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Slant Riemannian Maps from an Almost Contact Manifold

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Abstract. In the present paper, we introduce slant Riemannian maps from an almost contact manifold to Riemannian manifolds. We obtain the existence condition of slant Riemannian maps from an almost contact manifold to Riemannian manifolds. Moreover, we find the necessary and sufficient condition for slant Riemannian map to be totally geodesic and investigate the harmonicity of slant Riemannian maps from Sasakian manifold to Riemannian manifolds. Finally, we obtain a decomposition theorem for the total manifolds and also provide some examples of such maps.

1. Introduction

In 1992, Fisher introduced Riemannian maps between Riemannian manifolds in [7] as a generalization of the notions of isometric immersions and Riemannian submersions. Riemannian submersion between Riemannian manifolds equipped with differentiable structure were studied by Watson in [20]. Watson also showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases [20] and [6]. Since then almost Hermitian submersion have been extended to the almost contact manifolds [5], [10], locally conformal K*ä*hler manifolds [12] and quaternion K*ä*hler manifolds [11]. Given a $C^{\infty}-$ map F from a Riemannian manifold to a Riemannian manifold according to the conditions on the map F we call F a harmonic map [7], totally geodesic map [7], an isometric immersion [20], a Riemannian submersion [8] etc. The other basic maps for comparing geometric structures defined on Riemannian manifolds are Riemannian submersions and they were studied by O'Neill [14] and Gray [9]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their application in spacetime geometry, see [8]. N. Nore were studied second fundamental form of a map in [13] and some results were obtained for semi-slant submanifolds of Sasakian manifold in [3]. For application of manifolds and tensor, see [1]. Recently Sahin studied conformal Riemannian map [15], slant Riemannian Maps [16], anti-invariant Riemannian maps [17], biharmonic Riemannian maps [18] and semi-invariant Riemannian maps [19]. Motivated by above, we study slant Riemannian maps from almost contact manifolds,
In this paper, as another generalization of contact submersions, anti-invariant submersions and slant submersions we study slant Riemannian maps from an almost contact manifold to Riemannian manifolds. In section 2, we recall basic facts for Riemannian maps and an almost contact manifold. We also define slant Riemannian maps from an almost contact manifold. In section 3, we obtain a characterization of

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such maps and investigate the harmonicity of slant Riemannian maps from an almost contact manifold to Riemannian manifolds. Then we find necessary and sufficient conditions for slant Riemannian maps to be totally geodesic. In section 4, we obtain a decomposition theorem for the total manifold by using slant Riemannian maps from Sasakian manifold to Riemannian manifolds and also provide some examples.

2. Preliminaries

In this section, we are going to recall main definitions and properties of an almost contact manifold, Riemannian maps and slant Riemannian maps. Let $F : (M, g_m) \longrightarrow (N, g_n)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{m, n\}$, where $dimM = m$ and $dimN = n$. Then we denote the kernel space of F_* by $kerF_*$ and consider the orthogonal complementary space $\mathcal{H} = (kerF_*)^\perp$ to $kerF_*$ in TM. Thus the tangent bundle of M has the following decomposition

$$TM = kerF_* \oplus \mathcal{H}.$$

Where $kerF_* = D \oplus \xi$, D is a distribution in $kerF_*$ and ξ is orthogonal vector field to D in $kerF_*$. We denote the range of F_* by $rangeF_*$ and consider the orthogonal complementary space $(rangeF_*)^\perp$ to $rangeF_*$ in the tangent bundle TN of N. Since $0 < rankF < min\{m, n\}$, we always have $(rangeF_*)^\perp$. Thus the tangent bundle TN of N has following decomposition

