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On the Construction of q-Analogues for some Positive Linear Operators

Ersin Şimşek[a], Tuncay Tunç[a]

[a]*Department of Mathematics, Mersin University, Mersin, Turkey*

Abstract. In this paper, we propose a method based on generating functions for constructing q-analogues of some discrete type positive linear operators (e.g., q-Lupas, q-Bernstein, q-Meyer-König and Zeller and q-Bleimann-Butzer-Hahn operators etc.). In other words, general operators of the discrete type are constructed, and their respective formulae for central moments are thereby obtained. Finally, through the use of specific generating functions, we are able to provide some relevant exemplary applications of general operators.

1. Introduction

First, let us provide some background information regarding what we know about q-calculus formulae, the study of which was initiated by Euler in the eighteenth century. Following this, many remarkable results in the field were obtained in the nineteenth century. In 1908, F. H. Jackson [17] introduced q- functions. He was also the first to develop q-calculus in a systematic way. Below, we present the outlines of q-integers, q-factorials, q-binomial coefficients, and q-differentiations. The definitions used in this study are based on terminology and notations as is seen in [5], [18], [10], [27] and [28].

The required q-Calculus theorems and definitions are as outlined below, where $q > 0$. For $n \in \mathbb{N}$, the q-analogue of the integer n, called q-integer, is defined by

$$[n]_q := \frac{q^n - 1}{q - 1}, \qquad q \neq 1; \qquad [n]_1 := n.$$

Also $[0]_q := 0$. Similarly, the q-analogue of the factorial of n is defined by

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad n = 1, 2, 3, \cdots; \qquad [0]_q! := 1$$

Now, let us obtain the q-analogue of the Gauss binomial formula. The q-analogues of $(a + b)^n$ are given by

$$(a \oplus b)_q^n := \prod_{s=0}^{n-1} (a + q^s b); \qquad (a \oplus b)_q^0 := 1. \tag{1}$$

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Email addresses: simsek.ersin@gmail.com (Ersin Şimşek), ttunc77@gmail.com (Tuncay Tunç)

By simple calculations, it follows that

$$(a \oplus b)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} b^k a^{n-k},$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad 0 \leq k \leq n$$

is the q-binomial formula. All the concepts defined above, become their classical cases if q tends to 1.

The q-derivative of a function f, denoted by $D_q f$ is given by

$$\left(D_q f\right)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \qquad x \neq 0$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. If f is differentiable at a point $x \neq 0$, we have

$$\lim_{q \to 1} \left(D_q f\right)(x) = f'(x).$$

Let us define the q-partial derivatives of a function $f(x, y)$ of two variables. The q-partial derivative of $f(x, y)$ with respect to x is defined by

$$\frac{\partial_q f(x, y)}{\partial_q x} = \frac{f(qx, y) - f(x, y)}{(q-1)x}, \qquad x \neq 0.$$

Likewise, the q-partial derivative of $f(x; y)$ with respect to y can be defined.

The first q-analogue of positive linear operators (actually, Bernstein polynomials) was adapted by A. Lupas [19] in 1987. In 1997, G. M. Phillips [24] proposed another q-version of Bernstein polynomials. For interesting properties of q-Bernstein polynomials and their distinct variants refer to [13], [12], [15], [16], [23], [30] and, [22]. In addition to the information from the studies we have just referenced, there are a great number of mathematicians who constructed and investigated the q-analogues of positive linear operators of the discrete type, which are very important to approximation theory: Some of these are q-Baskakov operators [3], [4], [11], q-Meyer-König-Zeller operators [29], [9], [14] and q-Bleimann-Butzer-Hahn operators [2], [1], and [20].

The first general positive linear operators of the discrete type were constructed by V. A. Baskakov [6] in 1957. In 1966, F. Schurer [26] investigated some approximation properties of these operators. Our first encounter with a study on q-analogues of the operators based on generating functions was in C. Radu's paper [25] in 2009. In 2013, M. Mursaleen et al. constructed a new family of operators with the help of q-analogue of Chan–Chyan–Srivastava polynomials, and study the statistical approximation properties via A-statistical convergence [21]. In 2014, J. Choi and H. M. Srivastava established a set of three new families of generating functions for the generalized q-Gottlieb polynomials [8].

