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On Automorphisms and Structural Properties of Generalized Cayley Graphs

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Abstract. In this paper, generalized Cayley graphs are studied. It is proved that every generalized Cayley graph of order two times a prime is a Cayley graph. Special attention is given to generalized Cayley graphs on abelian groups. It is proved that every generalized Cayley graph on an abelian group with respect to an automorphism which acts as inversion is a Cayley graph if and only if the group is elementary abelian 2-group, or its Sylow 2-subgroup is cyclic. Necessary and sufficient conditions for a generalized Cayley graph to be unworthy are given.

1. Introduction

In this paper we consider generalized Cayley graphs, first introduced in [7].

Definition 1.1. *Let G be a group, S a subset of G and α an automorphism of G such that the following conditions are satisfied:*

(i) $\alpha^2 = 1$,

(ii) *if* $x \in G$ *then* $\alpha(x^{-1})x \notin S$,

(iii) *if* $x, y \in G$ *and* $\alpha(x^{-1})y \in S$ *then* $\alpha(y^{-1})x \in S$.

Then the generalized Cayley graph $X = GC(G, S, \alpha)$ *on G with respect to the ordered pair (S, α) is the graph with vertex set G, with two vertices $x, y \in V(X)$ being adjacent in X if and only if $\alpha(x^{-1})y \in S$. In other words, a vertex $x \in G$ is adjacent to all the vertices of the form $\alpha(x)s$, where $s \in S$.*

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Note that (ii) implies that X has no loops, and (iii) implies that X is undirected. Also, in view of (i), the condition (iii) is equivalent to $\alpha(S^{-1}) = S$. Namely, by letting $x = 1$ in (iii), we obtain $\alpha(S^{-1}) = S$, and conversely, if $\alpha(S^{-1}) = S$, then $\alpha(x^{-1})y \in S$ implies that $\alpha(y^{-1}\alpha(x)) = \alpha(y^{-1})x \in S$. If $\alpha = 1$ then we say that $GC(G, S, \alpha)$ is a *Cayley graph* and write simply $\mathrm{Cay}(G, S)$. Therefore every Cayley graph is also a generalized Cayley graph, but the converse is not true (see [7, Proposition 3.2]). A generalized Cayley graph $GC(G, S, \alpha)$ is connected if and only if S is a left generating set for the quasigroup $(G, *)$, where $f * g = \alpha(f)g$ for all $f, g \in G$ (see [7, Proposition 3.5]). Recall that a quasigroup $(G, *)$ is a set G with binary operation $*$, in which for arbitrary two elements $a, b \in G$, the equations $a * x = b$ and $y * a = b$ have uniquely defined solutions x and y. A subset S of a quasigroup G is said to be a *left generating set* of G if for each $g \in G$ there exist $s_1, \ldots, s_m \in S$ such that $g = (\ldots((s_1 * s_2) * s_3) * \ldots * s_m)$.

In [7] the properties of generalized Cayley graphs relative to canonical double covers (also called bipartite double covers) of graphs were studied. For graphs X and Y the *direct product* $X \times Y$ of X and Y is the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X \times Y$ if and only if x_1 is adjacent with x_2 in X and y_1 is adjacent with y_2 in Y. *Canonical double cover* $B(X)$ of a graph X is the direct product $X \times K_2$ (K_2 is the complete graph on two vertices). It is easily seen that $\mathrm{Aut}(B(X))$ contains a subgroup isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$. If $\mathrm{Aut}(B(X))$ is isomorphic to $\mathrm{Aut}(X) \times \mathbb{Z}_2$ then the graph X is called *stable*, otherwise it is called *unstable*. This concept was first defined by Marušič et al. [6] and studied later most notably by Surowski [9, 10], Wilson [11], Lauri et al. [5].

In [7, Proposition 3.3]) it is proved that every stable generalized Cayley graph is also a Cayley graph. Therefore, every generalized Cayley graph which is not Cayley graph is unstable. Recently, the existence of infinitely many vertex-transitive generalized Cayley graphs which are not Cayley graphs was proven (see [3]).

In this paper we continue studying the properties of generalized Cayley graphs started in [3]. Since for defining generalized Cayley graphs one needs a group automorphism of order two, in Section 2 we study some properties of such automorphisms. We first prove that for studying generalized Cayley graphs, it suffices to consider only the representatives of the conjugacy classes in $\mathrm{Aut}(G)$ (see Proposition 2.1). We also show how all generalized Cayley graphs on abelian groups of odd order can be represented in a simple way (see Proposition 2.4).