$$TN = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Now, a smooth map $F : (M^m, g_m) \to (N^n, g_n)$ is called Riemannian map at $x \in M$. If the horizontal restriction $F^h_{*x_m} : (kerF_{*x_m})^\perp \to (rangeF_{*x_m})$ is a linear isometry between the inner product space $((kerF_{*x_m})^\perp.g_m(x_m) \mid (kerF_{*x_m})^\perp)$ and $(rangeF_{*x_m}.g_n(x_n) \mid_(rangeF_{*x_m}))$, $x_n = F(x_m)$. Therefore Fisher stated in [15] that a Riemannian map is a map which is isometric in maximum possible domain. In another words, F_* satisfies the equation

$$g_m(X, Y) = g_n(F_*X, F_*Y), \tag{2.1}$$

for vector fields $X, Y \in \mathcal{H}$. It follows that isometric immersion and Riemannian submersion are particular Riemannian map with $kerF_* = 0$ and $(rangF_*)^\perp = 0$. It is known that a Riemannian map is a submersion and this fact implies that the rank of the linear map $F_{*x} : T_xM \longrightarrow T_{F(x)}N$ is constant for x in each connected component of M. Now, we recall a useful results which are related to the second fundamental form and the tension field of Riemannian map. Let (M, g_m) and (N, g_n) be a Riemannian manifolds and suppose that $F : M \to N$ is a smooth map between them. Then the differential F_* of F can be viewed a section of bundle $Hom(TM, F^{-1}TN) \longrightarrow M$, where $F^{-1}TN$ is the pullback bundle which has fibers $(F^{-1}TN)_x = T_{F(x)}, x \in M$. $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla^F_X F_*(Y) - F_*(\nabla^M_X Y), \tag{2.2}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric [2]. It is shown in [18] that the second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma(kerF_*)^\perp$, of a Riemannian map has no component in $rangF_*$. More precisely we have the following.

Lemma 2.1. *Let F be a Riemannian map from a Riemannian manifold (M, g_m) to a Riemannian manifold (N, g_n). Then*

$$g_n((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((kerF_*)^\perp). \tag{2.3}$$

As a result of Lemma (2.1), we obtain

$$(\nabla F_*)(X, Y) \in \Gamma((rangF_*)^\perp), \forall X, Y, \in \Gamma((kerF_*)^\perp). \tag{2.4}$$

For the tension field of a Riemannian map between Riemannian manifolds, we get the following:

Lemma 2.2. *Let $F : (M^m, g_m) \longrightarrow (N^n, g_n)$ be a Riemannian map between Riemannian manifolds. Then the tension field τ of F is*

$$\tau = -m_1 F_*(H) + m_2 H_2, \tag{2.5}$$

where $m_1 = dim(kerF_*), m_2 = rankF$, H and H_2 are the mean curvature vector fields of the distribution $kerF_*$ and $rangF_*$, respectively.

Let F be a Riemannian map from a Riemannian manifold (M, g_m) to a Riemannian manifold (N, g_n). Then we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{2.6}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{2.7}$$

for vector fields E,F on M, where ∇ is the Levi-Civita connection of g_m. In fact one can see that these tensor fields are O'Neill's tensor fields which are defined for Riemannian submersions. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric on $(\Gamma(TM), g_m)$ reversing the horizontal and vertical distributions. It is also easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field $\mathcal{T}$ satisfies

$$\mathcal{T}_U W = \mathcal{T}_W U, \forall U, W \in \Gamma(kerF_*). \tag{2.8}$$

On the other hand, from (2.4) and (2.5), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.9}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{2.10}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.11}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.12}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $V, W \in \Gamma(kerF_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.
From now on, for simplicity, we denote by ∇^n both the Levi-Civita connection of (N, g_n) and its pullback along F. Then for any vector field X on M and any section V of $(rangF_*)^\perp$, where $(rangF_*)^\perp$ is the subbundle of $F^{-1}(TM)$ with fibers $(F_*(T_xM)^\perp)$ orthogonal complement of $F_*(T_xM)$ for g_n over x, we have $\nabla^{F^\perp}_X V$ which is the orthogonal projection of ∇^n_X on $(F_*(TM))^\perp$. In [13] the author also showed that $\nabla^{F^\perp}$ is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F^\perp} g_2 = 0$. We now define $\mathcal{S}_V$ as

