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Endpoints of Multi-Valued Weakly Contraction in Complete Metric Spaces Endowed with Graphs

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Abstract. In this paper, we introduce a new concept of weak G-contraction for multi-valued mappings on a metric space endowed with a directed graph. Endpoint theorem of this mapping is established under some sufficient conditions in a complete metric space endowed with a directed graph. Our main results extend and generalize those fixed point in partially ordered metric spaces. Some examples supporting our main results are also given. Moreover, we apply our main results to obtain some coupled fixed point results in the context of complete metric spaces endowed with a directed graph which are more general than those in partially ordered metric spaces.

1. Introduction

Banach contraction principle [1] is one of the most fundamental result in metric fixed point theory. It plays very important role in studying the existence of solutions of various equation such as different equations, integral equations and system of linear equations. In 2001, Rhoades [2] extended this by introducing the concept of weakly contractive map and proved some results in metric spaces. Banach contraction principle has been widely generalized in different directions (see [3]-[10]). It was extended to multi-valued mappings by Nadler [11] in 1969.

Let $T : X \to 2^X$ be a multi-valued map. An element $x \in X$ is called a fixed point of T, if $x \in Tx$. An element $x \in X$is said to be an endpoint (or stationary point) of T, if $Tx = \{x\}$. Several authors paid attention to the existence of endpoints of multi-valued mapppings (see [12]-[16]).

Ran and Reurings [4] were the first who studied Banach contraction principle in partially ordered metric spaces and applied the obtain results to linear and nonlinear matrix equation. After that many authors extended those results and studied fixed point theorems in partially ordered metric spaces (see [4],[5],[6],[7]). In another way, Jachymski [17] extended this principle in the setting of metric space endowed with a graph. In this work, we aim to introduce and study a new concept of weak G-contraction for multi-valued mappings on a metric space endowed with a directed graph.

We first recall definitions of the following auxiliary functions. Let Ψ be all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

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(1) ψ is continuous and nondecreasing,

(2) $\psi(t) = 0$ if and only if $t = 0$,

In 2010, Harjani and Sadarangani [18] proved the following theorem

Theorem 1.1 ([18]). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies if x_n is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$ for all $n \in \mathbb{N}$. Let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(f(x), f(y))) \leq \psi(d(x, y)) - \phi(d(x, y)), \text{ for } x \leq y,$$

where $\psi, \phi \in \Psi$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$ then f has a fixed point.

In 2008, Jachymski [17] was the first who studied fixed point theorems for some contraction mappings in a metric space endowed with a graph. He proved some fixed point results of G-contractions in this setting.

Definition 1.2 ([17]). *Let (X, d) be a metric space and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$.*

We say that a mapping $f : X \to X$ is a *G-contraction* if f preserves edges of G, i.e.,

$$x, y \in X, (x, y) \in E(G) \Rightarrow (f(x), f(y)) \in E(G) \tag{1.1}$$

and there exists $\alpha \in (0, 1)$ such that

$$x, y \in X, (x, y) \in E(G) \Rightarrow d(f(x), f(y)) \leq \alpha d(x, y).$$

He showed in [17], under some certain properties on (X, d, G), a G-contraction $f : X \to X$ has a fixed point if and only if $X_f := \{x \in X : (x, f(x)) \in E(G)\}$ is nonempty. The mapping $f : X \to X$ satisfying the condition (1.1) is called a graph-perserving mapping.

Later, in 2013, Dinvari and Frigon [19] introduced a new concept of G-contraction which is weaker than that of Jachymski.

Definition 1.3 ([19]). *Let $T : X \to 2^X$ be a map with nonempty values. We say that T is a G-contraction(in the sense of Dinvari and Frigon) if there exists $\alpha \in (0, 1)$ such that for all $(x, y) \in E(G)$ and all $u \in Tx$, there exists $v \in Ty$ such that*

$$(u, v) \in E(G) \text{ and } d(u, v) \leq \alpha d(x, y).$$

They showed that a multi-valued G-contraction with closed values has a fixed point under some properties on a metric space which is weaker than Property(A) (see [19], Theorem 2.10 and Corollary 2.11).

Recently, Sultana and Vetrivel [20] introduced a notion of Mizoguchi-Takahashi multi-valued contraction in a metricc space endowed with a graph and they established its fixed point results.