In this study, we propose a method based on generating functions for constructing q-analogues of some discrete type positive linear operators (e.g., q-Lupas, q-Bernstein, q-Meyer-König and Zeller and q-Bleimann-Butzer-Hahn operators etc.). In other words, general operators of the discrete type are constructed, and their respective formulae for central moments are thereby obtained. Finally, through the use of specific generating functions, we are able to provide some relevant exemplary applications of general operators.

2. Construction of Generating Operators

In order to construct the sequence of positive linear operators with the aid of sequences of functions, we state the following:

Let $q > 0$ and $I \subset [0,\infty)$ be an interval. We assume that in the sequence

$$\left\{\varphi_{n,q}(x,u)\right\}_{n=1}^{\infty}$$

of real functions on $I \times [0,\infty)$, each function has the following conditions:

(i) $\varphi_{n,q}(x,0) \neq 0$ and $\varphi_{n,q}(x,1) = 1$ for every $n \in \mathbb{N}$ and $x \in I$.

(ii) $\left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0}$ exist and are continuous functions of x for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(iii) For all $k \in \mathbb{N}_0$, $x, u \geq 0$,

$$\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k} \geq 0, \qquad n \in \mathbb{N}.$$

The sequence $\left\{\varphi_{n,q}(x,u)\right\}$ generates a sequence of discrete type positive linear operators in the following way.

Expanding the function $\varphi_{n,q}(x,u)$ with $u \in [0,\infty)$, by using q-Taylor formula (see [10, p. 103]), we obtain

$$\varphi_{n,q}(x,u) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} u^k \tag{2}$$

and taking $u = 1$, we have by (i)

$$\sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} = 1. \tag{3}$$

Using the sequence $\left\{\varphi_{n,q}(x,u)\right\}$, we introduce the announced operators as follows:

$$E_{n,q}(f;x) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} f\left(\frac{[k]_q}{\alpha_{n,k,q}}\right) \tag{4}$$

where $\alpha_{n,k,q}$ are positive numbers. It is clear that the operators are linear and positive in view of (iii) on the space of bounded functions on I shown by $B(I)$. Also $\left\|E_{n,q}\right\| = 1$ by virtue of (3).

Lemma 2.1. *If the sequence $\left\{\varphi_{n,q}(x,u)\right\}$ has the conditions (i)-(iii), then, for all $m \in \mathbb{N}_0$*

$$\frac{\partial_q^m \varphi_{n,q}(x,u)}{\partial_q u^m} = \sum_{k=m}^{\infty} \frac{[k]_{q,m}}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} u^{k-m} \tag{5}$$

where $[k]_{q,m} = [k]_q[k-1]_q[k-2]_q \cdots [k-m+1]_q$.

Proof. The proof is by induction on m. For $m = 0$ the assertion is trivial. For $m = 1$ we have,

$$\begin{aligned}
\frac{\partial_q \varphi_{n,q}(x,u)}{\partial_q u} &= \frac{\varphi_{n,q}(x,qu) - \varphi_{n,q}(x,u)}{(q-1)u} \\
&= \sum_{k=1}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} \left(\frac{q^k-1}{q-1}\right) u^{k-1} \\
&= \sum_{k=1}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} [k]_q u^{k-1}.
\end{aligned}$$

For $m+1$, by assumption we obtain,

$$\frac{\partial_q^{m+1}\varphi_{n,q}(x,u)}{\partial_q u^{m+1}} = \frac{\partial_q}{\partial_q u}\left(\sum_{k=m}^{\infty}\frac{[k]_{q,m}}{[k]_q!}\left.\frac{\partial_q^k\varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} u^{k-m}\right)$$
$$= \sum_{k=m+1}^{\infty}\frac{[k]_{q,m+1}}{[k]_q!}\left.\frac{\partial_q^k\varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} u^{k-m-1}.$$

