soft pullbacks in **WSoftAct** is obtained.

It is remarkable to note that if one considers S to be the trivial monoid, then the category **SoftAct** is equivalent to the full subcategory $\overline{\textbf{SoftSet}}$ of **SoftSet** consisting of all soft sets $(F,A)_U$ satisfying $F(a) \neq \emptyset$ for all $a \in A$ and soft maps between them and hence all results here dealing with categorical properties of **SoftAct** is also valid for $\overline{\textbf{SoftSet}}$.

2. Preliminaries

In this section we give a brief account of some basic definitions about S-acts, soft sets, and soft S-acts needed in the sequel.

Let S be a monoid. A (*right*) *S-act* or *S-set* is a set A on which S acts unitarily from the right with the usual properties, that is to say $a(st) = (as)t$ and $a1 = a$, for all $a \in A$ and $s, t \in S$, where 1 denotes the identity of S. An element θ of an S-act is called a *zero* element if $\theta s = \theta$ for all $s \in S$. A one-element S-act is denoted by Θ_S, where $\Theta_S = \{\theta\}$. It should be noted that every set A can be made into an S-act with *trivial action*: $as = a$, for every $a \in A, s \in S$. An *S-homomorphism* (or an *S-map*) from an S-act A to an S-act B is a map $f : A \to B$ such that $f(as) = f(a)s$ for all $a \in A, s \in S$. Since the identity maps and the composition of two S-maps are S-maps, we have the category **Act-**S of all (right) S-acts and S-maps between them. Monomorphisms in **Act-**S are precisely injective S-maps. Also epimorphisms and surjective S-maps coincide. So isomorphisms of S-acts are exactly monomorphism as well as epimorphism S-maps which are bijective S-maps. We have the same situations in **Set**. An S-act B is said to be a homomorphic image of an S-act A if there is an epimorphism from A to B. Let A be an S-act. An equivalence relation ρ on A is called an *S-act congruence* (or simply a *congruence*) on A if $a\rho a'$ implies $as\rho a's$ for $a, a' \in A$, $s \in S$. If ρ is a congruence on A, then the factor set $A/\rho = \{[a]_\rho : a \in A\}$ is clearly an S-act, called the *factor act* of A by ρ, with the action given by $[a]_\rho s = [as]_\rho$, for $a \in A$, $s \in S$. For $H \subseteq A \times A$, $\rho(H)$ denotes the congruence on A generated by H, that is, the smallest congruence on A containing H (see [12, Lemma I.4.37]). The *product* of a family of S-acts is their cartesian product with componentwise action. Also the *coproduct* is their disjoint union with natural action. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. The symbol $\sqcup$ may be used for the coproduct of (not necessarily distinct) two S-acts. The reader is referred to [12] for some required definitions and basic categorical ingredients of **Act-**S needed in the sequel. Considering S to be the trivial monoid, the category **Acts-**S is equivalent to the category **Set** and so the categorical constructions are obtained similarly for sets. For required categorical definitions such as monomorphism, epimorphism, isomorphism, product, coproduct, equalizer, coequalizer, pullback and pushout, one may consult [3, 8]. Throughout the paper S stands for a monoid.

Here we recall the notion of soft set from [15]. Let U be a universe set and E be a set of parameters. Let $\mathrm{P}(U)$ denote the power set of U and $A \subseteq E$. A pair $(F,A)_U$, or briefly (F,A), is called a *soft set* over U if $F : A \to \mathrm{P}(U)$ is a set valued mapping. Let $(F,A)_U$ and $(G,B)_V$ be two soft sets. By a *soft map* from $(F,A)_U$ to $(G,B)_V$ is a pair (f,g) where $f : U \to V$ and $g : A \to B$ are two maps for which $f(F(a)) = G(g(a))$ for all $a \in A$.

The category of all soft sets and soft maps between them is denoted by **SoftSet**. For more information on this basic concept, see [14, 15].

Let us recall from [5] some definitions concerning soft S-acts.

Let $\mathcal{A}$ be an S-act and $(F, A)_{\mathcal{A}}$ be a soft set over $\mathcal{A}$. Then $(F, A)_{\mathcal{A}}$, or simply (F, A), is called a *soft S-act* or *soft S-set* over $\mathcal{A}$ if $F(a) \neq \emptyset$ is a subact of $\mathcal{A}$ for all $a \in A$. It is obvious that every soft set $(F, A)_U$ satisfying $F(a) \neq \emptyset$ for all $a \in A$ can be made into a soft S-act by considering U as an S-act with trivial action. The reader is referred to [5] to see some examples of soft S-acts.

Let (F, A) and (G, B) be two soft S-acts over S-acts $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $f : \mathcal{A} \to \mathcal{B}$ be an S-homomorphism and $g : A \to B$ be a map. Then (f, g) is called a *soft homomorphism* from $(F, A)_{\mathcal{A}}$ to $(G, B)_{\mathcal{B}}$ if $f(F(a)) = G(g(a))$ for all $a \in A$. Note that in [5], g is assumed to be onto in the definition of soft homomorphism whereas it is not supposed here. Also if $f(F(a)) \subseteq G(g(a))$ for all $a \in A$, then we say that (f, g) is a *weak soft homomorphism*. Clearly, every soft homomorphism is a weak soft homomorphism. Note that in some papers the weak condition "$G(g(a)) \subseteq f(F(a))$ for all $a \in A$" has been considered as the definition of soft mapping (see, for example, [18]). If $(f, g) : (F, A)_{\mathcal{A}} \to (G, B)_{\mathcal{B}}$ and $(h, k) : (G, B)_{\mathcal{B}} \to (H, C)_C$ are soft homomorphisms, then the *soft composition* of (f, g) and (h, k) is defined as $(h, k)(f, g) = (p, q) : (F, A)_{\mathcal{A}} \to (H, C)_C$ where $p = hf$ and $q = kg$. It is easily seen that (p, q) is a soft homomorphism. Also for every soft S-act (F, A) over an S-act $\mathcal{A}$, the identity soft homomorphism $id_{(F,A)_{\mathcal{A}}}$ is given as $(id_{\mathcal{A}}, id_A) : (F, A)_{\mathcal{A}} \to (F, A)_{\mathcal{A}}$. Note that for every $a \in A$, $id_{\mathcal{A}}(F(a)) = F(a) = F(id_A(a))$.

Example 2.1. Let $\mathcal{A}, \mathcal{A}'$ be two S-acts and $f : \mathcal{A} \to \mathcal{A}'$ be an S-map. Consider $\overline{\mathcal{A}}$ and $\overline{f}$ as the underling set of $\mathcal{A}$ and the underling map of f, respectively. Define $F : \overline{\mathcal{A}} \to \mathrm{P}(\mathcal{A})$ by $F(a) = \langle a \rangle$, for every $a \in \overline{\mathcal{A}}$. Then $(F, \overline{\mathcal{A}})_{\mathcal{A}}$ is a soft S-act. Similarly, take the soft S-act $(F', \overline{\mathcal{A}'})_{\mathcal{A}'}$. This implies that $(f, \overline{f}) : (F, \overline{\mathcal{A}})_{\mathcal{A}} \to (F', \overline{\mathcal{A}'})_{\mathcal{A}'}$ is a soft homomorphism. Indeed, for every $a \in \overline{\mathcal{A}}$, $f(F(a)) = f(\langle a \rangle) = f(aS) = f(a)S = \langle f(a) \rangle = \langle \overline{f}(a) \rangle = F'(\overline{f}(a))$.

The class of soft S-acts together with weak soft homomorphisms between them forms a category which is denoted by **WSoftAct**. Moreover, **SoftAct** stands for the subcategory of **WSoftAct** with the same objects as in **WSoftAct** together with all soft homomorphisms between them. It should be noted that **SoftAct** is concrete over **SoftSet**.

3. On Particular (Weak) Soft Homomorphisms

In this section we study soft monomorphisms and soft epimorphisms in **SoftAct**. Then we characterize all soft isomorphisms in terms of all isomorphisms in **Act-S** and **Set**. These notions in the weak form are also considered in **WSoftAct**. First we give some definitions.