$$\nabla^2_{F_*X} V = -\mathcal{S}_V F_* X + \nabla^{F^\perp}_X V, \tag{2.13}$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla^2_{F_*X} V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and F_*X and $\mathcal{S}_V F_* X$ at x depends only on V_x and $F_{*x}X_x$. By direct computation, we get

$$g_n(\mathcal{S}_V F_* X, F_* Y) = g_n(V, (\nabla F_*)(X, Y)), \tag{2.14}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $V \in \Gamma((rangF_*)^\perp)$. Since (∇F_*) is symmetric , it follows that $\mathcal{S}_V$ is a symmetric linear transformation of $rangF_*$.
Now, we define slant Riemannian map from almost contact manifold to Riemannian manifolds.

Definition 2.3. *Let F be a Riemannian map from an almost contact manifold $M(J, g_m, \eta, \xi)$ to a Riemannian manifold (N, g_n). If for any non zero vector $X \in \Gamma(kerF_*) - \{\xi\}$, the angle $\theta(X)$ between JX and the space $kerF_*$ is a constant, i.e. it is independent of the choice of the point $x \in M$ and choice of the tangent vector in $kerF_* - \{\xi\}$, then we say that F is a slant Riemannian map. In this case, the angle θ is called the slant angle of the slant Riemannian map.*

Since F is a subimmersion, it follows that the rank of F is constant on M, then the rank theorem for functions implies that $kerF_*$ is an integrable subbundle of TM, ([2], p. 205). Thus it follows from above definition that the leaves of the distribution $kerF_*$ of a slant Riemannian map are slant submanifolds of M for slant submanifolds, see [4]. Note that a slant Riemannian map is proper if it is not submersion.

2.1. *Almost contact manifold*

A $n-$ dimensional differential manifold M is said to have an almost contact structure $M(J, \xi, \eta)$ if it carries a tensor field J of type $(1, 1)$, a vector field ξ and $1-$form η on M respectively such that

$$J^2 = -I + \eta \otimes \xi, \; J\xi = 0, \; \eta \circ J = 0, \; \eta(\xi) = 1, \tag{2.15}$$

where I denotes identity tensor. An almost contact structure is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of J. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(JX, JY) = g(X, Y) - \eta(X)\eta(Y), \; g(X, \xi) = \eta(X). \tag{2.16}$$

Then $M(J, \xi, \eta, g_m)-$ structure is called an almost contact metric structure.
Define a tensor field Φ of type $(0, 2)$ by $\Phi(X, Y) = g(JX, Y)$. If $d\eta = \Phi$ then an almost contact metric structure is said to be normal contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_X J)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.17}$$

$$\nabla_X \xi = -JX, \tag{2.18}$$

where ∇ denotes the Levi-Civita connection of g. For a Sasakian manifold $M(, J, \xi, \eta, g_m)$, it is known that

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X. \tag{2.19}$$

3. Slant Riemannian Maps from an Almost Contact Manifold

In this section, as a generalization of almost contact submersions, slant submersions, anti-invariant Riemannian submersions and Riemannian maps from Hermitian manifold to a Riemannian manifold, we introduce slant Riemannian maps from an almost contact manifold to a Riemannian manifold. We first study the existence of such maps and investigate the effect of slant Riemannian maps on the geometry of the total manifold, the base manifold and themselves. Also, we study the geometry of leaves of distribution on the total manifold arisen from such maps. At last, we obtain necessary and sufficient condition for slant Riemannian maps from an almost contact manifold to a Riemannian manifold to be harmonic and totally geodesic.
Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n). Then for $X \in \Gamma(kerF_*)$, we write

$$JX = \phi X + \omega X, \tag{3.1}$$

where ϕX and ωX are vertical and horizontal part of JX. Similarly, for $Z \in \Gamma((kerF_*)^\perp)$, we have