□

Corollary 2.2. *Writing $u=1$ in (5) we have*

$$\left.\frac{\partial_q^m\varphi_{n,q}(x,u)}{\partial_q u^m}\right|_{u=1} = \sum_{k=m}^{\infty}\frac{[k]_{q,m}}{[k]_q!}\left.\frac{\partial_q^k\varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0}. \tag{6}$$

The test functions $e_{r,s,i}$ are given by

$$e_{r,s,i}(t) = t^r\left[\frac{1+(-1)^s t}{1-t^2}\right]^{ir}, \quad r\in\mathbb{N}_0, \quad t\in\mathbb{R}, \quad s,i\in\{0,1\}.$$

For $i=0$, while functions of $e_{r,s,0}$ are used as the test functions for Bernstein, Szasz-Mirakyan, Lupas and Baskakov operators, the functions of $e_{r,0,1}$ for Meyer-König and Zeller and the functions of $e_{r,1,1}$ for Butzer-Bleimann-Hahn operators are used.

In continuation of the relation for the numbers $\alpha_{n,k,q}$ indicated in (4), we assume the following:

$$e_{r,s,i}\left(\frac{[k]_q}{\alpha_{n,k,q}}\right) = \frac{[k]_q^r}{\alpha_{n,q}^r}, \qquad r\in\mathbb{N}_0. \tag{7}$$

Where $\alpha_{n,q}$ are positive numbers independent of k.

Definition 2.3. *For $m=0,1,\cdots,r$*

$$\mathbb{S}_q(r,m) = \frac{q^{-\binom{m}{2}}}{[m]_q!}\sum_{v=0}^{m}(-1)^v q^{\binom{v}{2}}\begin{bmatrix} m\\ v\end{bmatrix}_q [m-v]_q^r$$

are the q-Stirling polynomials of the second kind satisfying the equality

$$\mathbb{S}_q(r+1,m) = \mathbb{S}_q(r,m-1) + [m]_q\mathbb{S}_q(r,m),$$

for $r\geq 0$ and $m\geq 1$ with $\mathbb{S}_q(0,0)=1$, $\mathbb{S}_q(r,0)=0$ for $r>0$. Also $\mathbb{S}_q(r,m)=0$ for $m>r$. When $q=1$, $\mathbb{S}_q(r,m)=S(r,m)$ [7].

Theorem 2.4. *If the sequence $\left\{\varphi_{n,q}(x,u)\right\}$ satisfies the conditions (i)-(iii), for all $r\in\mathbb{N}_0$ and $n\in\mathbb{N}$, the following relation is true:*

$$E_{n,q}(e_{r,s,i};x) = \frac{1}{\alpha_{n,q}^r}\sum_{m=0}^{r} q^{\binom{m}{2}}\mathbb{S}_q(r,m)\left.\frac{\partial_q^m\varphi_{n,q}(x,u)}{\partial_q u^m}\right|_{u=1} \tag{8}$$

Proof. For $r=0$, (8) is obvious. Let $r>0$. Since

$$[k]_q^r = \sum_{m=0}^{r} q^{\binom{m}{2}}\mathbb{S}_q(r,m)[k]_{q,m}, \qquad r\in\mathbb{N}_0$$

([7, p. 989]), from (4) and (7), we have

$$\begin{aligned} E_{n,q}\left(e_{r,s,i};x\right) &= \frac{1}{\alpha_{n,q}^{r}}\sum_{k=0}^{\infty}\frac{1}{[k]_q!}\left.\frac{\partial_q^k\varphi_{n,q}\left(x,u\right)}{\partial_q u^k}\right|_{u=0}[k]_q^r \\ &= \frac{1}{\alpha_{n,q}^{r}}\sum_{k=0}^{\infty}\left.\frac{\partial_q^k\varphi_{n,q}\left(x,u\right)}{[k]_q!\,\partial_q u^k}\right|_{u=0}\sum_{m=0}^{r}q^{\binom{m}{2}}\mathbb{S}_q\left(r,m\right)[k]_{q,m} \\ &= \frac{1}{\alpha_{n,q}^{r}}\sum_{m=0}^{r}q^{\binom{m}{2}}\mathbb{S}_q\left(r,m\right)\sum_{k=m}^{\infty}\frac{[k]_{q,m}}{[k]_q!}\left.\frac{\partial_q^k\varphi_{n,q}\left(x,u\right)}{\partial_q u^k}\right|_{u=0}. \end{aligned}$$