Let $(f, g) : (F, A)_{\mathcal{A}} \to (G, B)_{\mathcal{B}}$ be a soft homomorphism. Then (f, g) is called a *soft monomorphism* if for any soft homomorphisms $(h, k), (h', k') : (H, C)_C \to (F, A)_{\mathcal{A}}$ with $(f, g)(h, k) = (f, g)(h', k')$, one deduces that $(h, k) = (h', k')$, i.e. $h = h'$ and $k = k'$. We have the dual definition for a *soft epimorphism*. By a *soft bimorphism* we mean a soft homomorphism which is both a soft monomorphism and a soft epimorphism. We say that (f, g) is a *soft retraction* if it has a right inverse, i.e. there exists a soft homomorphism $(f', g') : (G, B)_{\mathcal{B}} \to (F, A)_{\mathcal{A}}$ such that $(f, g)(f', g') = id_{(G,B)_{\mathcal{B}}}$. Dually, (f, g) is said to be a *soft coretraction* or *soft section* if it has a left inverse. Also a soft retraction as well as a soft section is called a *soft isomorphism*. We have also the same definitions for weak soft homomorphisms.

Obviously, a right and a left inverse of a soft isomorphism (f, g) coincide and so one can speak about the inverse of (f, g) which is unique and denoted by $(f, g)^{-1}$ which is indeed (f^{-1}, g^{-1}) (see Corollary 3.2). Clearly, every soft isomorphism is a soft bimorphism.

Theorem 3.1. *Let $(f, g) : (F, A)_{\mathcal{A}} \to (G, B)_{\mathcal{B}}$ be a soft homomorphism. Then (f, g) is a soft monomorphism* (*soft epimorphism*) *if and only if f and g are injective* (*surjective*).

Proof. If f and g are injective (surjective), then one can easily check that (f, g) is a soft monomorphism (soft epimorphism). Now we prove the converse of the assertion.

Assume that (f,g) is a soft monomorphism. First we show that f is injective. Take two S-maps $\alpha,\beta : C \to \mathcal{A}$ such that $f\alpha = f\beta$. Consider the soft S-act $(\Phi,\emptyset)_C$ where $\Phi : \emptyset \to P(C)$ is the empty map. Clearly, $(\alpha,\emptyset),(\beta,\emptyset) : (\Phi,\emptyset)_C \to (F,A)_{\mathcal{A}}$ are soft homomorphisms where $\emptyset : \emptyset \to A$ is the empty map. Also $(f,g)(\alpha,\emptyset) = (f,g)(\beta,\emptyset)$. The left cancellation property of (f,g) implies that $\alpha = \beta$, as desired. Now let $h,k : C \to A$ be two maps with $gh = gk$. Consider the soft S-act $(H,C)_{\mathcal{A}}$ where $H : C \to P(\mathcal{A})$ is given by $H(c) = F(h(c))$ for any $c \in C$. We claim that $F(h(c)) = F(k(c))$ for any $c \in C$. To this end, take a $c \in C$. Then

$$f(F(h(c))) = G(g(h(c))) = G(g(k(c))) = f(F(k(c))),$$

and hence $F(h(c)) = F(k(c))$ by injectivity of f. This implies that $(id_{\mathcal{A}},h),(id_{\mathcal{A}},k) : (H,C)_{\mathcal{A}} \to (F,A)_{\mathcal{A}}$ are soft homomorphisms. Moreover, $(f,g)(id_{\mathcal{A}},h) = (f,g)(id_{\mathcal{A}},k)$. Since (f,g) is left cancellable, $h = k$ which shows that g is injective.

Finally, let (f,g) be a soft epimorphism. Suppose that g is not surjective. So there is a $u \in B$ such that $u \notin Im(g)$. Define $h,k : B \to B \cup \{1,2\}$ by

$$h(x)=\begin{cases} x & x \neq u \\ 1 & x = u, \end{cases}$$

and

$$k(x)=\begin{cases} x & x \neq u \\ 2 & x = u. \end{cases}$$

Clearly, $hg = kg$. Now consider the map $H : B \cup \{1,2\} \to P(\mathcal{B} \cup \{\theta\})$, where θ is a zero element externally adjoint to B, defined as

$$H(x)=\begin{cases} G(b) & x = h(b) \text{ or } k(b) \\ \{\theta\} & \text{otherwise.} \end{cases}$$

Then $(H,B \cup \{1,2\})_{\mathcal{B}\cup\{\theta\}}$ is a soft S-act. It is easy to see that $(\mathcal{J},h),(\mathcal{J},k) : (G,B)_{\mathcal{B}} \to (H,B \cup \{1,2\})_{\mathcal{B}\cup\{\theta\}}$ are soft homomorphisms, where $\mathcal{J} : \mathcal{B} \to \mathcal{B} \cup \{\theta\}$ is the inclusion map. Also $(\mathcal{J},h)(f,g) = (\mathcal{J},k)(f,g)$. It follows from the right cancellation property of (f,g) that $h = k$ which is a contradiction. It remains to show that f is surjective. Let $\alpha,\beta : \mathcal{B} \to C$ be two S-maps such that $\alpha f = \beta f$. Consider the soft S-act $(H,B)_C$ where $H : B \to P(C)$ is defined by $H(b) = \alpha(G(b))$ for any $b \in B$. We show that $\alpha(G(b)) = \beta(G(b))$ for any $b \in B$. For this, take any $b \in B$. Since g is surjective, $b = g(a)$ for some $a \in A$. Now we get

$$\alpha(G(b)) = \alpha(G(g(a))) = \alpha(f(F(a))) = \beta(f(F(a))) = \beta(G(g(a))) = \beta(G(b)).$$

This gives that $(\alpha,id_B),(\beta,id_B) : (G,B)_{\mathcal{B}} \to (H,B)_C$ are soft homomorphisms. Moreover, $(\alpha,id_B)(f,g) = (\beta,id_B)(f,g)$ whence $\alpha = \beta$ by the right cancellation property of (f,g). □

The following result characterizes all soft isomorphisms in terms of all isomorphisms in **Act-**S and **Set**.

Corollary 3.2. *Let $(f,g) : (F,A)_{\mathcal{A}} \to (G,B)_{\mathcal{B}}$ be a soft homomorphism. Then (f,g) is a soft isomorphism in* **SoftAct** *if and only if f and g are isomorphisms in* **Act-**S *and* **Set***, respectively. In this case, $(f,g)^{-1} = (f^{-1},g^{-1})$.*

Proof. The first assertion follows from Theorem 3.1. It remains to verify that (f^{-1},g^{-1}) is a soft homomorphism. Using the fact that (f,g) is a soft homomorphism, for every $b \in B$ we have

$$f^{-1}(G(b)) = f^{-1}(G(id_B(b))) = f^{-1}(G(gg^{-1}(b))) = f^{-1}(G(g(g^{-1}(b))))$$

$$= f^{-1}(f(F(g^{-1}(b)))) = (f^{-1}f)(F(g^{-1}(b))) = id_{\mathcal{A}}(F(g^{-1}(b))) = F(g^{-1}(b)),$$

as required. □

A category is called *balanced* if every bimorphism is an isomorphism. It is known that the categories **Act-**S and **Set** are balanced. In view of Theorem 3.1, the same result holds for the category **SoftAct**, i.e. soft bimorphisms and soft isomorphisms coincide.

As for weak soft monomorphisms and weak soft epimorphisms, we have

Proposition 3.3. *Let* $(f,g) : (F,A)_{\mathcal{A}} \to (G,B)_{\mathcal{B}}$ *be a weak soft homomorphism. The following assertions hold:*
(i) *If* f *and* g *are injective, then* (f,g) *is a weak soft monomorphism.*
(ii) *If* (f,g) *is a weak soft monomorphism, then* f *is injective.*
(iii) (f,g) *is a weak soft epimorphism if and only if* f *and* g *are surjective.*
(iv) *If* (f,g) *is a weak soft isomorphism, then* f *and* g *are isomorphisms in* **Act-**S *and* **Set**, *respectively.*

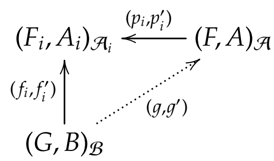
Proof. The statements (i), (ii) and (iii) follow from the same methods as in the proof of Theorem 3.1 for the weak soft case, except for surjectivity of f in the necessary direction of (iii) that requires to modify the corresponding proof of Theorem 3.1 by considering the soft S-act $H : B \to P(C)$ defined by $H(b) = C$ for any $b \in B$. This implies that $(\alpha, id_B), (\beta, id_B) : (G,B)_{\mathcal{B}} \to (H,B)_C$ are weak soft homomorphisms which gives the assertion.