$$JZ = BZ + CZ, \tag{3.2}$$

where BZ and CZ are vertical and horizontal part of JZ.
Using equations (2.9), (2.10), (2.17), (2.19), (3.1) and (3.2), we get

$$(\nabla_X \omega)Y = \mathcal{C}\mathcal{T}_X Y - \mathcal{T}_X \phi Y, \tag{3.3}$$

$$(\nabla_X \phi)Y = \mathcal{B}\mathcal{T}_X Y - \mathcal{T}_X \omega Y + R(\xi, X)Y, \tag{3.4}$$

where ∇ is the Levi-Civita connection on M and

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \hat{\nabla}_X Y,$$

$$(\nabla_X\phi)Y = \hat{\nabla}_X\phi Y - \phi\hat{\nabla}_X Y,$$

for $X, Y \in \Gamma(kerF_*)$.
Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n). Then we say that ω is parallel with respect to the Levi-Civita connection ∇ on $kerF_*$ if its covariant derivative with respect to ∇ vanishes, i.e., we get

$$(\nabla_X\omega)Y = \nabla_X\omega Y - \omega(\nabla_X Y) = 0,$$

for $X, Y \in \Gamma(kerF_*)$.
Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n). Then $\omega(kerF_*)$ is a subspace of $(kerF_*)^\perp)$. Thus it follows that $(kerF_{*x}) \oplus \omega(kerF_{*x})$ is invariant with respect to J. Then for every $x \in M$, there exist an invariant subspace μ_x of $((kerF_{*x})^\perp)$ such that

$$T_xM = kerF_{*x} \oplus \omega(kerF_{*x}) \oplus \mu_x.$$

Theorem 3.1. *Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n) such that $\xi \in \Gamma(kerF_*)$. Then, F is a slant Riemannian map if and only if there exist a constant $\lambda \in [0, 1]$ such that*

$$\phi^2 X = -\lambda(X - \eta(X)\xi), \tag{3.5}$$

for $X \in \Gamma(kerF_*)$. If F is slant Riemannian map, then $\lambda = \cos^2\theta$

Lemma 3.2. *Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n) with slant angle θ. Then, we get*

$$g_m(\phi X, \phi Y) = cos^2\theta(g_m(X, Y) - \eta(X)\eta(Y)), \tag{3.6}$$

$$g_m(\omega X, \omega Y) = sin^2\theta(g_m(X, Y) - \eta(X)\eta(Y)), \tag{3.7}$$

for any $X, Y \in \Gamma(kerF_*)$.

The proof of the above Theorem and Lemma is exactly the same with slant immersions [3] for Sasakian case. Therefore we omit its proof.
Also, using equation (3.6), we have $\{e_1, sec\theta\phi e_1, e_2, sec\theta\phi e_2, ..., e_k, sec\theta\phi e_k, \xi\}$ is an orthonormal frame for $\Gamma(kerF_*)$. On the other hand using equation (3.7), we can easily conclude that $\{csc\theta\omega e_1, csc\theta\omega e_2, ..., csc\theta\omega e_r\}$ is orthonormal frame for $\Gamma(\omega(kerF_*))$. As in slant immersions, we call the frame $\{e_1, sec\theta\phi e_1, e_2, sec\theta\phi e_2, ..., e_k, sec\theta\phi e_k, csc\theta\omega e_1, csc\theta\omega e_2, ..., csc\theta\omega e_r, \xi\}$ an adopted frame for slant Riemannian maps. We note that since the distribution $kerF_*$ is integrable it follows that $\mathcal{T}_X Y = \mathcal{T}_Y X$ for $X, Y \in \Gamma(kerF_*)$. Then the following Lemma can be obtain by using Theorem (3.1).