Definition 1.4. *A multi-valued mapping $T : X \to CB(X)$ is said to be Mizoguchi-Takahashi G contraction if for any $x, y \in X$, $x \neq y$ with $(x, y) \in E(G)$:*

(i) $H(T(x), T(y)) \leq \alpha(d(x, y))d(x, y)$ *where* $\alpha : [0, \infty) \to [0, 1)$ *such that* $\limsup_{s \to t^+} \alpha(s) < 1$ *for any* $t \in [0, \infty)$, *and*

(ii) If $u \in T(x)$ and $v \in T(y)$ are such that $d(u, v) \leq d(x, y)$, then $(u, v) \in E(G)$.

In this work, by combination the concept of weak contraction for single-valued mappings and Mizoguchi-Takahashi G-contraction given by Harjani and Sadarangani [18] and Sultana and Vetrivel [20], we introduce a new type of G-contraction for multi-valued mappings, called G-weakly contraction, and prove the existence of endpoints for this type of mappings.

2. Preliminaries

In this section, we give some basic, useful definitions and some known results that will be used in the other sections.

A partial order is a binary relation $\leq$ over the set X which satisfies the followings conditions:

1. $x \leq x$ (reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity)

for all $x, y \in X$. A set with a partial order $\leq$ is called a partially ordered set. We write $x \prec y$ if $x \leq y$ and $x \neq y$.

Definition 2.1. *Let $(X, \leq)$ be a partial order set and $T : X \to X$. Then T is said to be nondecreasing if*

$$x \leq y \Rightarrow Tx \leq Ty.$$

Let $G = (V(G), E(G))$ be a directed graph where $V(G)$ is a set of vertices of graph and $E(G)$ is a set of its edges. Assume that G has no parallel edges. We denote by G^{-1} the directed graph obtained from G by reversing the definition of edges. That is

$$E(G^{-1}) = \{(x, y) : (y, x) \in E(G)\}.$$

Definition 2.2. *Let X be a nonempty set and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$. Then G is said to be transitive if*

$$(x, y), (y, z) \in E(G) \Rightarrow (x, z) \in E(G).$$

Definition 2.3. *Let X be a nonempty set and $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and let $T : X \to X$. Then T is said to be graph-preserving if*

$$(x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G).$$

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed and bounded subsets of X. For $x \in X$ and $A, B \in CB(X)$, define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

$$\delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\}.$$

Denote H the Pompeiu-Hausdorff metric induced by d, see [24], that is

$$H(A, B) = \max\{\sup_{u\in A} d(u, B), \sup_{v\in B} d(v, A)\}.$$

3. Main Results

We introduce a new type of G-contraction in this section and prove its endpoint theorem in a complete metric space endowed with a directed graph. We begin with the following definitions.

Definition 3.1. *Let (X, d) be a metric space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = X$ and $T : X \to CB(X)$. T is said to be G-continuous with respect to δ if for any sequence $\{x_n\}$ in X and $x \in X$ such that $(x_n, x_{n+1}) \in E(G)$ and $d(x_n, x) \to 0$, then $\delta(Tx_n, Tx) \to 0$.*

Example 3.2. *Let $X = [1, \infty)$ with usual metric d and let G be a directed graph with $V(G) = X$ and $E(G) = \{(1 + \frac{1}{n}, 1 + \frac{1}{n+1}) | n \in \mathbb{N}\}$. Define $T : X \to 2^X$ by $Tx = [1 + \frac{1}{x}, 2]$. Then T is G-continuous with respect to δ.*

Remark 3.3. *Let (X, d) be a metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and $T : X \to CB(X)$. Suppose T is G-continuous on X with respect to δ. Then we have*

1. *if $(x, x) \in E(G)$ then Tx is a singleton set, and*
2. *if $\delta(Tx_n, Tx) \to 0$ then Tx is a singleton set.*

Proof. *(1) If $(x, x) \in E(G)$, then the sequence $x_n = x$ for all $n \in \mathbb{N}$ satisfies the conditions $(x_n, x_{n+1}) \in E(G)$ and $d(x_n, x) \to 0$. This implies that $\delta(Tx, Tx) = 0$, so Tx is a singleton set.*
(2) Suppose that $\delta(Tx_n, Tx) \to 0$. Let $y_n \in Tx_n$ for all $n \in \mathbb{N}$. If $y, z \in Tx$, we have

$$d(y_n, y) \leq \delta(Tx_n, Tx) \to 0 \text{ and}$$
$$d(y_n, z) \leq \delta(Tx_n, Tx) \to 0.$$