(iv) Suppose that $(f,g) : (F,A)_{\mathcal{A}} \to (G,B)_{\mathcal{B}}$ is a weak soft isomorphism. It follows that (f,g) has an inverse, say (f',g'), in **WSoftAct**. Thus $(f,g)(f',g') = id_{(G,B)_{\mathcal{B}}} = (id_{\mathcal{B}}, id_B)$ and $(f',g')(f,g) = id_{(F,A)_{\mathcal{A}}} = (id_{\mathcal{A}}, id_A)$. This implies that $ff' = id_{\mathcal{B}}, f'f = id_{\mathcal{A}}, gg' = id_B, g'g = id_A$; showing that f and g are isomorphisms in **Act-**S and **Set**, respectively. □

4. On (W-)Soft Products and (W-)Soft Coproducts

In this section products and coproducts in the categories **SoftAct** and **WSoftAct** are investigated.

Let $\{(F_i,A_i)_{\mathcal{A}_i}\}_{i\in I}$ be a family of soft S-acts. A *soft product* $\prod_{i\in I}(F_i,A_i)_{\mathcal{A}_i}$ of this family is a soft S-act $(F,A)_{\mathcal{A}}$ together with soft homomorphisms $(p_i,p'_i) : (F,A)_{\mathcal{A}} \to (F_i,A_i)_{\mathcal{A}_i}$ such that for any soft S-act $(G,B)_{\mathcal{B}}$ and soft homomorphisms $(f_i,f'_i) : (G,B)_{\mathcal{B}} \to (F_i,A_i)_{\mathcal{A}_i}$, there exists a unique soft homomorphism $(g,g') : (G,B)_{\mathcal{B}} \to (F,A)_{\mathcal{A}}$ such that for every $i \in I$ the following diagram commutes:



Replacing "soft homomorphism" by "weak soft homomorphism" in the above definition, we get the notion of *w-soft product* in **WSoftAct** which is represented as $\prod^{w}_{i\in I}(F_i,A_i)_{\mathcal{A}_i}$. Dually, a *soft coproduct* $\coprod_{i\in I}(F_i,A_i)_{\mathcal{A}_i}$ and a *w-soft coproduct* $\coprod^{w}_{i\in I}(F_i,A_i)_{\mathcal{A}_i}$ of the family $\{(F_i,A_i)_{\mathcal{A}_i}\}_{i\in I}$ in **SoftAct** and **WSoftAct** are defined, respectively.

If $I = \emptyset$, the soft product of $\{(F_i,A_i)_{\mathcal{A}_i}\}_{i\in I}$ in **SoftAct** is a *terminal object*, that is, an object T with only one morphism from every object to T; and dually, the soft coproduct of $\{(F_i,A_i)_{\mathcal{A}_i}\}_{i\in I}$ in **SoftAct** is an *initial object* in **SoftAct**. In **Set** the empty set $\emptyset$ is an initial object and any one-element set is a terminal object. In **Act-**S the one-element act Θ_S is a terminal object. But, **Act-**S has no initial object because we can always consider a two-element act $\Theta_S \sqcup \Theta_S$ in **Act-**S and thus any act has at least two S-maps into $\Theta_S \sqcup \Theta_S$. As for **SoftAct** we have the following:

Proposition 4.1. *The category* **SoftAct** *has a terminal object but not an initial object. The same situation holds in* **WSoftAct**.

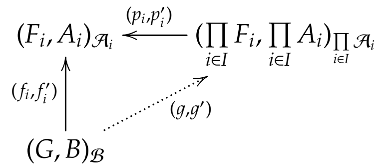
Proof. First we show that $(T, \{*\})_{\Theta_S}$ is a terminal object in **SoftAct**, where $\Theta_S = \{\theta\}$, and $T : \{*\} \to \mathrm{P}(\Theta_S)$ is given by $T(*) = \Theta_S$. Take any soft S-act $(F, A)_{\mathcal{A}}$. Then the pair $(t, t') : (F, A)_{\mathcal{A}} \to (T, \{*\})_{\Theta_S}$ is the unique soft homomorphism, where $t : \mathcal{A} \to \Theta_S$ and $t' : A \to \{*\}$ are defined by $t(\alpha) = \theta$ and $t'(a) = *$, for all $\alpha \in \mathcal{A}, a \in A$, respectively. Indeed, for every $a \in A$, $t(F(a)) = \Theta_S = T(*) = T(t'(a))$. Also it is easily seen that (t, t') is unique. Now we claim that **SoftAct** has no initial object. First note that the pair $(G, \{1, 2\})_{\Theta_S \sqcup \Theta_S}$ with $\Theta_S \sqcup \Theta_S = \{(\theta, 1), (\theta, 2)\}$, $G : \{1, 2\} \to \mathrm{P}(\Theta_S \sqcup \Theta_S)$ given by $G(i) = \Theta_S \times \{i\}, i = 1, 2$, is a soft S-act. For every soft S-act $(F, A)_{\mathcal{A}}$, there always exist at least two soft homomorphisms $(f_i, g_i) : (F, A)_{\mathcal{A}} \to (G, \{1, 2\})_{\Theta_S \sqcup \Theta_S}, i = 1, 2$, where $f_i : \mathcal{A} \to \Theta_S \sqcup \Theta_S$ and $g_i : A \to \{1, 2\}$ are defined by $f_i(\alpha) = (\theta, i)$ and $g_i(a) = i$, respectively. For this, let $a \in A$. Then $f_i(F(a)) = \{(\theta, i)\} = \Theta_S \times \{i\} = G(i) = G(g_i(a))$. Analogously, the second assertion is also obtained. □

Proposition 4.1 states that **(W)SoftAct** has empty (w-)soft products but not empty (w-)soft coproducts.

The following result presents a connection between w-soft products in **WSoftAct** and the products in **Act-**S and **Set**.

Theorem 4.2. **WSoftAct** *has arbitrary w-soft products:*
Let $\{(F_i, A_i)_{\mathcal{A}_i}\}_{i\in I}$ *be a (non-empty) family of soft S-acts. Then the soft S-act* $(\prod_{i\in I} F_i, \prod_{i\in I} A_i)_{\prod_{i\in I} \mathcal{A}_i}$ *is a* $\overset{w}{\prod_{i\in I}} (F_i, A_i)_{\mathcal{A}_i}$ *of this family, where* $\prod_{i\in I} F_i : \prod_{i\in I} A_i \to \mathrm{P}(\prod_{i\in I} \mathcal{A}_i)$ *is given by* $(\prod_{i\in I} F_i)((a_i)_{i\in I}) = \prod_{i\in I}(F_i(a_i))$ *for all* $a_i \in A_i, i \in I$.

Proof. If $I = \emptyset$, then the w-soft product exists by Proposition 4.1. Let I be non-empty. Note that for every $i \in I$, $F_i(a_i)$ is a non-empty subact of $\mathcal{A}_i$ and then $(\prod_{i\in I} F_i)((a_i)_{i\in I}) = \prod_{i\in I}(F_i(a_i))$ is a non-empty subact of $\prod_{i\in I} \mathcal{A}_i$. Thus $(\prod_{i\in I} F_i, \prod_{i\in I} A_i)_{\prod_{i\in I} \mathcal{A}_i}$ is a soft S-act. Also the pair $(p_i, p'_i) : (\prod_{i\in I} F_i, \prod_{i\in I} A_i)_{\prod_{i\in I} \mathcal{A}_i} \to (F_i, A_i)_{\mathcal{A}_i}$ is a soft homomorphism and then weak soft homomorphism, where $p_i : \prod_{i\in I} \mathcal{A}_i \to \mathcal{A}_i$ and $p'_i : \prod_{i\in I} A_i \to A_i$ are the i-th projection maps. Indeed, $p_i((\prod_{i\in I} F_i)((a_i)_{i\in I})) = p_i(\prod_{i\in I}(F_i(a_i))) = F_i(a_i) = F_i(p'_i((a_i)_{i\in I}))$. Suppose that $(G, B)_{\mathcal{B}}$ is any soft S-act and $(f_i, f'_i) : (G, B)_{\mathcal{B}} \to (F_i, A_i)_{\mathcal{A}_i}$ is a weak soft homomorphism. Define $g : \mathcal{B} \to \prod_{i\in I} \mathcal{A}_i$ by $g(\beta) = (f_i(\beta))_{i\in I}$. Clearly, g is an S-map. Similarly, define $g' : B \to \prod_{i\in I} A_i$ by $g'(b) = (f'_i(b))_{i\in I}$. We show that (g, g') is the unique weak soft homomorphism such that for all $i \in I$ the following diagram commutes:



For every $b \in B$, we have

$$\begin{aligned} g(G(b)) = \{g(\beta) : \beta \in G(b)\} = \{(f_i(\beta))_{i\in I} : \beta \in G(b)\} &\subseteq \prod_{i\in I}(f_i(G(b))) \\ \subseteq \prod_{i\in I}(F_i(f'_i(b))) = (\prod_{i\in I} F_i)((f'_i(b))_{i\in I}) = (\prod_{i\in I} F_i)g'(b). \end{aligned}$$

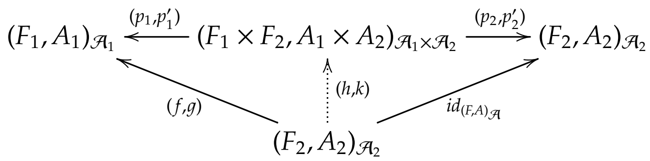
This shows that (g, g') is a weak soft homomorphism. Also for all $i \in I$, $(p_i, p'_i)(g, g') = (p_i g, p'_i g') = (f_i, f'_i)$. Suppose that $(h, h') : (G, B)_{\mathcal{B}} \to (\prod_{i\in I} F_i, \prod_{i\in I} A_i)_{\prod_{i\in I} \mathcal{A}_i}$ is a weak soft homomorphism such that for every $i \in I$, $(p_i, p'_i)(h, h') = (f_i, f'_i)$ and hence $p_i g = f_i = p_i h$ and $p'_i g' = f'_i = p'_i h'$. It follows from the universal property of products in **Act-**S and **Set** that $g = h$ and $g' = h'$, respectively. Therefore, $(g, g') = (h, h')$. This completes the proof. □

Here we show that, in contrast to w-soft products (Theorem 4.2), soft products in **SoftAct** are not constructed via products in **Act-**S and **Set** in general.

By a *universal soft S-act* we mean a soft S-act $(F, A)_{\mathcal{A}}$ for which $F(a) = \mathcal{A}$ for all $a \in A$.

Proposition 4.3. *Let $(F_1, A_1)_{\mathcal{A}_1}$ and $(F_2, A_2)_{\mathcal{A}_2}$ be two universal soft S-acts for which $\mathcal{A}_1$ is an epimorphic image of $\mathcal{A}_2$ and $|\mathcal{A}_1| \geq 2$. Then the soft S-act $(F_1 \times F_2, A_1 \times A_2)_{\mathcal{A}_1 \times \mathcal{A}_2}$ given by $(F_1 \times F_2)(a_1, a_2) = F_1(a_1) \times F_2(a_2)$ for every $(a_1, a_2) \in A_1 \times A_2$ with soft homomorphisms $(p_i, p'_i) : (F_1 \times F_2, A_1 \times A_2)_{\mathcal{A}_1 \times \mathcal{A}_2} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1, 2$, where $p_i : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}_i$ and $p'_i : A_1 \times A_2 \to A_i$ are the i-th projection maps, is not a soft product of $(F_1, A_1)_{\mathcal{A}_1}$ and $(F_2, A_2)_{\mathcal{A}_2}$.*

Proof. Suppose on the contrary that $(F_1 \times F_2, A_1 \times A_2)_{\mathcal{A}_1 \times \mathcal{A}_2}$ is a soft product of $(F_1, A_1)_{\mathcal{A}_1}$ and $(F_2, A_2)_{\mathcal{A}_2}$. Using the assumption, there exists an epimorphism $f : \mathcal{A}_2 \to \mathcal{A}_1$. Take an arbitrary mapping $g : A_2 \to A_1$. Then (f, g) is a soft homomorphism from $(F_2, A_2)_{\mathcal{A}_2}$ to $(F_1, A_1)_{\mathcal{A}_1}$ because $fF_2(a) = f(\mathcal{A}_2) = \mathcal{A}_1 = F_1g(a)$ for every $a \in A_2$. It follows from the universality of soft products that there exists a unique soft homomorphism $(h, k) : (F_2, A_2)_{\mathcal{A}_2} \to (F_1 \times F_2, A_1 \times A_2)_{\mathcal{A}_1 \times \mathcal{A}_2}$ such that the following diagram commutes:



This implies that $h(\alpha) = (f(\alpha), \alpha)$ and $k(a) = (g(a), a)$ for all $\alpha \in \mathcal{A}_2, a \in A_2$. Let $a \in A_2$. We have

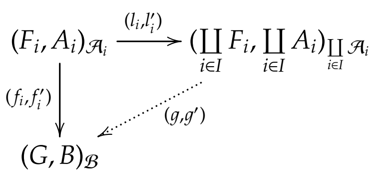
$$\mathcal{A}_1 \times \mathcal{A}_2 = F_1(g(a)) \times F_2(a) = (F_1 \times F_2)(g(a), a) = (F_1 \times F_2)k(a) = hF_2(a) = h(\mathcal{A}_2),$$

and so $|\mathcal{A}_1 \times \mathcal{A}_2| = |h(\mathcal{A}_2)| = |\{(f(\alpha), \alpha) : \alpha \in \mathcal{A}_2\}| = |\mathcal{A}_2|$ whence $|\mathcal{A}_1| = 1$ which is a contradiction. □

In the following, a dual result of Theorem 4.2 for soft coproducts in **(W)SoftAct** is obtained.

Theorem 4.4. **(W)SoftAct** *has non-empty* (*w*-)*soft coproducts:*
Let $\{(F_i, A_i)_{\mathcal{A}_i}\}_{i \in I}$ be a non-empty family of soft S-acts. Then the soft S-act $(\coprod_{i \in I} F_i, \coprod_{i \in I} A_i)_{\coprod_{i \in I} \mathcal{A}_i}$ is a $\coprod_{i \in I} (F_i, A_i)_{\mathcal{A}_i}$ of this family, where $\coprod_{i \in I} F_i : \coprod_{i \in I} A_i \to \mathrm{P}(\coprod_{i \in I} \mathcal{A}_i)$ is given by $(\coprod_{i \in I} F_i)(a_i, i) = F_i(a_i) \times \{i\}$ for all $a_i \in A_i, i \in I$. The same construction holds for w-soft coproducts.

Proof. Since $F_i(a_i)$ is a non-empty subact of $\mathcal{A}_i$ for every $i \in I$, $(\coprod_{i \in I} F_i)(a_i, i) = F_i(a_i) \times \{i\}$ is a non-empty subact of $\coprod_{i \in I} \mathcal{A}_i$. Thus $(\coprod_{i \in I} F_i, \coprod_{i \in I} A_i)_{\coprod_{i \in I} \mathcal{A}_i}$ is a soft S-act. Also the pair $(l_i, l'_i) : (F_i, A_i)_{\mathcal{A}_i} \to (\coprod_{i \in I} F_i, \coprod_{i \in I} A_i)_{\coprod_{i \in I} \mathcal{A}_i}$ is a soft homomorphism, where $l_i : \mathcal{A}_i \to \coprod_{i \in I} \mathcal{A}_i$ and $l'_i : A_i \to \coprod_{i \in I} A_i$ are the i-th injection maps. Indeed, $(\coprod_{i \in I} F_i)l'_i(a_i) = (\coprod_{i \in I} F_i)(a_i, i) = F_i(a_i) \times \{i\} = l_iF_i(a_i)$ for every $a_i \in A_i, i \in I$. Suppose that $(G, B)_{\mathcal{B}}$ is a soft S-act and for $i \in I$, $(f_i, f'_i) : (F_i, A_i)_{\mathcal{A}_i} \to (G, B)_{\mathcal{B}}$ is a soft homomorphism. Define $g : \coprod_{i \in I} \mathcal{A}_i \to \mathcal{B}$ by $g(\alpha_i, i) = f_i(\alpha_i)$, for all $\alpha_i \in \mathcal{A}_i$. Clearly, g is an S-map. Similarly, define $g' : \coprod_{i \in I} A_i \to B$ by $g'(a_i, i) = f'_i(a_i)$. We show that (g, g') is the unique soft homomorphism such that for all $i \in I$ the following diagram commutes:



For every $a_i \in A_i, i \in I$, we have

$$g((\coprod_{i \in I} F_i)(a_i, i)) = g(F_i(a_i) \times \{i\}) = f_i(F_i(a_i)) = G(f'_i(a_i)) = G(g'(a_i, i)).$$

This shows that (g, g') is a soft homomorphism. Also for all $i \in I$, $(g, g')(l_i, l'_i) = (gl_i, g'l'_i) = (f_i, f'_i)$. Suppose that $(h, h') : (\coprod_{i\in I} F_i, \coprod_{i\in I} A_i)_{\coprod_{i\in I}\mathcal{A}_i} \to (G, B)_{\mathcal{B}}$ is a soft homomorphism such that for every $i \in I$, $(h, h')(l_i, l'_i) = (f_i, f'_i)$ and hence $gl_i = f_i = hl_i$ and $g'l'_i = f'_i = h'l'_i$. It follows from the universal property of coproducts in **Act-**S and **Set** that $g = h$ and $g' = h'$, respectively. Consequently, $(g, g') = (h, h')$, as required. □

Remark 4.5. Since coproducts in a category are unique up to isomorphisms, using Corollary 3.2 and Theorem 4.4, if $(F, A)_{\mathcal{A}}$ is a soft coproduct of $\{(F_i, A_i)_{\mathcal{A}_i}\}_{i\in I}$ in **SoftAct**, then A and $\mathcal{A}$ are isomorphic to $\coprod_{i\in I} A_i$ and $\coprod_{i\in I} \mathcal{A}_i$ in **Set** and **Act-**S, respectively. But we don not know if the same fact holds for the soft products.