Lemma 3.3. *Let F be a slant Riemannian map from an almost contact manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n). if ω is parallel with respect to ∇ on $kerF_*$, then we have*

$$\mathcal{T}_{\phi X}\phi X = -cos^2\theta(\mathcal{T}_X X - \eta(X)\mathcal{T}_X\xi) \tag{3.8}$$

Proof. If ω is parallel, from (3.3), we get

$$C\mathcal{T}_X Y = \mathcal{T}_X\phi Y, \tag{3.9}$$

for $X, Y \in \Gamma(kerF_*)$. Interchange X and Y in equation (3.9), we obtain

$$\mathcal{T}_X\phi Y = \mathcal{T}_Y\phi X, \tag{3.10}$$

Substituting Y by ϕX in above equation and then using Theorem (3.1), we obtain the required formula.
Now, we give necessary and sufficient condition for F to be harmonic.

Theorem 3.4. *Let F be a slant Riemannian map from an almost contact manifold $M(J,\xi,\eta,g_m)$ to a Riemannian manifold (N,g_n). Then F is harmonic if and only if*

$$\mathcal{T}_{\phi e_i}\phi e_i = -cos^2\theta(\mathcal{T}_{e_i}e_i - \eta(e_i)\mathcal{T}_{e_i}\xi),$$

$$trace\,|_{\omega(kerF_*)}\ {}^*F_*(\mathcal{S}_{E_j}F_*(.)) \in \Gamma(\mu),$$

$$trace\,|_{\mu}\ {}^*F_*(\mathcal{S}_{E_j}F_*(.)) \in \Gamma(\omega(kerF_*)),$$

where $\{e_1, sec\theta\phi e_1, e_2, sec\theta\phi e_2, ..., e_k, sec\theta\phi e_k, \xi\}$ is an orthonormal frame for $\Gamma(kerF_*)$ and $\{E_s\}$ is an orthonormal frame of $\Gamma((rangF_*)^\perp)$.

Proof. Let a canonical orthonormal frame $\{e_1, sec\theta\phi e_1, e_2, sec\theta\phi e_2, ..., e_k, sec\theta\phi e_k, csc\theta\omega e_1, csc\theta\omega e_2, ..., csc\theta\omega e_r, \xi, \bar{e_1}, \bar{e_2}, ..., \bar{e_m}\}$ such that $\{e_1, sec\theta\phi e_1, e_2, sec\theta\phi e_2, ..., e_n, sec\theta\phi e_n, \xi\}$ is an orthonormal basis of $kerF_*$ and $\{\bar{e_1}, \bar{e_2}, ..., \bar{e_m}\}$ of μ, where θ is the slant angle. Then F is harmonic if and only if

$$\sum_{i=1}^{k}(\nabla F_*)(e_i,e_i) + sec^2\sum_{i=1}^{k}(\nabla F_*)(\phi e_i,\phi e_i) + (\nabla F_*)(\xi,\xi) + csc^2\sum_{i=1}^{r}(\nabla F_*)(\omega e_i,\omega e_i) + \sum_{j=1}^{m}(\nabla F_*)(\bar{e}_j,\bar{e}_j) = 0. \quad (3.11)$$

Using equations (2.2) and (2.9), we get

$$\sum_{i=1}^{k}(\nabla F_*)(e_i,e_i) + sec^2\sum_{i=1}^{k}(\nabla F_*)(\phi e_i,\phi e_i) = -F_*(\mathcal{T}_{e_i}e_i + sec^2\mathcal{T}_{\phi e_i}\phi e_i - \eta(e_i)\mathcal{T}_{e_i}\xi). \quad (3.12)$$