It follows that $y = z$. Hence Tx is singleton. □

Definition 3.4. *Let (X, d) be a metric space, $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and $T : X \to B(X)$. T is said to be a G-weakly contraction if there exist $\psi, \phi \in \Psi$ with*

$$\psi(\delta(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$ such that $(x, y) \in E(G)$
and if $u \in Tx$ and $v \in Ty$ satisfying

$$d(u, v) \leq d(x, y)$$

then $(u, v) \in E(G)$.

The following property is useful for our study.
Property A([17]) For any sequence x_n in X, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ where $x \in X$, then there is a subsequence x_{n_k} of $\{x_n\}$ with $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Theorem 3.5. *Let (X, d) be a complete metric space and $G = (V(G), E(G))$ a directed graph such that $V(G) = X$ and G is transitive. If $T : X \to B(X)$ is a multi-valued mapping satisfying the following properties:*

(1) there exists $x_0 \in X$ such that $(x_0, y) \in E(G)$, for some $y \in Tx_0$;

(2) T is a G-weakly contraction;

(3) Suppose either T is G-continuous or X has the property A;

then T has an endpoint.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$. By (2), we obtain

$$\psi(\delta(Tx_0, Tx_1)) \leq \psi(d(x_0, x_1)) - \phi(d(x_0, x_1)) \tag{1.2}$$

We choose $x_2 \in Tx_1$. Observe that $d(x_1, x_2) \leq \delta(Tx_0, Tx_1)$. If $d(x_0, x_1) = 0$, then $x_0 = x_1 \in Tx_0$, so x_0 is a fixed point of T. Suppose $d(x_0, x_1) > 0$. Since ψ is a nondecreasing, we have

$$\begin{aligned}\psi(d(x_1, x_2)) &\leq \psi(\delta(Tx_0, Tx_1)) \\ &\leq \psi(d(x_0, x_1)) - \phi(d(x_0, x_1)) \\ &< \psi(d(x_0, x_1)).\end{aligned}$$

This implies $d(x_0, x_1) \leq d(x_1, x_2)$. Because T is a G-weakly contraction, we have $(x_1, x_2) \in E(G)$. Again, choosing $x_3 \in Tx_2$, we observe that $d(x_2, x_3) \leq \delta(Tx_1, Tx_2)$. If $d(x_1, x_2) = 0$, then $x_1 \in Tx_1$, so T has a fixed point. Suppose $d(x_1, x_2) > 0$. Since ψ is a nondecreasing, we obtain

$$\begin{aligned}\psi(d(x_2, x_3)) &\leq \psi(\delta(Tx_1, Tx_2))\\ &\leq \psi(d(x_1, x_2)) - \phi(d(x_1, x_2))\\ &< \psi(d(x_1, x_2)),\end{aligned}$$

hence $d(x_2, x_3) \leq d(x_1, x_2)$. Since T is a G-weakly contraction, we have $(x_2, x_3) \in E(G)$. By induction, we obtain a sequence $\{x_n\} \in X$ such that $x_{n+1} \in Tx_n$ and $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since $\{d(x_n, x_{n+1})\}$ is a nonnegative and nonincreasing sequence, there exists $r \geq 0$ such that

$$d(x_n, x_{n+1}) \to r \text{ as } n \to \infty.$$

Since T is a G-weakly contraction and $(x_n, x_{n+1}) \in E(G)$, we obtain

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &\leq \psi(\delta(Tx_n, Tx_{n+1}))\\ &\leq \psi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})),\end{aligned}$$

when $n \to \infty$, we get $\psi(r) \leq \psi(r) - \phi(r)$. It implies $\phi(r) = 0$. Since $\phi \in \Psi$, we get $r = 0$. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{1.3}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$, subsequence $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon. \tag{1.4}$$

Let $n(k)$ be the smallest integer with $m(k) < n(k)$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ but

$$d(x_{m(k)}, x_{n(k)-1}) < \epsilon. \tag{1.5}$$

By (1.3), (1.4) and (1.5), we have

$$\begin{aligned}\epsilon &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})\\ &< \epsilon + d(x_{n(k)-1}, x_{n(k)}).\end{aligned}$$