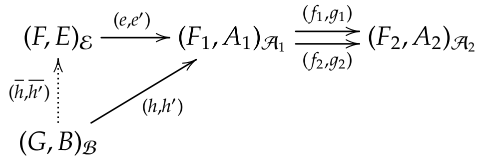
5. On Soft Equalizers and Soft Coequalizers

This section is devoted to study soft equalizers and soft coequalizers in **SoftAct**. We obtain the interrelations of these notions in **SoftAct** with their correspondings in **Act-**S and **Set**. But, considering **WSoftAct**, these connections remain as an open problem.

By a *soft equalizer* of a pair of soft homomorphisms $(F_1, A_1)_{\mathcal{A}_1} \underset{(f_2,g_2)}{\overset{(f_1,g_1)}{\rightrightarrows}} (F_2, A_2)_{\mathcal{A}_2}$ we mean a pair $((F, E)_{\mathcal{E}}, (e, e'))$, where $(F, E)_{\mathcal{E}}$ is a soft S-act and $(e, e') : (F, E)_{\mathcal{E}} \to (F_1, A_1)_{\mathcal{A}_1}$ is a soft homomorphism such that

(i) $(f_1, g_1)(e, e') = (f_2, g_2)(e, e')$.

(ii) For any soft homomorphism $(h, h') : (G, B)_{\mathcal{B}} \to (F_1, A_1)_{\mathcal{A}_1}$ with $(f_1, g_1)(h, h') = (f_2, g_2)(h, h')$, there exists a unique soft homomorphism $(\overline{h}, \overline{h'}) : (G, B)_{\mathcal{B}} \to (F, E)_{\mathcal{E}}$ such that the following diagram commutes:



Replacing "soft homomorphism" by "weak soft homomorphism" in the above definition, we get the notion of *w-soft equalizer* in **WSoftAct**. The notions of *soft coequalizer* and *w-soft coequalizer* in **SoftAct** and **WSoftAct**, respectively, can be defined dually.

The following result presents an equivalent condition for existing soft equalizers in **SoftAct**.

Theorem 5.1. *The soft equalizer of soft homomorphisms* $(F_1, A_1)_{\mathcal{A}_1} \underset{(f_2,g_2)}{\overset{(f_1,g_1)}{\rightrightarrows}} (F_2, A_2)_{\mathcal{A}_2}$ *in* **SoftAct** *is a pair* $((F, E)_{\mathcal{E}}, (e, e'))$, *where* $(e, e') : (F, E)_{\mathcal{E}} \to (F_1, A_1)_{\mathcal{A}_1}$ *is a soft homomorphism for which* $(\mathcal{E}, e)$ *and* (E, e') *are the (existing) equalizers of* $\mathcal{A}_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathcal{A}_2$ *and* $A_1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} A_2$ *in* **Act-**S *and* **Set**, *respectively, if and only if* $F(a) = F_1(a) \subseteq \mathcal{E}$ *for any* $a \in E$.

Proof. First note that $\mathcal{E} = \{\alpha \in \mathcal{A}_1 : f_1(\alpha) = f_2(\alpha)\}$ and $E = \{a \in A_1 : g_1(a) = g_2(a)\}$. Also e and e' are the natural embeddings of $\mathcal{E}$ and E into $\mathcal{A}_1$ and A_1, respectively.

sufficiency. Using the assumption, $(F, E)_{\mathcal{E}}$ is clearly a soft S-act where $F : E \to \mathrm{P}(\mathcal{E})$ is given by $F = F_1|_E$. Moreover, $(e, e') : (F, E)_{\mathcal{E}} \to (F_1, A_1)_{\mathcal{A}_1}$ is a soft homomorphism. Indeed, $e(F(a)) = F(a) = F_1(a) = F_1(e'(a))$ for any $a \in E$. Since $f_1 e = f_2 e$ and $g_1 e' = g_2 e'$, we get $(f_1, g_1)(e, e') = (f_2, g_2)(e, e')$. Suppose that $(h, h') : (G, B)_{\mathcal{B}} \to (F_1, A_1)_{\mathcal{A}_1}$ is a soft homomorphism such that $(f_1, g_1)(h, h') = (f_2, g_2)(h, h')$. Thus $f_1 h = f_2 h$ and $g_1 h' = g_2 h'$. This clearly gives that $Im(h) \subseteq \mathcal{E}$ and $Im(h') \subseteq E$. Considering the S-map $\overline{h} = h : \mathcal{B} \to \mathcal{E}$ and the map $\overline{h'} = h' : B \to E$, the pair $(\overline{h}, \overline{h'}) : (G, B)_{\mathcal{B}} \to (F, E)_{\mathcal{E}}$ is a soft homomorphism because

$$\overline{h}(G(b)) = h(G(b)) = F_1(h'(b)) = F(h'(b)) = F(\overline{h'}(b)),$$

for every $b \in B$. Furthermore,

$$(e,e')(\overline{h},\overline{h'}) = (e,e')(h,h') = (eh, e'h') = (h,h') = (\overline{h},\overline{h'}),$$

i.e. the following diagram is commutative:

$$\begin{array}{ccccc} (F,E)_{\mathcal{E}} & \xrightarrow{(e,e')} & (F_1,A_1)_{\mathcal{A}_1} & \underset{(f_2,g_2)}{\overset{(f_1,g_1)}{\rightrightarrows}} & (F_2,A_2)_{\mathcal{A}_2} \\ {\scriptstyle (\overline{h},\overline{h'})}\uparrow & \nearrow{\scriptstyle (h,h')} & & & \\ (G,B)_{\mathcal{B}} & & & & \end{array}$$

It remains to prove that $(\overline{h},\overline{h'})$ is unique. For this, let $(k,k') : (G,B)_{\mathcal{B}} \to (F,E)_{\mathcal{E}}$ be any soft homomorphism with $(e,e')(k,k') = (h,h')$. Hence, $k = ek = h = \overline{h}$ and $k' = e'k' = h' = \overline{h'}$ which implies that $(k,k') = (\overline{h},\overline{h'})$.

Necessity. Let $a \in E$. Since $F(a)$ is a non-empty subact of $\mathcal{E}$ and (e,e') is a soft homomorphism, $F(a) = e(F(a)) = F_1(e'(a)) = F_1(a)$. □

The coequalizer of a pair of S-maps $\mathcal{A}_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathcal{A}_2$ in **Act-**S is a pair (C,c), where $C = \mathcal{A}_2/\rho$ for an S-act congruence ρ on $\mathcal{A}_2$ generated by $\mathcal{H} = \{(f_1(\alpha), f_2(\alpha)) : \alpha \in \mathcal{A}_1\}$, and $c : \mathcal{A}_2 \to C$ is the canonical epimorphism. We have the same situation for a coequalizer (C,c') of a pair of maps $A_1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} A_2$ in **Set**. In fact, $C = A_2/\theta$ for an equivalence relation θ on A_2 generated by $H = \{(g_1(a), g_2(a)) : a \in A_1\}$, and $c' : A_2 \to C$ is the canonical surjection. Under these notations, the soft coequalizer in **SoftAct** is characterized as follows.

Theorem 5.2. **SoftAct** *has soft coequalizers:*
The soft coequalizer of soft homomorphisms $(F_1,A_1)_{\mathcal{A}_1} \underset{(f_2,g_2)}{\overset{(f_1,g_1)}{\rightrightarrows}} (F_2,A_2)_{\mathcal{A}_2}$ *in* **SoftAct** *is a pair* $((F,C)_C, (c,c'))$, *where* $(c,c') : (F_2,A_2)_{\mathcal{A}_2} \to (F,C)_C$ *is a soft homomorphism for which* (C,c) *and* (C,c') *are the coequalizers of* $\mathcal{A}_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} \mathcal{A}_2$ *and* $A_1 \underset{g_2}{\overset{g_1}{\rightrightarrows}} A_2$ *in* **Act-**S *and* **Set**, *respectively, and* $F([a]_\theta) = c(F_2(a))$ *for every* $a \in A_2$.