Also, from Lemma (2.1), we have $csc^2\sum_{i=1}^{r}(\nabla F_*)(\omega e_i,\omega e_i) + \sum_{j=1}^{m}(\nabla F_*)(\bar{e}_j,\bar{e}_j) \in \Gamma((rangeF_*)^\perp)$. So, we can conclude that

$$csc^2\sum_{i=1}^{r}(\nabla F_*)(\omega e_i,\omega e_i) + \sum_{j=1}^{m}(\nabla F_*)(\bar{e}_j,\bar{e}_j) = csc^2\sum_{i=1}^{r}\sum_{s=1}^{t}g_n((\nabla F_*)(\omega e_i,\omega e_i),E_s)E_s)$$
$$+ \sum_{j=1}^{m}\sum_{s=1}^{t}g_n((\nabla F_*)(\omega\bar{e}_j,\omega\bar{e}_j),E_s)E_s, \quad (3.13)$$

where $\{E_s\}$ is orthonormal basis of $\Gamma((rangeF_*)^\perp)$. Then using equation (2.14), we obtain

$$csc^2\sum_{i=1}^{r}(\nabla F_*)(\omega e_i,\omega e_i) + \sum_{j=1}^{m}(\nabla F_*)(\bar{e}_j,\bar{e}_j) = csc^2\sum_{i=1}^{r}\sum_{s=1}^{t}g_n(\mathcal{S}_{E_s}F_*(\omega e_i).(\omega e_i))E_s$$
$$+ \sum_{j=1}^{m}\sum_{s=1}^{t}g_n(\mathcal{S}_{E_s}F_*(\bar{e}_j).F_*(\bar{e}_j))E_s. \quad (3.14)$$

From the adjoint of equation (3.12) and (3.14), we obtain our result.

Lemma 3.5. *Let F be a slant Riemannian map from an almost contact manifold $M(J,\xi,\eta,g_m)$ to a Riemannian manifold (N,g_n). if ω is parallel with respect to ∇ on $kerF_*$, then (3.8) is satisfied.*

It is remarkable that the equality (3.8)(as a result of above lemma parallel ω) is enough for a slant submersion to be harmonic however for a slant Riemannian map this case is not valid anymore.
In this part, we now investigate necessary and sufficient condition for a slant Riemannian map F to be totally geodesic. We recall that a differentiable map F between Riemannian manifolds (M,g_m) and (M,g_n) is called a totally geodesic map if $(\nabla F_*)(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$

Theorem 3.6. *Let F be a slant Riemannian map from a Sasakian manifold $M(J,\xi,\eta,g_m)$ to a Riemannian manifold (N,g_n). Then F is totally geodesic if and only if*

$$g_n((\nabla F_*)(U,\omega V),F_*(\mathcal{C}X)) - g_n((\nabla F_*)(U,\omega\phi V),F_*(X)) = g_m(\mathcal{T}_U\omega V,F_*(\mathcal{B}X)),$$

$$g_n(\nabla^F_X F_*(\omega\phi U),F_*(Y)) - g_n((\nabla^F_X F_*)(\omega U),F_*(\mathcal{C}Y)) = g_m(\mathcal{A}_X\omega U,F_*(\mathcal{B}Y)),$$

and

$$\nabla^F_X F_*(Y) + F_*(\mathcal{C}(\mathcal{A}_X\mathcal{B}Y + \mathcal{H}\nabla^m_X\mathcal{C}Y) + \omega(\mathcal{V}\nabla^m_X\mathcal{B}Y + \mathcal{A}_X\mathcal{C}Y)) \in \Gamma(kerF_*),$$

for $X,Y \in \Gamma((kerF_*)^\perp)$ and $U,V \in D$ where ∇^m is the Levi-Civita connection of M.

Proof. The decomposition of the total manifold of a slant Riemannian map, follows that F is totally geodesic if and only if $g_n(\nabla F_*(U,V),F_*(X)) = 0$, $g_n(\nabla F_*(X,U),F_*(Y)) = 0$ and $(\nabla F_*)(X,Y) = 0$ for $U,V \in D$ and $X,Y \in \Gamma((kerF_*)^\perp)$. Since F is a slant Riemannian map, using (2.1) we get

$$g_n(\nabla F_*(U,V),F_*(X)) = -g_m(\nabla^m_U V,X) \tag{3.15}$$

Since M is Sasakian manifold. Then using equations (3.1) and (3.2), we obtain

$$g_n((\nabla F_*)(U,V),F_*(X)) = -cos^2 g_m(\nabla^m_U V,X) + g_m(\nabla^m_U\omega\phi V,X) - g_m(\nabla^m_U\omega V,\mathcal{B}X) - g_m(\nabla^m_U\omega V,\mathcal{C}X).$$