Taking $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon.$$

From

$$\begin{aligned}&d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}), \text{ and}\\ &d(x_{m(k)-1}, x_{n(k)-1}) \leq d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}),\end{aligned}$$

by letting $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon.$$

Since $m(k) < n(k)$ and $(x_i, x_{i+1}) \in E(G)$, by transitivity of $E(G)$, we obtain $(x_{m(k)}, x_{n(k)}) \in E(G)$. It follows, by (2), that

$$\begin{aligned}\psi(d(x_{m(k)}, x_{n(k)})) &\leq \psi(\delta(d(x_{m(k)-1}, x_{n(k)-1})))\\ &\leq \psi(d(x_{m(k)-1}, x_{n(k)-1})) - \phi(d(x_{m(k)-1}, x_{n(k)-1})).\end{aligned}$$

Taking $k \to \infty$, we obtain

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon).$$

Then $\phi(\epsilon) = 0$, i.e., $\epsilon = 0$, which is a contradiction. So $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. We shall show that $\delta(z, Tz) = 0$.

Case (i): If X has Property A.

Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, z) \in E(G)$. It follows that

$$\begin{aligned}\delta(z, Tz) &\leq d(z, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) + \delta(Tx_{n_k}, Tz)\\ &= d(z, x_{n_k+1}) + \delta(Tx_{n_k}, Tz).\end{aligned}$$

Suppose that $\delta(z, Tz) > 0$, we can choose $N \in \mathbb{N}$ such that for any $k > N$,

$$0 \leq \delta(z, Tz) - d(z, x_{n_k+1}) \leq \delta(Tx_{n_k}, Tz)$$

Since $\psi \in \Psi$ and $(x_{n_k}, z) \in E(G)$, we have

$$\begin{aligned}\psi(\delta(z, Tz) - d(z, x_{n_k+1})) &\leq \psi(\delta(Tx_{n_k}, Tz))\\ &\leq \psi(d(x_{n_k}, z)) - \phi(d(x_{n_k}, z)).\end{aligned}$$

Letting $k \to \infty$, we obtain

$$0 < \psi(\delta(z, Tz)) \leq \psi(0) - \phi(0) = 0,$$

which is a contradiction. Hence $\delta(z, Tz) = 0$ i.e., $Tz = \{z\}$, so z is an endpoint of T.

Case (ii): T is G-continuous.

Since $x_n \to z$ and $(x_n, x_{n+1}) \in E(G)$, we obtain

$$\begin{aligned}\delta(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tx_n) + \delta(Tx_n, Tz)\\ &= d(z, x_{n+1}) + \delta(Tx_n, Tz) \to 0.\end{aligned}$$

Hence $\delta(z, Tz) = 0$ that is $Tz = \{z\}$, so z is an endpoint of T. $\square$

The following example is an illustration of Theorem 3.5.

Example 3.6. *Let $X = [0, 1]$ and let G be a directed graph with $V(G) = X$ and $E(G) = \{(\frac{1}{m}, \frac{1}{n}) : m, n \in \mathbb{N}$ and $2 \leq m < n\} \cup \{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$. Then G is transitive. Define $T : X \to B(X)$ by*

$$Tx = \begin{cases} \{0\} & \text{if } x = 0;\\ \{\frac{1}{2}, 1\} & \text{if } x = 1;\\ \{\frac{1}{2x+1}, \frac{1}{2x}\} & \text{otherwise.}\end{cases}$$

Let $\psi(t) = t$ and $\phi(t) = \frac{t}{16}$, then $\psi, \phi \in \Psi$. We see that $(\frac{1}{4}, \frac{1}{8}) \in E(G)$ where $\frac{1}{8} \in T(\frac{1}{4}) = \{\frac{1}{9}, \frac{1}{8}\}$. Let $x, y \in X$ be such that $(x, y) \in E(G)$.