Proof. We show that $(F,C)_C$ is a soft S-act. Consider every $a \in A_2$. Note that $F([a]_\theta) = c(F_2(a))$ is a non-empty subact of C because $F_2(a)$ is a non-empty subact of $\mathcal{A}_2$, where θ is an equivalence relation on A_2 generated by $H = \{(g_1(a), g_2(a)) : a \in A_1\}$. Also F is well-defined. For this, let $[a]_\theta = [b]_\theta$ for $a, b \in A_2$. This implies that $a = b$ and then we are done, or there exist $p_1, \ldots, p_n, q_1, \ldots, q_n \in A_2$ where for $i = 1, \ldots, n$, $(p_i, q_i) \in H$ or $(q_i, p_i) \in H$, such that

$$\begin{array}{ccccc} a = p_1 & q_2 = p_3 & & \ldots & q_n = b. \\ & q_1 = p_2 & q_3 = p_4 \ldots & & \end{array}$$

We claim that for each $i = 1, \ldots, n$, $F_2(p_i)/\rho = F_2(q_i)/\rho$. With no loss of generality, one can assume that $p_i = g_1(a_i)$ and $q_i = g_2(a_i)$ for some $a_i \in A_1$. Since ρ is an S-act congruence on $\mathcal{A}_2$ generated by $\mathcal{H} = \{(f_1(\alpha), f_2(\alpha)) : \alpha \in \mathcal{A}_1\}$, for all $\alpha \in F_1(a_i)$ we have $(f_1(\alpha), f_2(\alpha)) \in \rho$. This gives that $[f_1(\alpha)]_\rho = [f_2(\alpha)]_\rho$ and then

$$f_1(F_1(a_i))/\rho = \{[f_1(\alpha)]_\rho : \alpha \in F_1(a_i)\} = \{[f_2(\alpha)]_\rho : \alpha \in F_1(a_i)\} = f_2(F_1(a_i))/\rho.$$

Hence,

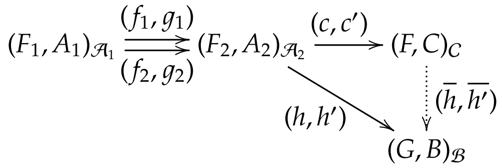
$$F_2(p_i)/\rho = F_2(g_1(a_i))/\rho = f_1(F_1(a_i))/\rho = f_2(F_1(a_i))/\rho = F_2(g_2(a_i))/\rho = F_2(q_i)/\rho.$$

Now we get

$$F([a]_\theta) = c(F_2(a)) = F_2(a)/\rho = F_2(p_1)/\rho = F_2(q_1)/\rho = F_2(p_2)/\rho$$

$$= F_2(q_2)/\rho = \cdots = F_2(q_n)/\rho = F_2(b)/\rho = c(F_2(b)) = F([b]_\theta).$$

Moreover, for each $a \in A_2$, $F(c'(a)) = F([a]_\theta) = c(F_2(a))$. Then (c, c') is a soft homomorphism. Since $cf_1 = cf_2$ and $c'g_1 = c'g_2$, we have $(c, c')(f_1, g_1) = (c, c')(f_2, g_2)$. Suppose that $(h, h') : (F_2, A_2)_{\mathcal{A}_2} \to (G, B)_{\mathcal{B}}$ is a soft homomorphism such that $(h, h')(f_1, g_1) = (h, h')(f_2, g_2)$. Thus $hf_1 = hf_2$ and $h'g_1 = h'g_2$. It follows from the universality of coequalizers in **Act-**S and **Set** that there exist a unique S-map $\overline{h} : \mathcal{C} \to \mathcal{B}$ and a unique map $\overline{h'} : C \to B$ such that $\overline{h}c = h$ and $\overline{h'}c' = h'$. This gives that $(\overline{h}, \overline{h'})(c, c') = (h, h')$, i.e. the following diagram commutes:



We show that $(\overline{h}, \overline{h'})$ is a soft homomorphism. For every $a \in A_2$, we have

$$\overline{h}(F([a]_\theta)) = \overline{h}(c(F_2(a))) = (\overline{h}c)(F_2(a)) = h(F_2(a))$$

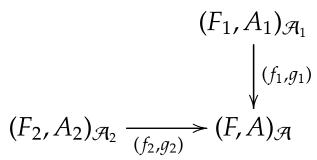
$$= G(h'(a)) = G(\overline{h'}c'(a)) = G(\overline{h'}([a]_\theta)).$$

It remains to prove that $(\overline{h}, \overline{h'})$ is unique. To this end, let $(k, k') : (F, C)_C \to (G, B)_{\mathcal{B}}$ be any soft homomorphism with $(k, k')(c, c') = (h, h')$. Then $kc = h$ and $k'c' = h'$. Thus $\overline{h}c = kc$ and $\overline{h'}c' = k'c'$. Since c and c' are epimorphisms, $\overline{h} = k$ and $\overline{h'} = k'$. Hence, $(\overline{h}, \overline{h'}) = (k, k')$. □

6. On (W-)Soft Pullbacks and (W-)Soft Pushouts

In this section we study the notions of pullbacks and pushouts in **SoftAct** and **WSoftAct**. First we construct w-soft pullbacks in **WSoftAct** under a condition. Thereafter, it is shown that **SoftAct** has soft pushouts and a particular kind of them concerning amalgamated coproducts is characterized as well.

A *soft pullback* of soft homomorphisms

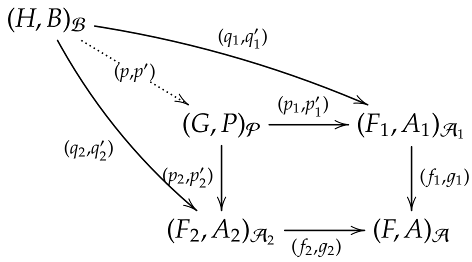


in **SoftAct** is a pair $((G, P)_{\mathcal{P}}, ((p_1, p'_1), (p_2, p'_2)))$, where $(p_i, p'_i) : (G, P)_{\mathcal{P}} \to (F_i, A_i)_{\mathcal{A}_i}$, $i = 1, 2$, are soft homomorphisms such that

(i) $(f_1, g_1)(p_1, p'_1) = (f_2, g_2)(p_2, p'_2)$, and

(ii) the following universal property is fulfilled in **SoftAct**:

For any pair $((H, B)_{\mathcal{B}}, ((q_1, q'_1), (q_2, q'_2)))$ with soft homomorphisms $(q_i, q'_i) : (H, B)_{\mathcal{B}} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1, 2$, and $(f_1, g_1)(q_1, q'_1) = (f_2, g_2)(q_2, q'_2)$ there exists a unique soft homomorphism $(p, p') : (H, B)_{\mathcal{B}} \to (G, P)_{\mathcal{P}}$ such that $(p_i, p'_i)(p, p') = (q_i, q'_i)$, $i = 1, 2$, i.e. the following diagram commutes:

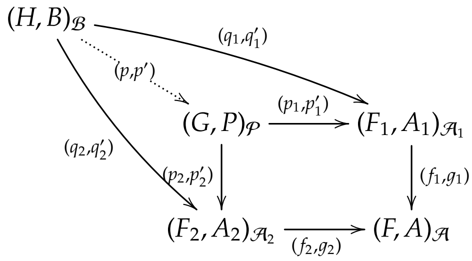


Replacing "soft homomorphism" by "weak soft homomorphism" in the above definition, one gets the notion of *w-soft pullback* in **WSoftAct**. Dually, the concepts of *soft pushout* and *w-soft pushout* are defined in **SoftAct** and **WSoftAct**, respectively.

In the following, a result concerning w-soft pullbacks in **WSoftAct** is obtained.

Theorem 6.1. *The w-soft pullback of weak soft homomorphisms* $(f_i, g_i) : (F_i, A_i)_{\mathcal{A}_i} \to (F, A)_{\mathcal{A}}, i = 1, 2$, *in* **WSoftAct** *is a pair* $((G, P)_{\mathcal{P}}, ((p_1, p_1'), (p_2, p_2')))$, *where* $(\mathcal{P}, (p_1, p_2))$ *and* $(P, (p_1', p_2'))$ *are the (existing) pullbacks of* f_1, f_2 *and* g_1, g_2 *in* **Act-***S and* **Set**, *respectively, and* $(p_i, p_i') : (G, P)_{\mathcal{P}} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1, 2$, *are weak soft homomorphisms for which* $G(a_1, a_2) = F_1(a_1) \times F_2(a_2)$ *provided that* $F_1(a_1) \times F_2(a_2) \subseteq \mathcal{P}$ *for all* $(a_1, a_2) \in P$.