The most natural choice of an automorphism of order 2 is the mapping that inverts all elements, that is $\iota : x \mapsto x^{-1}$. This mapping is an automorphism of a group G if and only if G is abelian group. Therefore, we study generalized Cayley graphs on abelian groups arising from the inversion automorphism $\iota : x \mapsto x^{-1}$ in Section 3. We prove that in some cases, all graphs constructed in this way turn out to be Cayley graphs (but not necessarily on an abelian group). We prove the following theorem.

Theorem 1.2. *Let G be an abelian group and let ι be the inversion automorphism of G. Then every generalized Cayley graph on G with respect to ι is a Cayley graph if and only if one of the following holds:*

(i) G is an elementary abelian 2-group;

(ii) the Sylow 2-subgroup of G is cyclic.

In Section 4 we consider generalized Cayley graphs of order twice a prime, and prove that there are no non-Cayley graphs among them.

Theorem 1.3. *Every generalized Cayley graph of order $2p$ is a Cayley graph.*

Necessary and sufficient conditions for a generalized Cayley graph to be unworthy are given in Section 5 (a graph is called *unworthy* if it has two vertices with the same neighbours). Using this it is proved that for an abelian group G, involutory automorphism α of G, and the maximal allowed set S in Definition 1.1, the graph $GC(G, S, \alpha)$ is isomorphic to the lexicographic product of a complete graph and an empty graph, see Corollary 5.4.

2. Group Automorphisms of Order Two

For defining a generalized Cayley graph on a group G, one needs an automorphism of G of order two. Therefore it is important to understand the structure of such group automorphisms. We start this section with the following proposition, which tells us that for studying generalized Cayley graphs on a group G, it is sufficient to consider only the representatives of conjugacy classes in $\mathrm{Aut}(G)$.

Proposition 2.1. *$GC(G, S, \alpha) \cong GC(G, \varphi(S), \varphi\alpha\varphi^{-1})$ for any $\varphi \in \mathrm{Aut}(G)$.*

Proof. Let us first prove that $\varphi(S)$ and $\varphi\alpha\varphi^{-1}$ satisfy the conditions from Definition 1.1.

(i) $(\varphi\alpha\varphi^{-1})^2 = \varphi\alpha\varphi^{-1}\varphi\alpha\varphi^{-1} = id$.

(ii) Suppose that there exists $g \in G$ such that $(\varphi\alpha\varphi^{-1})(g^{-1})g \in \varphi(S)$. Then $\varphi^{-1}((\varphi\alpha\varphi^{-1})(g^{-1})g) \in S$, and consequently $\alpha((\varphi^{-1}(g))^{-1})\varphi^{-1}(g) \in S$, a contradiction.

(iii) $(\varphi\alpha\varphi^{-1})(\varphi(S)) = \varphi(\alpha(S)) = \varphi(S^{-1}) = \varphi(S)^{-1}$.

Mapping φ is clearly a bijective mapping from G to G. Let $\{x, y\}$ be an arbitrary edge in $GC(G, S, \alpha)$. Then $y = \alpha(x)s$ for some $s \in S$. Further we have

$$\varphi(y) = \varphi(\alpha(x)s) = \varphi(\alpha(x))\varphi(s) = (\varphi\alpha)(x)\varphi(s) = (\varphi\alpha\varphi^{-1})(\varphi(x))\varphi(s).$$

This implies that $\varphi(x)$ and $\varphi(y)$ are adjacent in $GC(G, \varphi(S), \varphi\alpha\varphi^{-1})$. Similarly, one can see that if $\varphi(x)$ and $\varphi(y)$ are adjacent in $GC(G, \varphi(S), \varphi\alpha\varphi^{-1})$, then x and y are adjacent in $GC(G, S, \alpha)$. Therefore φ is in fact an isomorphism between $GC(G, S, \alpha)$ and $GC(G, \varphi(S), \varphi\alpha\varphi^{-1})$. □

For a group G and $\alpha \in \mathrm{Aut}(G)$, the set $\mathrm{Fix}(\alpha)$ is defined as $\mathrm{Fix}(\alpha) = \{g \in G \mid \alpha(g) = g\}$. We let $\omega_\alpha : G \to G$ be the mapping defined by $\omega_\alpha(x) = \alpha(x)x^{-1}$ and let $\omega_\alpha(G) = \{\omega_\alpha(g) \mid g \in G\}$. Notice that Definition 1.1(ii) is equivalent to $\omega_\alpha(G) \cap S = \emptyset$, that is $\omega_\alpha(G)$ is the set of forbidden elements for the generating set of the graph $GC(G, S, \alpha)$. In the following proposition some properties of these sets are given. The proof is straightforward and is omitted.