The desired result follows (6). □

Corollary 2.5. *Due to obvious equality (8) for $r = 0, 1, 2$, we have*

$$\begin{aligned} &E_{n,q}\left(e_{0,s,i};x\right) = 1; \\ &E_{n,q}\left(e_{1,s,i};x\right) = \frac{1}{\alpha_{n,q}}\left.\frac{\partial_q\varphi_{n,q}\left(x,u\right)}{\partial_q u}\right|_{u=1}; \\ &E_{n,q}\left(e_{2,s,i};x\right) = \frac{1}{\alpha_{n,q}^{2}}\left.\left\{q\frac{\partial_q^2\varphi_{n,q}\left(x,u\right)}{\partial_q u^2}+\frac{\partial_q\varphi_{n,q}\left(x,u\right)}{\partial_q u}\right\}\right|_{u=1}. \end{aligned}$$

Corollary 2.6. *Let $j \in \mathbb{N}_0$. Then the j-th central moments of $E_{n,q}$ operators has the representation*

$$\mu_{n,j,q}(x) = \sum_{l=0}^{j}(-1)^l\begin{bmatrix} j \\ l \end{bmatrix}_q q^{\binom{l}{2}}e_{1,s,i}^{l}(x)E_{n,q}(e_{j-l,s,i};x),$$

where $\mu_{n,j,q}(x) := E_{n,q}((e_{1,s,i}(\cdot) - e_{1,s,i}(x))^j;x)$.

3. Applications of $E_{n,q}$ Operators

In this section, we provide some generating functions, in the sense mentioned in section 2, in order to generate some well-known positive linear operators in q-Calculus, such as q-Bernstein, q-Lupas, q-Meyer-König-Zeller and q-Bleimann-Butzer-Hahn operators.

Firstly, we define the q-analogue of $(a + b + c)^n$, which we will use to construct the q-Bernstein operators.

Definition 3.1. *For $a, b, c \in \mathbb{R}$, we define*

$$(a \boxplus b \boxplus c)_q^n := \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a \oplus b)_q^{n-k}\, c^k; \qquad (a \boxplus b \boxplus c)_q^0 := 1 \tag{9}$$

and

$$(a \boxminus b \boxplus c)_q^n := \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a \ominus b)_q^{n-k}\, c^k$$

where $(a \ominus b)_q^n := (a \oplus (-b))_q^n$.

If we take $a = 0$ or $c = 0$, the formula (9) can be written in the form (1):

$$(a \boxplus b \boxplus 0)_q^n := (a \oplus b)_q^n\,, \qquad (0 \boxplus b \boxplus c)_q^n := (c \oplus b)_q^n\,.$$

Note that the order of summation changed in the case $a = 0$. If we take $b = 0$ then,

$$(a \boxplus 0 \boxplus c)_q^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} c^k := (a + c)_{\tilde{q}}^n ,$$

where $(a + c)_{\tilde{q}}^n$ are defined in equation (26,12) in [18].

Example 3.2 (q-Bernstein Operators). *For $n \in \mathbb{N}$ and $q \in (0, 1)$, we consider the function*

$$\varphi_{n,q}(x, u) := (1 \boxminus x \boxplus xu)_q^n , \qquad x \in [0, 1] \tag{10}$$

where $(1 \boxminus x \boxplus xu)_q^n := (1 \boxplus (-x) \boxplus xu)_q^n$. It is easy to check that the sequence $\left\{\varphi_{n,q}(x, u)\right\}$ satisfies the condition (i) in section 2. By the definition of q-partial derivatives, we obtain

$$\frac{\partial_q \varphi_{n,q} (x, u)}{\partial_q u} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1 \ominus x)_q^{n-k} [k]_q u^{k-1} = [n]_q x (1 \boxminus x \boxplus xu)_q^{n-1} .$$