If $(x, y) = (\frac{1}{n}, 0)$, then $T(\frac{1}{n}) = \{\frac{1}{2n+1}, \frac{1}{2n}\}$ and $T(0) = \{0\}$, we have

$$\begin{aligned}\psi(\delta(Tx, Ty)) = \frac{1}{2n} &\leq \Big(\frac{1}{n} - 0\Big) - \frac{1}{16}\Big(\frac{1}{n} - 0\Big)\\ &= \psi(d(x, y)) - \phi(d(x, y))\end{aligned}$$

and we see that $(0, \frac{1}{2n+1}), (0, \frac{1}{2n}) \in E(G)$.
If $(x, y) = \left(\frac{1}{m}, \frac{1}{n}\right)$ *where* $m, n \in \mathbb{N}$ *and* $2 \leq m < n$, *then* $Tx = \{\frac{1}{2m+1}, \frac{1}{2m}\}$, $Ty = \{\frac{1}{2n+1}, \frac{1}{2n}\}$, *we have*

$$\begin{aligned}\psi(\delta(Tx, Ty)) &= \frac{1}{2m} - \frac{1}{2n+1} \\ &\leq \Big(\frac{1}{m} - \frac{1}{n}\Big) - \frac{1}{16}\Big(\frac{1}{m} - \frac{1}{n}\Big) \\ &= \psi(d(x, y)) - \phi(d(x, y))\end{aligned}$$

and we see that $(\frac{1}{2m+1}, \frac{1}{2n+1}), (\frac{1}{2m+1}, \frac{1}{2n}), (\frac{1}{2m}, \frac{1}{2n+1}), (\frac{1}{2m}, \frac{1}{2n}) \in E(G)$. *Therefore T is a G-weakly contraction. It is easy to see that X has Property A. Hence all conditions of Theorem 3.5 are satisfied and it is seen that* $T(0) = \{0\}$.

Remark 3.7. *The mapping T in Example 3.6 is not a contraction because for any* $k \in (0, 1)$,

$$H(T(0), T(1)) = 1 \nleq kd(0, 1).$$

Therefore, Nadler's Theorem [11] cannot be applied.

Theorem 3.8. *Let* (X, d) *be a complete metric space with a partially order* $\preceq$. *If* $T : X \to B(X)$ *is a multi-valued mapping satisfying the following properties:*

(1) there exist $\psi, \phi \in \Psi$ *with*

$$\psi(\delta(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$ *such that* $x \neq y$ *and* $x \preceq y$
and if $u \in Tx$ *and* $v \in Ty$ *are such that*

$$d(u, v) \leq d(x, y)$$

then $u \preceq v$;

(2) there exists $x_0 \in X$ *such that* $x_0 \preceq y$, *for some* $y \in Tx_0$;

(3) For any sequence $(x_n)_{n\in\mathbb{N}}$ *in X with* $x_n \to x$ *and* $x_n \preceq x_{n+1}$ *for* $n \in \mathbb{N}$, *then there is a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *with* $x_{n_k} \preceq x$ *for* $k \in \mathbb{N}$.

Then T has an endpoint.

Proof. Denote $E(G) = \{(x, y) \in X \times X | x \preceq y \text{ and } x \neq y\}$. It easy to see that $E(G)$ is transitive. Let $x, y \in X$ and $(x, y) \in E(G)$. So $x \preceq y$ and $x \neq y$. By (1), we obtain

$$\psi(\delta(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)).$$

Now let $u \in Tx, v \in Ty$ are such that $d(u, v) \leq d(x, y)$, so $u \preceq v$, that is $(u, v) \in E(G)$. Therefore T is a G-weakly contraction. By (2), there exists $x_0 \in X$ such that $x_0 \preceq y$ for some $y \in Tx_0$, then $(x_0, y) \in E(G)$. The condition (3) implies that X has Property A. It follows directly from Theorem 3.5 that T has an endpoint. □

The following result is directly obtained by Theorem 3.5 because a graph preserving mapping T is G-weakly contraction.

Corollary 3.9. *Let* (X, d) *be a complete metric space having Property A and* $G = (V(G), E(G))$ *a directed graph such that* $V(G) = X$ *and* $E(G)$ *is transitive. If* $T : X \to X$ *is a single-valued mapping satisfying*

(1) there exists $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;

(2) T is graph-preserving;

(3) there exist $\psi, \phi \in \Psi$ with

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$ such that $(x, y) \in E(G)$,

then there exists $u \in X$ such that $u = Tu$.

The following theorem is an application of Corollary 3.9.