Proposition 2.2. *Let G be a group and $\alpha \in \mathrm{Aut}(G)$ such that $\alpha^2 = 1$. Then the following hold:*

(a) $\mathrm{Fix}(\alpha)$ is a subgroup of G;

(b) If G is an abelian group then $\omega_\alpha(G)$ is a subgroup of G;

(c) $\alpha(x) = x^{-1}$, for every $x \in \omega_\alpha(G)$.

Observe that the set $\omega_\alpha(G)$ is not always a subgroup of G. For example, if $G = A_4$ and α acts on G as a conjugation by $(1\,2)$, then $\omega_\alpha(G) = \{id, (1\,2\,3), (1\,3\,2), (1\,2\,4), (1\,4\,2), (1\,2)(3\,4)\}$, which is not a subgroup of A_4.

An action of a group A on a group G is called *coprime* if $(|A|, |G|) = 1$. If G is of odd order, and α is an involutory automorphism of G, then the action of $A = \langle\alpha\rangle$ on G is coprime. Then by [4, 8.2.7], it follows that $G = \mathrm{Fix}(\alpha)\omega_\alpha(G)$. Moreover, if G is abelian, then we have the following result (this result was first proved by Miller [8] in 1909).

Proposition 2.3. [4, 8.4.2] *If G is an abelian group of odd order then $G = \mathrm{Fix}(\alpha) \times \omega_\alpha(G)$.*

Proposition 2.3 enables us to describe generalized Cayley graphs on an abelian group of odd order in the following way.

Proposition 2.4. *Let X be a generalized Cayley graph on an abelian group of odd order. Then X is isomorphic to the graph Y given with*

1. *$V(Y) = G_1 \times G_2$, where G_1 and G_2 are abelian groups of odd order;*

2. *$E(Y) = \{\{(g_1, g_2), (g_1s_1, g_2^{-1}s_2)\} \mid (g_1, g_2) \in G_1 \times G_2,\ (s_1, s_2) \in \overline{S}\}$ where $\overline{S} \subseteq (G_1 \setminus \{1\}) \times G_2$ such that $(s_1, s_2) \in \overline{S} \Leftrightarrow (s_1^{-1}, s_2) \in \overline{S}$.*

Proof. Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on an abelian group G of odd order. By Proposition 2.3 it follows that $G = G_1 \times G_2$, where $G_1 = \mathrm{Fix}(\alpha)$ and $G_2 = \omega_\alpha(G)$. Let φ be a natural isomorphism between G and $G_1 \times G_2$, that is $\varphi(g) = (g_1, g_2)$, where $g = g_1g_2$, $g \in G, g_1 \in G_1, g_2 \in G_2$. Let $\overline{S} = \varphi(S)$. Recall that by Definition 1.1(ii) we have $S \cap \omega_\alpha(G) = \emptyset$. Hence, for every $s \in S$, we have $s = s_1s_2$, where $s_1 \in G_1 \setminus \{1\}$ and $s_2 \in G_2$, implying that $\overline{S} \subseteq (G_1 \setminus \{1\}) \times G_2$. Also, since $s \in S \Leftrightarrow \alpha(s^{-1}) \in S$, it follows that $(s_1, s_2) \in \overline{S} \Leftrightarrow (s_1^{-1}, s_2) \in \overline{S}$.

Let $\{g, \alpha(g)s\}$ be an edge of X. Then $g = g_1g_2$, and $s = s_1s_2$, where $g_1, s_1 \in G_1$, $g_2, s_2 \in G_2$ and $s \in S$. This further implies that $\alpha(g)s = \alpha(g_1g_2)s_1s_2 = g_1g_2^{-1}s_1s_2 = (g_1s_1)(g_2^{-1}s_2)$. Hence we obtain $\varphi(g) = (g_1, g_2)$ and $\varphi(\alpha(g)s) = (g_1s_1, g_2^{-1}s_2)$. Therefore, φ is an isomorphism between X and Y. □