Now, using equations (2.2) and (2.10), we have

$$g_n((\nabla F_*)(U,V),F_*(X)) = -sec^2\{-g_m(\mathcal{T}_U\omega V,\mathcal{B}X) + g_n((\nabla F_*)(U,\omega V),F_*(\mathcal{C}X)) - g_n((\nabla F_*)(U,\omega\phi V),F_*(X))\}. \tag{3.16}$$

Also in a similar way, we have

$$g_n((\nabla F_*)(X,U),F_*(Y)) = -sec^2\{-g_m(\mathcal{A}_X\omega U,\mathcal{B}Y + g_n(\nabla^F_X F_*(\omega\phi U),F_*(\mathcal{C}Y)) - g_n(\nabla^F_X F_*(\omega U),F_*(Y))\}. \tag{3.17}$$

Now, using equations (2.2) and (2.17), we obtain

$$(\nabla F_*)(X,Y) = \nabla^F_X F_*(Y) + F_*(J\nabla^F_X JY - J(\nabla_X J)Y), \tag{3.18}$$

for $X,Y \in \Gamma((kerF_*)^\perp)$.
Then using equations (3.1), (3.2),(2.9)-(2.14) and (2.16), we get

$$\begin{aligned}(\nabla F_*)(X,Y) = {} & \nabla^F_X F_*(Y) + F_*(\mathcal{B}\mathcal{A}_X\mathcal{B}Y + \mathcal{C}\mathcal{A}_X\mathcal{B}Y + \phi\mathcal{V}\nabla^F_X\mathcal{B}Y + \omega\mathcal{V}\nabla^F_X\mathcal{B}Y + \mathcal{B}\mathcal{H}\nabla^F_X\mathcal{C}Y + \mathcal{C}\mathcal{H}\nabla^F_X\mathcal{C}Y \\ & + \phi\mathcal{A}_X\mathcal{C}Y + \omega\mathcal{A}_X\mathcal{C}Y).\end{aligned}$$

Since $\mathcal{B}\mathcal{A}_X\mathcal{B}Y + \phi\mathcal{V}\nabla^F_X\mathcal{B}Y + \mathcal{B}\mathcal{H}\nabla^F_X\mathcal{C}Y + \phi\mathcal{A}_X\mathcal{C}Y \in \Gamma(kerF_*)$. We obtain

$$(\nabla F_*)(X,Y) = \nabla^F_X F_*(Y) + F_*(\mathcal{C}\mathcal{A}_X\mathcal{B}Y + \omega\mathcal{V}\nabla^F_X\mathcal{B}Y + \mathcal{C}\mathcal{H}\nabla^F_X\mathcal{C}Y + \omega\mathcal{A}_X\mathcal{C}Y). \tag{3.19}$$

We obtain result from (3.16), (3.17) and (3.19).

4. Decomposition Theorem for Slant Riemannian Maps from an Almost Contact Manifold to Riemannian Manifolds

In this section, we find the necessary and sufficient condition for the total manifold of slant Riemannian map to be locally product Riemannian manifold. Finally, we give some examples of slant Riemannian maps.

Theorem 4.1. *Let F be a slant Riemannian map from a Sasakian manifold $M(J, \xi, \eta, g_m)$ to a Riemannian manifold (N, g_n). Then M is locally product Riemannian manifold if and only if*

$$g_n((\nabla F_*)(U, \omega V), F_*(CX)) - g_n((\nabla F_*)(U, \omega\phi V), F_*(X)) = g_m(\mathcal{T}_U \omega V, F_*(\mathcal{B}X)),$$

$$g_n(\nabla^F_X F_*(\omega\phi U), F_*(Y)) - g_n((\nabla^F_X F_*)(\omega U), F_*(CY)) = g_n((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)),$$

for $U, V \in \Gamma(kerF_*)$ and $X, Y \in \Gamma((kerF_*)^\perp)$.