By induction, we obtain that

$$\frac{\partial_q^k \varphi_{n,q} (x, u)}{\partial_q u^k} = [n]_{q,k} x^k (1 \boxminus x \boxplus xu)_q^{n-k} , \qquad k \in \mathbb{N}.$$

If we write $u = 0$ in the last equality, then we get

$$\left.\frac{\partial_q^k \varphi_{n,q} (x, u)}{\partial_q u^k}\right|_{u=0} = [n]_{q,k} x^k \prod_{s=0}^{n-k-1} (1 - q^s x). \tag{11}$$

Since the right hand side of (11) is a polynomial of x, then the condition (ii) holds and since $q \in (0, 1)$ and $x \in [0, 1]$, then the condition (iii) is satisfied too, thereby allowing for the functions $\varphi_{n,q} (x, u)$ defined by (10) to generate some positive and linear operators.

Writing (11) and considering $\alpha_{n,k,q} = [n]_q$ in the operators $E_{n,q}$ given by (4), we have

$$\begin{aligned} E_{n,q} (f ; x) &= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q} (x, u)}{\partial_q u^k}\right|_{u=0} f\left(\frac{[k]_q}{\alpha_{n,k,q}}\right) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right) \end{aligned}$$

and since $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q := 0$ for $k > n$ we obtain q-Bernstein operators $B_{n,q}$ defined by G. M. Phillips [24]: For $f \in B[0, 1]$,

$$B_{n,q}(f ; x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1 - q^s x) f\left(\frac{[k]_q}{[n]_q}\right).$$

From Corollary 5.2 we have

$$\begin{aligned} &B_{n,q} (e_{0,s,0}; x) = 1; \\ &B_{n,q} (e_{1,s,0}; x) = x; \\ &B_{n,q} (e_{2,s,0}; x) = x^2 + \frac{x (1 - x)}{[n]_q}. \end{aligned}$$

Remark 3.3. *Note that G. M. Phillips obtained q-analogues of Bernstein polynomials and related formulae of moments by using the q-difference; but we obtained some results by a different way, using the q-partial derivatives of generating functions given by (10).*

Example 3.4 (q-Bleimann-Butzer-Hahn Operators). *For $n \in \mathbb{N}$ and $q \in (0,1)$, we consider the function*

$$\varphi_{n,q}(x,u) = \frac{(1 \oplus xu)_q^n}{(1 \oplus x)_q^n}, \qquad x \in [0,\infty). \tag{12}$$

It is easy to check that the sequence $\left\{\varphi_{n,q}(x,u)\right\}$ satisfies the condition (i) in section 2. By the definition of q-partial derivatives, we obtain

$$\frac{\partial_q \varphi_{n,q}(x,u)}{\partial_q u} = [n]_q x \frac{\prod_{s=0}^{n-2}(1+q^s qxu)}{(1 \oplus x)_q^n} = [n]_q x \frac{(1 \oplus qxu)_q^{n-1}}{(1 \oplus x)_q^n}.$$

By induction, we obtain that

$$\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k} = [n]_{q,k} x^k q^{k(k-1)/2} \frac{\left(1 \oplus q^k xu\right)_q^{n-k}}{(1 \oplus x)_q^n}, \qquad k \in \mathbb{N}.$$

If we write $u = 0$ in the last equality, then we get

$$\left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} = \frac{1}{(1 \oplus x)_q^n}[n]_{q,k} q^{k(k-1)/2} x^k. \tag{13}$$

Since the right hand side of (13) is a rational function of x which does not have any singular points in $[0,1]$, then the condition (ii) holds and since $q \in (0,1)$ and $x \in [0,\infty)$, then the condition (iii) is satisfied too, there by the functions $\varphi_{n,q}(x,u)$ defined by (12) generate some positive and linear operators.