If G is an abelian group of even order, and $G = E \times O$, where E has 2-power order, and O has odd order, then $\mathrm{Aut}(G) = \mathrm{Aut}(E) \times \mathrm{Aut}(O)$. If α is an involutory automorphism of G, then $\alpha = \alpha_1 \cdot \alpha_2$, where α_1 is an involutory automorphism of E and α_2 is an involutory automorphism of O. By Proposition 2.3 it follows that $O = \mathrm{Fix}(\alpha_2) \times \omega_{\alpha_2}(O)$, and consequently α centralizes $\mathrm{Fix}(\alpha_2)$ and acts by inversion on $\omega_{\alpha_2}(O)$. The difficulty now lies in describing the action of α_1 on E. It is clear that $E = \mathbb{Z}_{2^{k_1}} \times \ldots \times \mathbb{Z}_{2^{k_m}}$, for $1 \le k_1 \le \ldots \le k_m$. Then the action of α_1 on E can be given in terms of an $m \times m$ integer matrix A_α (see [2] for more details). Since α is an involution, it follows that $A_\alpha^2 = I$, where the equalities in i-th row are modulo 2^{k_i}. If $k_1 = \ldots = k_m = 1$, then E is an elementary abelian 2-group, and there are $\lfloor \frac{m}{2} \rfloor$ conjugacy classes of involutions in $\mathrm{Aut}(E) = GL(n, 2)$. Namely, since $A_\alpha^2 = I$, it follows that $x^2 - 1$ is the minimal polynomial of A_α. This implies that $\lambda = 1$ is the unique eigenvalue of A_α. Moreover, A_α can be written in Jordan form with all ones on the diagonal, and with every Jordan block of size 1 or 2. There must be at least one Jordan block of size 2, otherwise α acts trivially on E. Hence, the number of representatives of conjugacy classes of involutions in $\mathrm{Aut}(E) = GL(n, 2)$ is $\lfloor \frac{m}{2} \rfloor$. Similarly, if $E \cong \mathbb{Z}_{2^k}$, then α acts on E as a multiplication by a, where $a \in \{\pm 1, 2^k \pm 1\}$. If E is neither an elementary abelian 2-group nor a cyclic group, then the description of the action of α on E is more complicated.

3. Generalized Cayley Graphs with Respect to the Inversion Automorphism

Throughout this section we assume that G is an abelian group, and that ι is the inversion automorphism of G, that is $\iota(x) = x^{-1}$ $(\forall x \in G)$. Before stating the main results of this section, let us recall the definition of generalized dihedral groups. For an abelian group G, the *generalized dihedral group* $\mathrm{Dih}(G)$ is the semidirect product $G \rtimes \mathbb{Z}_2$, with $\mathbb{Z}_2$ acting on G by inverting elements. More precisely, for $i \in \mathbb{Z}_2$ and $g_1, g_2 \in G$ we have

$$\begin{aligned} (g_1, i) \circ (g_2, 0) &= (g_1g_2, i) \\ (g_1, i) \circ (g_2, 1) &= (g_1^{-1}g_2, i + 1). \end{aligned}$$

(Note that in a standard definition of the generalized dihedral group $\mathrm{Dih}(G) = G \rtimes \mathbb{Z}_2$ the group operation is defined by $(g_1, i) \circ (g_2, j) = (g_1\varphi(i)(g_2), i + j)$, where $\varphi \colon \mathbb{Z}_2 \to \mathrm{Aut}(G)$ is a group homomorphism mapping $0 \in \mathbb{Z}_2$ to the identity automorphism of G and mapping $1 \in \mathbb{Z}_2$ to $\iota \in \mathrm{Aut}(G)$. However, it is not difficult to check that our definition is equivalent to this standard definition.)

The following theorem shows that every generalized Cayley graph with respect to the inversion automorphism on an abelian group of even order with cyclic Sylow 2-subgroup is isomorphic to a Cayley graph on a generalized dihedral group.

Theorem 3.1. *Let n be a non-negative integer and let G be a finite abelian group of odd order. Then the generalized Cayley graph $GC(\mathbb{Z}_{2^n} \times G, S, \iota)$ is isomorphic to a Cayley graph on* $\mathrm{Dih}(\mathbb{Z}_{2^{n-1}} \times G)$.

Proof. Let $X = GC(\mathbb{Z}_{2^n} \times G, S, \iota)$. If $n = 0$, then Definition 1.1(ii) implies that $S = \emptyset$, and therefore X has no edges. Hence we assume that $n \ge 1$. For $(x, g) \in \mathbb{Z}_{2^n} \times G$, define a mapping $\varphi : \mathbb{Z}_{2^n} \times G \to (\mathbb{Z}_{2^{n-1}} \times G) \times \mathbb{Z}_2$ with

$$\varphi((x, g)) = \left(\left(\left\lfloor \frac{x}{2} \right\rfloor, g\right), x \bmod 2\right).$$