Proof. First recall that $P = \{(a_1, a_2) \in A_1 \times A_2 : g_1(a_1) = g_2(a_2)\}$ and $\mathcal{P} = \{(\alpha_1, \alpha_2) \in \mathcal{A}_1 \times \mathcal{A}_2 : f_1(\alpha_1) = f_2(\alpha_2)\}$. Also $p_i, p_i', i = 1, 2$, are the restrictions to P and $\mathcal{P}$ of the i-th projections from $A_1 \times A_2$ and $\mathcal{A}_1 \times \mathcal{A}_2$ onto A_i and $\mathcal{A}_i$, respectively. Since $f_1 p_1 = f_2 p_2$ and $g_1 p_1' = g_2 p_2'$, we have $(f_1, g_1)(p_1, p_1') = (f_2, g_2)(p_2, p_2')$. It follows from the assumption that $(G, P)_{\mathcal{P}}$ is a soft S-act. Also $p_1(G(a_1, a_2)) = p_1(F_1(a_1) \times F_2(a_2)) = F_1(a_1) = F_1(p_1'(a_1, a_2))$, i.e. (p_1, p_1') and similarly (p_2, p_2') are soft homomorphisms and then weak soft homomorphisms. Now consider any soft S-act $(H, B)_{\mathcal{B}}$ and weak soft homomorphisms $(q_i, q_i') : (H, B)_{\mathcal{B}} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1, 2$, with $(f_1, g_1)(q_1, q_1') = (f_2, g_2)(q_2, q_2')$. This implies that $f_1 q_1 = f_2 q_2$ and $g_1 q_1' = g_2 q_2'$. In view of the universality of pullbacks in **Act-**S and **Set**, there exist a unique S-map $p : \mathcal{B} \to \mathcal{P}$ and a unique map $p' : B \to P$ such that $p_i p = q_i$ and $p_i' p' = q_i'$ for $i = 1, 2$. Therefore, (p, p') satisfies $(p_i, p_i')(p, p') = (q_i, q_i'), i = 1, 2$, i.e. the following diagram commutes:



It must be proved that (p, p') is a weak soft homomorphism. To this end, note that $p(\beta) = (q_1(\beta), q_2(\beta))$ for all $\beta \in \mathcal{B}$, and $p'(b) = (q_1'(b), q_2'(b))$ for all $b \in B$. Let $b \in B$. Then we have

$$p(H(b)) = \{p(\beta) : \beta \in H(b)\} = \{(q_1(\beta), q_2(\beta)) : \beta \in H(b)\} \subseteq q_1(H(b)) \times q_2(H(b))$$

$$\subseteq F_1(q_1'(b)) \times F_2(q_2'(b)) = G(q_1'(b), q_2'(b)) = G(p'(b)).$$

Finally, it remains to show the uniqueness of (p, p'). Suppose that (q, q') is a weak soft homomorphism such that $(p_i, p_i')(q, q') = (q_i, q_i'), i = 1, 2$. Then $p_i q = q_i$ and $p_i' q' = q_i'$ for $i = 1, 2$. It follows from the uniqueness of p and p' that $p = q$ and $p' = q'$. This gives that $(p, p') = (q, q')$. Consequently, (p, p') is the unique soft weak homomorphism with the required properties. Then the assertion holds. □

Here we investigate soft pushouts in **SoftAct**. To this aim, let us first recall the construction of pushouts in the categories **Act-**S and **Set**.

The pushout of two S-maps $f_i : \mathcal{A} \to \mathcal{A}_i, i = 1,2$, in **Act-**$S$ is a pair $(\boldsymbol{Q}, (q_1, q_2))$, where $\boldsymbol{Q} = (\mathcal{A}_1 \sqcup \mathcal{A}_2)/\rho$ and ρ is the congruence relation on $\mathcal{A}_1 \sqcup \mathcal{A}_2$ generated by all pairs $(l_1 f_1(\alpha), l_2 f_2(\alpha))$, $\alpha \in \mathcal{A}$, where $l_i : \mathcal{A}_i \to \mathcal{A}_1 \sqcup \mathcal{A}_2, i = 1,2$, are the injections, and $q_i = \pi l_i : \mathcal{A}_i \to \boldsymbol{Q}, i = 1,2$, where $\pi : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \boldsymbol{Q}$ is the canonical epimorphism. There is the same situation for a pushout $(Q, (q'_1, q'_2))$ of two maps $g_i : A \to A_i, i = 1,2$, in **Set**. More exactly, $Q = (A_1 \sqcup A_2)/\theta$ and θ is the equivalence relation on $A_1 \sqcup A_2$ generated by all pairs $(l'_1 g_1(a), l'_2 g_2(a))$, $a \in A$, where $l'_i : A_i \to A_1 \sqcup A_2, i = 1,2$, are the injections, and $q'_i = \pi' l'_i : A_i \to Q, i = 1,2$, where $\pi' : A_1 \sqcup A_2 \to Q$ is the canonical epimorphism.

It is well-known that there is a canonical construction of pushouts via coproducts and coequalizers in a general category (see, for example, [3]). Now, under the above notations for pushouts in **Act-**S and **Set** and by using the fact that **SoftAct** has non-empty soft coproducts and soft coequalizers (Theorems 4.4 and 5.2), the following result is obtained.

Theorem 6.2. **SoftAct** *has soft pushouts:*
The soft pushout of soft homomorphisms $(f_i, g_i) : (F, A)_{\mathcal{A}} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1,2$, *in* **SoftAct** *is a pair*

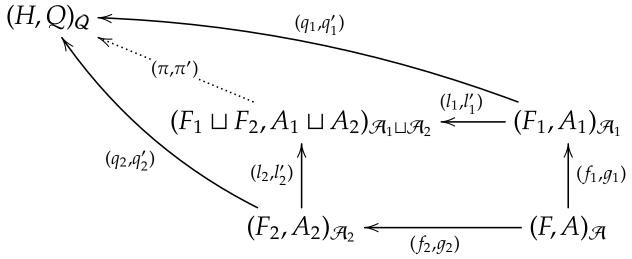
$$((H, Q)_{\boldsymbol{Q}}, ((q_1, q'_1), (q_2, q'_2))),$$

where $(\boldsymbol{Q}, (q_1, q_2))$ *and* $(Q, (q'_1, q'_2))$ *are the pushouts of* f_1, f_2 *and* g_1, g_2 *in* **Act-**S *and* **Set**, *respectively, and* $(q_i, q'_i) : (F_i, A_i)_{\mathcal{A}_i} \to (H, Q)_{\boldsymbol{Q}}, i = 1,2$, *are soft homomorphisms for which* $H([l_i(a_i)]_\theta) = \pi((F_1 \sqcup F_2)(l_i(a_i)))$ *for any* $a_i \in A_i, i = 1,2$.

Proof. In view of Theorem 4.4, there exists the soft coproduct $(F_1 \sqcup F_2, A_1 \sqcup A_2)_{\mathcal{A}_1 \sqcup \mathcal{A}_2}$ of the soft S-acts $(F_i, A_i)_{\mathcal{A}_i}, i = 1,2$. Consider the pair of soft homomorphisms:

$$(F, A)_{\mathcal{A}} \quad \underset{(l_2 f_2, l'_2 g_2)}{\overset{(l_1 f_1, l'_1 g_1)}{\rightrightarrows}} \quad (F_1 \sqcup F_2, A_1 \sqcup A_2)_{\mathcal{A}_1 \sqcup \mathcal{A}_2},$$

where for $i = 1,2$, $l_i : \mathcal{A}_i \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ and $l'_i : A_i \to A_1 \sqcup A_2$ are the injections. Using Theorem 5.2, the soft coequalizer of the above pair is given by $((H, Q)_{\boldsymbol{Q}}, (\pi, \pi'))$, where $\boldsymbol{Q} = (\mathcal{A}_1 \sqcup \mathcal{A}_2)/\rho$ for a congruence relation ρ on $\mathcal{A}_1 \sqcup \mathcal{A}_2$ generated by all pairs $(l_1 f_1(\alpha), l_2 f_2(\alpha))$, $\alpha \in \mathcal{A}$, and $Q = (A_1 \sqcup A_2)/\theta$ for an equivalence relation θ on $A_1 \sqcup A_2$ generated by all pairs $(l'_1 g_1(a), l'_2 g_2(a))$, $a \in A$, and $\pi : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \boldsymbol{Q}$ and $\pi' : A_1 \sqcup A_2 \to Q$ are the canonical maps; and $H([l_i(a_i)]_\theta) = \pi((F_1 \sqcup F_2)(l_i(a_i)))$ for any $a_i \in A_i, i = 1,2$. Now it follows from the construction of pushouts via coproducts and coequalizers that the pair $((H, Q)_{\boldsymbol{Q}}, ((q_1, q'_1), (q_2, q'_2)))$ is the soft pushout of soft homomorphisms $(f_i, g_i) : (F, A)_{\mathcal{A}} \to (F_i, A_i)_{\mathcal{A}_i}, i = 1,2$, where for $i = 1,2$, $q_i = \pi l_i : \mathcal{A}_i \to \boldsymbol{Q}$ and $q'_i = \pi' l'_i : A_i \to Q$:



Furthermore, it is clear that $(\boldsymbol{Q}, (q_1, q_2))$ and $(Q, (q'_1, q'_2))$ are the pushouts of f_1, f_2 and g_1, g_2 in **Act-**S and **Set**, respectively. □

Finally, we study a particular kind of soft pushouts in **SoftAct** and give an explicit characterization for it. Let us list some preliminaries.

Let $(F,U)_{\mathcal{U}}$ and $(G,A)_{\mathcal{A}}$ be two soft S-acts. We say that $(F,U)_{\mathcal{U}}$ is a *soft subact* of $(G,A)_{\mathcal{A}}$ if $U \subseteq A$, $\mathcal{U}$ is a subact of $\mathcal{A}$ and $F(u) = G(u)$ for all $u \in U$.

A special case of the pushout in **Act**-S, where $f_1 = f_2 = j : \mathcal{U} \hookrightarrow \mathcal{A}$ is the inclusion map, is denoted by $\mathcal{A} \coprod^{\mathcal{U}} \mathcal{A}$, the so called *amalgamated coproduct*. One can show that $\mathcal{A} \coprod^{\mathcal{U}} \mathcal{A}$ may be realized as the set $((\mathcal{A} \setminus \mathcal{U}) \times \{1,2\}) \cup \mathcal{U}$, where the natural action on $\mathcal{U}$ is extended to the remaining elements by defining

$$(\alpha, i)s = \begin{cases} (\alpha s, i) & \alpha s \in \mathcal{A} \setminus \mathcal{U} \\ \alpha s & \alpha s \in \mathcal{U} \end{cases}$$

for every $\alpha \in \mathcal{A} \setminus \mathcal{U}, s \in S$, and $i = 1,2$ (see [12, Proposition II.2.26]). Analogously, the pushout $A \coprod^U A$ of an inclusion map $g_1 = g_2 = j' : U \hookrightarrow A$ in **Set** is the set $((A \setminus U) \times \{1,2\}) \cup U$. Furthermore, if $(F,U)_{\mathcal{U}}$ is a soft subact of $(G,A)_{\mathcal{A}}$, then $(j,j') : (F,U)_{\mathcal{U}} \to (G,A)_{\mathcal{A}}$ is a soft monomorphism and the soft pushout of $(f_1,g_1) = (f_2,g_2) = (j,j')$ is denoted by $(G,A)_{\mathcal{A}} \coprod^{(F,U)_{\mathcal{U}}} (G,A)_{\mathcal{A}}$. Now we have:

Proposition 6.3. *The soft pushout* $(G,A)_{\mathcal{A}} \coprod^{(F,U)_{\mathcal{U}}} (G,A)_{\mathcal{A}}$ *is the pair*

$$((H, A \textstyle\coprod^U A)_{\mathcal{A} \coprod^{\mathcal{U}} \mathcal{A}}, ((q_1, q_1'), (q_2, q_2'))),$$

where $(A \coprod^U A, q_1', q_2')$ *and* $(\mathcal{A} \coprod^{\mathcal{U}} \mathcal{A}, q_1, q_2)$ *are the pushouts in* **Set** *and* **Act**-S*, respectively, and* $H(a,i) = q_i(G(a))$ *for every* $a \in A \setminus U$, $i = 1,2$, $H(a) = F(a)$ *for every* $a \in U$.

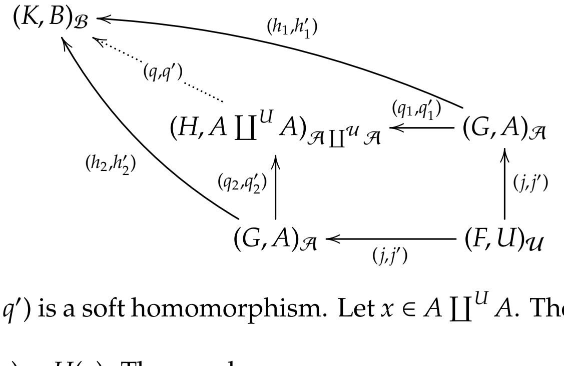
Proof. Note that the S-maps q_i for $i = 1,2$ are given by:

$$q_i(\alpha) = \begin{cases} (\alpha, i) & \alpha \in \mathcal{A} \setminus \mathcal{U} \\ \alpha & \alpha \in \mathcal{U}. \end{cases}$$

Also the maps q_i', $i = 1,2$, are defined in the same way. More exactly, we have

$$q_i'(a) = \begin{cases} (a, i) & a \in A \setminus U \\ a & a \in U. \end{cases}$$

It is easily seen that $(H, A \coprod^U A)_{\mathcal{A} \coprod^{\mathcal{U}} \mathcal{A}}$ is a soft S-act. We show that (q_i, q_i'), $i = 1,2$ are soft homomorphisms. Let $a \in A$. If $a \in A \setminus U$, then $Hq_1'(a) = H(a,1) = q_1(G(a))$. Now let $a \in U$. Then $F(a)$ is a subact of $\mathcal{U}$ which implies that $F(a) = q_1(F(a))$. Hence, we obtain that $Hq_1'(a) = H(a) = F(a) = q_1(F(a)) = q_1(G(a))$. This shows that (q_1, q_1') is a soft homomorphism. Similarly, (q_2, q_2') is also a soft homomorphism. Moreover, $(q_1, q_1')(j,j') = (q_1 j, q_1' j') = (q_2 j, q_2' j') = (q_2, q_2')(j,j')$. Now consider any soft S-act $(K,B)_{\mathcal{B}}$ and soft homomorphisms $(h_i, h_i') : (G,A)_{\mathcal{A}} \to (K,B)_{\mathcal{B}}$, $i = 1,2$, with $(h_1, h_1')(j,j') = (h_2, h_2')(j,j')$. This implies that $h_1 j = h_2 j$ and $h_1' j' = h_2' j'$. In view of the universality of pushouts in **Act**-S and **Set**, there exist a unique S-map $q : \mathcal{A} \coprod^{\mathcal{U}} \mathcal{A} \to \mathcal{B}$ and a unique map $q' : A \coprod^U A \to B$ such that $qq_i = h_i$ and $q'q_i' = h_i'$ for $i = 1,2$. Therefore, (q, q') is a unique pair satisfying $(q,q')(q_i, q_i') = (h_i, h_i')$, $i = 1,2$, i.e. the following diagram commutes:



It suffices to prove that (q, q') is a soft homomorphism. Let $x \in A \coprod^U A$. There are the following possible cases:

(i) If $x \in U$, then $G(x) = F(x) = H(x)$. Thus we have

$$K(q'(x)) = K(q'(q'_1(x))) = K(q'q'_1(x)) = K(h'_1(x)) = h_1(G(x))$$

$$= (qq_1)(F(x)) = q(q_1(F(x)) = q(F(x)) = q(H(x)).$$

(ii) If $x = (a, i), a \in A \setminus U, i = 1, 2$, then we get

$$K(q'(x)) = K(q'(a, i)) = K(q'(q'_i(a)) = K(q'q'_i(a)) = K(h'_i(a))$$

$$= h_i(G(a)) = (qq_i)(G(a)) = q(q_i(G(a))) = q(H(a, i)) = q(H(x)).$$

This completes the proof. □

At the end of the paper, we ask the following:

Question. What about the interrelations of w-soft (co)equalizers and w-soft pushouts in **WSoftAct** and soft products and soft pullbacks in **SoftAct** with their counterparts in **Act-***S* and **Set**?

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