By writing (13) and considering $\alpha_{n,k,q} = q^k[n-k+1]_q$ in the operators $E_{n,q}$ given by (4), we have

$$\begin{aligned} E_{n,q}(f;x) &= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left.\frac{\partial_q^k \varphi_{n,q}(x,u)}{\partial_q u^k}\right|_{u=0} f\left(\frac{[k]_q}{\alpha_{n,k,q}}\right) \\ &= \frac{1}{(1 \oplus x)_q^n} \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k f\left(\frac{[k]_q}{q^k[n-k+1]_q}\right) \end{aligned}$$

and since $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q := 0$ for $k > n$ we obtain q-Bleimann-Butzer and Hahn operators $H_{n,q}$ defined in [5, p. 63] : For $f \in B[0,\infty)$,

$$H_{n,q}(f;x) = \frac{1}{(1 \oplus x)_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k f\left(\frac{[k]_q}{q^k[n-k+1]_q}\right).$$

From Corollary 5.2, we have

$$\begin{aligned} &H_{n,q}(e_{0,1,1};x) = 1; \\ &H_{n,q}(e_{1,1,1};x) = \frac{[n]_q}{[n+1]_q} \frac{x}{1+x} \\ &H_{n,q}(e_{2,1,1};x) = \frac{[n]_q[n-1]_q}{[n+1]_q^2} \frac{q^2x^2}{(1+x)(1+qx)} + \frac{[n]_q}{[n+1]_q^2} \frac{x}{1+x}. \end{aligned}$$

Example 3.5 (q-Lupas Operators). *Let $n \in \mathbb{N}$. If we consider $\alpha_{n,k,q} = [n]_q$ and*

$$\varphi_{n,q}(x,u) := \frac{((1-x)\oplus xu)_q^n}{((1-x)\oplus x)_q^n}, \quad x \in [0,1]; \quad q > 0$$

in the operators $E_{n,q}$ as defined by (4), then $E_{n,q}$ become the q-Lupas operators $L_{n,q}$ constructed by A. Lupas [19] as follows: For $f \in B[0,1]$

$$L_{n,q}(f;x) = \frac{1}{((1-x)\oplus x)_q^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} x^k (1-x)^{n-k} f\left(\frac{[k]_q}{[n]_q}\right)$$

From Corollary 5.2 we have

$$L_{n,q}(e_{0,s,0};x) = 1;$$
$$L_{n,q}(e_{1,s,0};x) = x;$$
$$L_{n,q}(e_{2,s,0};x) = x^2 + \frac{x(1-x)}{[n]_q}\left(\frac{1-x+q^n x}{1-x+qx}\right).$$

Example 3.6 (q-Meyer-König and Zeller Operators). *Let $n \in \mathbb{N}$. If we consider $\alpha_{n,k,q} = q^{-n}[k+n]_q$ and*

$$\varphi_{n,q}(x,u) := \frac{(1\ominus x)_q^{n+1}}{(1\ominus xu)_q^{n+1}}, \quad x \in [0,1), \quad q \in [0,1)$$

in the operators $E_{n,q}$ as defined by (4), then $E_{n,q}$ become the q-Meyer-König and Zeller operators $M_{n,q}$ constructed in [9] as follows: For $f \in B[0,1)$

$$M_{n,q}(f;x) = (1\ominus x)_q^{n+1} \sum_{k=0}^{\infty} \begin{bmatrix} k+n \\ k \end{bmatrix}_q x^k f\left(\frac{[k]_q}{q^{-n}[k+n]_q}\right).$$

From Corollary 5.2 we have

$$M_{n,q}(e_{0,0,1};x) = 1;$$
$$M_{n,q}(e_{1,0,1};x) = \frac{[n+1]_q}{[n]_q}\frac{xq^n}{(1-xq^{n+1})};$$
$$M_{n,q}(e_{2,0,1};x) = \frac{[n+1]_q[n+2]_q}{[n]_q^2}\frac{x^2q^{2n+1}}{(1-xq^{n+1})(1-xq^{n+2})} + \frac{[n+1]_q}{[n]_q^2}\frac{xq^{2n}}{(1-xq^{n+1})}.$$

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