It is not difficult to verify that φ is bijection. It is also clear that $((\mathbb{Z}_{2^{n-1}} \times G) \times \mathbb{Z}_2, \circ)$ with the operation $\circ$ defined in the following way:

$$\begin{aligned}((x_1, g_1), i) \circ ((x_2, g_2), 0) &= ((x_1 + x_2, g_1 g_2), i)\\ ((x_1, g_1), i) \circ ((x_2, g_2), 1) &= ((-x_1 + x_2, g_1^{-1} g_2), i + 1)\end{aligned}$$

is the generalized dihedral group $\mathrm{Dih}(\mathbb{Z}_{2^{n-1}} \times G)$. (The group G is abelian, but we write its operation in a multiplicative way.) Observe that $((x, g), 0)^{-1} = ((-x, g^{-1}), 0)$ and $((x, g), 1)^{-1} = ((x, g), 1)$. It is also not difficult to verify that for $x_1, x_2 \in \mathbb{Z}_{2^{n-1}}$ and $g_1, g_2 \in G$,

$$((x_1, g_1), 0)^{-1} \circ ((x_2, g_2), 1) = ((x_1, g_1), 1)^{-1} \circ ((x_2, g_2), 0) = ((x_1 + x_2, g_1 g_2), 1) \tag{1}$$

Definition 1.1(ii) implies that for any $(x, g) \in \mathbb{Z}_{2^n} \times G$, $\iota\left((x, g)^{-1}\right) \cdot (x, g) = (x, g) \cdot (x, g) = (2x, g^2) \notin S$. Since G is of odd order we have $G = \{g^2 \mid g \in G\}$. Therefore, if $s = (x, g) \in S$, then x is odd and $\varphi(s) = \left(\left(\frac{x-1}{2}, g\right), 1\right)$.

We claim that φ is isomorphism between X and $Y = \mathrm{Cay}(\mathrm{Dih}(\mathbb{Z}_{2^{n-1}} \times G), \varphi(S))$. Let (x_1, g_1) and (x_2, g_2) be two adjacent vertices of X. Then $(x_1 + x_2, g_1 g_2) \in S$. We have already seen that for each element in S its first coordinate is an odd element from $\mathbb{Z}_{2^n}$, hence either ($x_1 \equiv 0 \bmod 2, x_2 \equiv 1 \bmod 2$) or ($x_1 \equiv 1 \bmod 2, x_2 \equiv 0 \bmod 2$).

Consider first the case when $x_1 \equiv 0 \bmod 2$ and $x_2 \equiv 1 \bmod 2$. Observe that now $\left\lfloor \frac{x_1}{2} \right\rfloor + \left\lfloor \frac{x_2}{2} \right\rfloor = \left\lfloor \frac{x_1+x_2}{2} \right\rfloor$. Then $\varphi((x_1, g_1)) = \left(\left(\left\lfloor \frac{x_1}{2} \right\rfloor, g_1\right), 0\right)$ and $\varphi((x_2, g_2)) = \left(\left(\left\lfloor \frac{x_2}{2} \right\rfloor, g_2\right), 1\right)$ and using (1), we obtain

$$\begin{aligned}\varphi((x_1, g_1))^{-1} \circ \varphi((x_2, g_2)) &= \left(\left(\left\lfloor \frac{x_1}{2} \right\rfloor, g_1\right), 0\right)^{-1} \circ \left(\left(\left\lfloor \frac{x_2}{2} \right\rfloor, g_2\right), 1\right) =\\ &\left(\left(\left\lfloor \frac{x_1}{2} \right\rfloor + \left\lfloor \frac{x_2}{2} \right\rfloor, g_1 g_2\right), 1\right) = \left(\left(\left\lfloor \frac{x_1 + x_2}{2} \right\rfloor, g_1 g_2\right), 1\right) = \varphi((x_1 + x_2, g_1 g_2)) \in \varphi(S).\end{aligned}$$

Similarly, if $x_1 \equiv 1 \bmod 2$ and $x_2 \equiv 0 \bmod 2$, then again using (1), we obtain

$$\varphi((x_1, g_1))^{-1} \circ \varphi((x_2, g_2)) = \left(\left(\left\lfloor \frac{x_1}{2} \right\rfloor, g_1\right), 1\right)^{-1} \circ \left(\left(\left\lfloor \frac{x_2}{2} \right\rfloor, g_2\right), 0\right) = \varphi((x_1 + x_2, g_1 g_2)) \in \varphi(S).$$

Therefore $\varphi((x_1, g_1))^{-1} \circ \varphi((x_2, g_2)) \in \varphi(S)$ and hence $\varphi(x_1, g_1)$ and $\varphi(x_2, g_2)$ are adjacent in Y.

Suppose now that $\varphi(x_1, g_1)$ and $\varphi(x_2, g_2)$ are adjacent in Y for some $x_1, x_2 \in \mathbb{Z}_{2^n}$ and $g_1, g_2 \in G$. Then $\varphi((x_1, g_1))^{-1} \circ \varphi((x_2, g_2)) \in \varphi(S)$, and we conclude that x_1 and x_2 have different parity. Further, using (1), we obtain that $\varphi((x_1, g_1))^{-1} \circ \varphi((x_2, g_2)) = \varphi((x_1 + x_2, g_1 g_2))$. Therefore, $(x_1 + x_2, g_1 g_2) \in S$, and hence (x_1, g_1) and (x_2, g_2) are adjacent in X. This shows that φ is an isomorphism between X and Y, which concludes the proof. □

Theorem 3.1 shows that every generalized Cayley graph on an abelian group with cyclic Sylow 2-subgroup and with respect to the inversion automorphism is a Cayley graph. The following two examples show that this does not hold for abelian groups in general.