Proof. For $U \in \Gamma(kerF_*)$ and $X, Y \in \Gamma((kerF_*)^\perp)$, from equations (2.17), (3.1) , (3.2) and Theorem (3.1), we obtain

$$g_m(\nabla^m_X Y, U) = -cos^2 g_m(Y, \nabla^m_X U) + g_m(Y, \nabla^m_X \omega\phi U) - g_m(\mathcal{B}Y, \nabla^m_X \omega U) - g_m(CY, \nabla^m_X \omega U). \tag{4.1}$$

Since F is Riemannian map and using equation (2.2), we have

$$\begin{aligned} g_m(\nabla^m_X Y, U) = sec^2\{&g_n(F_*(Y), \nabla^m_X F_*(\omega\phi U)) - g_n(F_*(Y), (\nabla F_*)(X, \omega\phi U)) + g_n((\nabla F_*)(X, \omega U), F_*(CY)) \\ &- g_n((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)) - g_n(F_*(CY), \nabla^m_X F_*(\omega U))\}. \end{aligned} \tag{4.2}$$

Using Lemma (2.1), we get

$$g_m(\nabla^m_X Y, U) = sec^2\{g_n((F_*(Y), \nabla^m_X F_*(\omega\phi U)) - g_n((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)) - g_n(F_*(CY), \nabla^m_X F_*(\omega U))\}. \tag{4.3}$$

Thus we obtain our proof from (3.16) and (4.3).

Example 4.2. *We consider R^{2n+1} with Cartesian coordinates $(x_i, y_i, z_i)(i = 1, ..., n)$ and its usual contact form*

$$\eta = \frac{1}{2}(dz - \sum y_i dx_i).$$

The characteristic vector field ξ is given by $2\frac{\partial}{\partial z}$ and its Riemannian metric g and its tensor field J are given by

$$g = \frac{1}{4}(\eta \oplus \eta + \sum((dx_i)^2 + (dy_i)^2), \; J = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_j & 0 \end{pmatrix}, i = 1, ..., n$$

This gives a contact structure on R^{2n+1}. The vector fields $E_i = 2\frac{\partial}{\partial y_i}, E_{n+i} = 2\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z}$, ξ form a $J-$ basis for the contact metric structure. On the other hand, it can be shown that $R^{2n+1}(J, \xi, \eta, g)$ is a Sasakian manifold.

Example 4.3. *R^5 has got a Sasakian structure as in the preceding Example 4.2. Let $F : R^5 \longrightarrow R^2$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (x_1 - 2\sqrt{2}x_2 + y_1, 2x_1 - 2\sqrt{2}x_2 + y_1)$. Then, by direct calculations*

$$kerF_* = span\{V_1 = 2E_1 + \frac{1}{\sqrt{2}}E_4, V_2 = E_2, V_3 = E_5 = \xi\}.$$

Then it is easy to see that F is a Riemannian Map. Moreover, $JV_1 = 2E_3 - \frac{1}{\sqrt{2}E_2}$ and $JV_2 = E_4$ imply that $|g(JV_1, V_2)| = \frac{1}{\sqrt{2}}$. So F is a slant Riemannian map with slant angle $\theta = \frac{\pi}{4}$

Example 4.4. *Every proper slant submersion with the slant angle θ is a slant Riemannian map with $(rangF_*)^\perp = \{0\}$*

Example 4.5. *Every anti-invariant Riemannian submersion from an almost contact manifold onto a Riemannian manifold is a slant Riemannian map with the slant angle $\theta = \frac{\pi}{2}$ and $(rangF_*)^\perp = \{0\}$.*

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