Example 3.2. *Let $G = \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, $m \geq 1$, $n \geq 2$, $S = \{(1, 0), (0, 1), (1, 1)\}$ and let ι be the inversion automorphism of G. Then $GC(G, S, \iota)$ is not vertex-transitive.*

Proof. Let us consider the triangles contained in $X = GC(G, S, \iota)$. Suppose that the vertices $a, b, c \in G$ form a triangle. Each edge of this triangle is generated by a different element from S, since one element of S generates a perfect matching of the graph. Therefore, without loss of generality we may assume that $a + b = (1, 0)$, $b + c = (1, 1)$, $c + a = (0, 1)$. From this we obtain $b = (1, 0) - a$, $c = (1, 1) - b = (0, 1) + a$ and $a = (0, 1) - c = -a$, and hence $2a = 0$. Therefore, each triangle in X contains one element of order 2 from G. If $m \geq 3$ or $n \geq 3$ then the vertex $(2, 2)$ does not lie on a triangle, hence X is not vertex-transitive in this case. If $n = 2$ and $m \in \{1, 2\}$, then it is not difficult to see that there exist at least two vertices of X that belong to different number of triangles in X. This concludes the proof. □

Example 3.3. *Let* $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}$, $k \geq 1$, $S = \{(1,0,0),(0,1,0),(1,1,1)\}$ *and let* ι *be the inversion automorphism of* G. *Then* $GC(G,S,\iota)$ *is not vertex-transitive.*

Proof. It is easy to verify that the vertex $(0,0,0)$ does not lie on a triangle, whereas the vertex $(0,0,k)$ lies on the triangle $[(0,0,k),(1,0,-k),(0,1,k+1)]$. Therefore $GC(G,S,\iota)$ is not vertex-transitive. □

Before we state the main result of this section, we need the following lemma about the direct product of generalized Cayley graphs. Recall that the direct product $X \times Y$ of graphs X and Y is the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $X \times Y$ if and only if $\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in E(Y)$.

Lemma 3.4. *Let* $X = GC(G_1, S_1, \alpha_1)$, $Y = GC(G_2, S_2, \alpha_2)$, *and let* α *be the automorphism of* $G_1 \times G_2$ *defined by* $\alpha(g_1, g_2) = (\alpha_1(g_1), \alpha_2(g_2))$ *for* $g_1 \in G_1$ *and* $g_2 \in G_2$. *Then* $X \times Y \cong GC(G_1 \times G_2, S_1 \times S_2, \alpha)$.

Proof. Let us first verify that $GC(G_1 \times G_2, S_1 \times S_2, \alpha)$ is well-defined, that is, that $S_1 \times S_2$ and α satisfy Definition 1.1. It is clear that α is an automorphism of $G_1 \times G_2$, and that $\alpha^2 = 1$. We have $\omega_\alpha(G_1 \times G_2) = \left\{\alpha((g_1,g_2)^{-1})(g_1,g_2) \mid g_1 \in G_1, g_2 \in G_2\right\} = \omega_{\alpha_1}(G_1) \times \omega_{\alpha_2}(G_2)$. Since $S_1 \cap \omega_{\alpha_1}(G_1) = \emptyset$ and $S_2 \cap \omega_{\alpha_2}(G_2) = \emptyset$, it follows that $(S_1 \times S_2) \cap \omega_\alpha(G) = \emptyset$, and therefore Definition 1.1(ii) is satisfied. It is straightforward to verify that $\alpha(S^{-1}) = \alpha_1(S_1^{-1}) \times \alpha_2(S_2^{-1}) = S_1 \times S_2 = S$, and consequently Definition 1.1(iii) is satisfied.

Let (x_1, y_1) and (x_2, y_2) be two vertices of $X \times Y$. Then

$$\begin{array}{ll} & \{(x_1,y_1),(x_2,y_2)\} \in E(X \times Y) \\ \Leftrightarrow & \{x_1,x_2\} \in E(X) \text{ and } \{y_1,y_2\} \in E(Y) \\ \Leftrightarrow & \alpha_1(x_1^{-1})x_2 \in S_1 \text{ and } \alpha_2(y_1^{-1})y_2 \in S_2 \\ \Leftrightarrow & \alpha((x_1,y_1)^{-1}) \cdot (x_2,y_2) \in S_1 \times S_2 = S \\ \Leftrightarrow & \{(x_1,y_1),(x_2,y_2)\} \in E(GC(G_1 \times G_2, S_1 \times S_2, \alpha)) \end{array}$$

This shows that the mapping $\varphi : VG_1 \times G_2 \to G_1 \times G_2$ defined with $\varphi(x,y) = (x,y)$ is an isomorphism between $X \times Y$ and $GC(G_1 \times G_2, S_1 \times S_2, \alpha)$. □

Theorem 3.1 shows that every generalized Cayley graph on an abelian group with cyclic Sylow 2-subgroup with respect to the inversion automorphism is also a Cayley graph. The same result holds if G is an elementary abelian 2-group. Namely, in this case the inversion automorphism is the identity mapping, since each element of G is of order 2. We are now ready to present the proof of Theorem 1.2.

Theorem 1.2. *Let* G *be an abelian group and let* ι *be the inversion automorphism of* G. *Then every generalized Cayley graph on* G *with respect to* ι *is a Cayley graph if and only if one of the following holds:*

(i) G *is an elementary abelian* 2*-group;*

(ii) the Sylow 2-subgroup of G *is cyclic.*

Proof. If G is an elementary abelian 2-group, then ι is the identity map, and therefore $GC(G,S,\iota) \cong Cay(G,S)$. If the Sylow 2-subgroup of G is cyclic, then $G \cong \mathbb{Z}_{2^n} \times H$, where H is an abelian group of odd order. By Theorem 3.1 we conclude that $GC(G,S,\alpha)$ is a Cayley graph.

Suppose now that G is not an elementary abelian 2-group and that the Sylow 2-subgroup of G is not cyclic. First consider the case when the Sylow 2-subgroup of G is elementary abelian. Then, since G is not elementary abelian, the order of G must be divisible by some odd number and hence $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2k+1} \times H$, where k is a positive integer, and H is an abelian group. Let $S_1 = \{(1,0,0),(0,1,0),(1,1,1)\}$, $S_2 = H \setminus \{1_H\}$, $S = S_1 \times S_2$ and let ι_1 be the restriction of ι to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}$ and ι_2 the restriction of ι to H. Then, by Lemma 3.4, $GC(G,S,\iota) \cong GC(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}, S_1, \iota_1) \times GC(H, S_2, \iota_2)$. Since the direct product of two graphs is vertex-transitive if and only if both factors are vertex-transitive (see [1, Theorem 8.19]), and since by Example 3.3, $GC(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}, S_1, \iota_1)$ is not vertex-transitive, it follows that $GC(G,S,\iota)$ is not vertex-transitive, and consequently it is not a Cayley graph.

If the Sylow 2-subgroup of G is not elementary abelian nor cyclic, then $G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n} \times H$ for $m \geq 1$ and $n \geq 2$ and an abelian group H. Using Example 3.2 and Lemma 3.4 we construct a non-vertex-transitive generalized Cayley graph on G. □

4. Generalized Cayley Graphs of Order $2p$

In this section we consider generalized Cayley graphs of order $2p$, where p is a prime. We are now going to prove Theorem 1.3.

Theorem 1.3. *Every generalized Cayley graph of order $2p$ is a Cayley graph.*

Proof. Let G be a group of order $2p$. Then $G \cong \mathbb{Z}_{2p}$ or $G \cong D_{2p}$. Suppose first that $G \cong \mathbb{Z}_{2p}$. Then there are only two automorphisms of G of order 2, namely the identity mapping, or the inversion automorphism ι. For $\alpha = 1$, $GC(G, S, \alpha) \cong \mathrm{Cay}(G, S)$ by definition. For $\alpha = \iota$, $GC(\mathbb{Z}_{2p}, S, \alpha)$ is a Cayley graph on the group $D_{2p} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_2$ by Theorem 3.1.

Suppose now that $G \cong D_{2p} = \langle \tau, \rho \mid \tau^2 = \rho^p = 1,\ \tau\rho\tau = \rho^{-1} \rangle$. If $p = 2$ then the claim clearly holds, therefore we will assume that $p > 2$. Let α be an automorphism of D_{2p} of order 2. Since automorphisms preserve the order of the elements, it follows that $\alpha(\rho) = \rho^k$, where $(k, p) = 1$, and $\alpha(\tau) = \tau\rho^l$. The fact that α is an involution gives us the following restrictions on k and l:

$$\rho = \alpha(\alpha(\rho)) = \rho^{k^2} \Rightarrow k \equiv \pm 1 \pmod{p}$$

$$\tau = \alpha(\alpha(\tau)) = \alpha(\tau\rho^l) = \alpha(\tau)\alpha(\rho^l) = \tau\rho^{l(k+1)} \Rightarrow l(k+1) \equiv 0 \pmod{p}.$$

Moreover, if $k = 1$ then $l = 0$ and α is the identity. Therefore, we can assume that $\alpha(\rho) = \rho^{-1}$ and $\alpha(\tau) = \tau\rho^l$ for some $l \in \mathbb{Z}_p$. It is now easy to see that α is an inner automorphism of D_{2p} which acts as conjugation by $\tau\rho^{\frac{l}{2}}$. Now applying [7, Proposition 3.6] it follows that every generalized Cayley graph with respect to α is a Cayley graph. This concludes the proof. □

5. Unworthy Generalized Cayley Graphs

Recall that a graph X is said to be *unworthy* if there exist two vertices of X with the same neighbourhood in X, and worthy otherwise. This section deals with the question which generalized Cayley graphs are unworthy. In order to answer this question, for a group G, a subset S of G and $\alpha \in \mathrm{Aut}(G)$ such that the conditions from Definition 1.1 are satisfied, we define

$$K = \{\alpha(g) \mid gS = S\} = \{g \mid \alpha(g)S = S\}.$$

Observe that K is a subgroup of G.

Proposition 5.1. *The vertices x and y in $X = GC(G, S, \alpha)$ have the same neighbours in X if and only if $x^{-1}y \in K$.*

Proof. Suppose first that $x^{-1}y \in K$, that is, $y = xk$ for some $k \in K$. Then the neighbourhood of x is $\alpha(x)S$ and the neighbourhood of y is $\alpha(y)S = \alpha(xk)S = \alpha(x)\alpha(k)S = \alpha(x)S$. Therefore x and y have the same neighbours.

Conversely, if x and y have the same neighbours, then $\alpha(x)S = \alpha(y)S$, which implies $S = \alpha(x^{-1}y)S$. Therefore, $x^{-1}y \in K$, and the result follows. □

Corollary 5.2. *The graph $GC(G, S, \alpha)$ is unworthy if and only if $K \neq \{1_G\}$.*

The following proposition shows that an unworthy generalized Cayley graph can be decomposed into the lexicographic product of a worthy graph and an empty graph. (The *lexicographic product* of graphs X and Y is the graph $X[Y]$ with vertex set $V(X) \times V(Y)$, where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $\{x_1, x_2\} \in E(X)$ or $x_1 = x_2$ and $\{y_1, y_2\} \in E(Y)$.) Let X_K be the quotient graph of X with respect to the partition $\{xK \mid x \in G\}$.

Proposition 5.3. *Let $X = GC(G, S, \alpha)$ be unworthy. Then $X \cong X_K[\overline{K_n}]$, where $n = |K|$.*

Proof. Suppose that the generalized Cayley graph $X = GC(G, S, \alpha)$ is unworthy. By Proposition 5.1, two vertices of X have the same neighbours if and only if they belong to the same left coset of K in G. Then S is a union of several left cosets of K, and all the vertices in the same left coset of K have the same neighbours. It is now easy to see that $X \cong X_K[\overline{K_n}]$. □

Corollary 5.4. *If G is an abelian group and $S = G \setminus \omega_\alpha(G)$ then $GC(G, S, \alpha) \cong K_m[\overline{K_n}]$, where $n = \omega_\alpha(G)$, and $m = |G|/n$.*

Proof. Since G is abelian, by Proposition 2.2(ii), $\omega_\alpha(G)$ is a subgroup of G. We have $K = \{\alpha(g) \mid g\cdot(G\setminus\omega_\alpha(G)) = G \setminus \omega_\alpha(G)\} = \{\alpha(g) \mid g \cdot \omega_\alpha(G) = \omega_\alpha(G)\} = \{\alpha(g) \mid g \in \omega_\alpha(G)\}$. By Proposition 2.2(iii), $\alpha(g) = g^{-1}$ for each $g \in \omega_{\alpha(G)}$, and therefore $K = \omega_\alpha(G)$.

We claim that $X_K \cong K_m$. The number of vertices in X_K is equal to the index of K in G, which is $|G|/|K| = |G|/n = m$. Let xK and yK be two different vertices of X_K, that is $x^{-1}y \notin K$. This implies that $\alpha(x)$ and y are adjacent in X. Then, since G is abelian, $\alpha(x) = x\alpha(x)x^{-1} \in xK$. Therefore, vertices xK and yK are adjacent in X_K, and consequently $X_K \cong K_m$. The claim now follows by Proposition 5.3. □

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