Theorem 3.10 ([18]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping. Suppose that there exists $x_0 \in X$ with $x_0 \preceq Tx_0$. Suppose also that there exists $\psi, \phi \in \Psi$ satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$ with $x \neq y$ and $x \preceq y$. Suppose that either T is continuous or X has the following property: if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. Then there exists $x \in X$ such that $x = Tx$.

Proof. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X | x \preceq y$ and $x \neq y\}$. Therefore, the result is obtained directly by Corollary 3.9.

4. Application to Coupled Fixed Point

In this section, we apply our main results to obtain a coupled fixed point theorem for a single-valued mapping $F : X \times X \to X$ when X is a complete metric space endowed with a directed graph. We first recall some basic definitions of coupled fixd point and mixed monotone mappings.

Definition 4.1 ([8]). *Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a given mapping. The mapping F is said to have mixed monotone property on X if it is monotone nondecreasing in x and monotone nonincreasing in y, that is,*

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 4.2 ([8]). *Let $F : X \times X \to X$ be a given mappings. A point $(x, y) \in X \times X$ is called a coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.*

In 2014, Chifu and Petrusel [21] introduced the concept of edge preserving of $F : X \times X \to X$ as the following.

Definition 4.3. *We say that $F : X \times X \to X$ is edge preserving if*

$$(x, u) \in E(G), (y, v) \in E(G^{-1}) \Rightarrow (F(x, y), F(u, v)) \in E(G) \text{ and } (F(y, x), F(v, u)) \in E(G^{-1}).$$

For a metric space (X, d) with a directed graph $G = (V(G), E(G))$ where $V(G) = X$, we let $G' = (V(G'), E(G'))$ be a directed graph defined on product $X \times X$ as follows: $V(G') = X \times X$ and

$$E(G') = \{((x, y), (u, v)) : (x, u) \in E(G), (y, v) \in E(G^{-1})\}.$$

It is noted that the mapping $\eta : (X \times X) \times (X \times X) \to (X \times X)$ given by

$$\eta((x, y), (u, v)) = d(x, u) + d(y, v),$$

for all $(x, y), (u, v) \in X \times X$, is a metric on the product $X \times X$.
Now, define the mapping T_F by

$$T_F(x, y) = (F(x, y), F(y, x)) \text{ for all } (x, y) \in X \times X.$$

It is known that (X, d) is complete if and only if $(X \times X, \eta)$ is complete, and $(x, y) \in X \times X$ is a coupled fixed point of F if and only if (x, y) is a fixed point of T_F. □

Before obtaining a coupled fixed point theorem by applications of the main results in Section 3, we first give the following useful fact.

Lemma 4.4. *If F is edge preserving, then T_F is graph-preserving.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $((x_1, y_1), (x_2, y_2)) \in E(G_{X\times X})$, so $(x_1, x_2) \in E(G')$ and $(y_1, y_2) \in E(G)$. Therefore,

$$(F(x_1, y_1), F(x_2, y_2)) \in E(G) \text{ and } (F(y_1, x_1), F(y_2, x_2)) \in E(G^{-1}).$$

So,

$$((F(x_1, y_1), F(y_1, x_1)), (F(x_2, y_2), F(y_2, x_2)) \in E(G').$$

Thus,

$$(T_F(x_1, y_1), T_F(x_2, y_2)) \in E(G').$$

Therefore, T_F is graph-preserving. □

Now, we prove a main results of this section.

Theorem 4.5. *Let (X, d) be a complete metric space and $G = (V(G), E(G)$ a directed graph such that $V(G) = C$ and G is transitive. Let $F : X\times X \to X$ be edge preserving. Suppose that there exist $x_0, y_0 \in X$ such that $(x_0, F(x_0, y_0)) \in E(G)$ and $(y_0, F(y_0, x_0)) \in E(G^{-1})$ and there exist $\psi, \phi \in \Psi$ for which F satisfies*

$$\psi\left(\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2}\right) \le \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) - \phi\left(\frac{d(x, u) + d(y, v)}{2}\right) \tag{4.1}$$

for all $x, y, u, v \in X$ with $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. Suppose that either

(1) F is continuous or ;

(2) X has the following properties:

(a) if a sequence $\{x_n\} \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$ and

(b) if a sequence $\{y_n\} \to y$ and $(y_{n+1}, y_n) \in E(G)$ for all $n \in \mathbb{N}$, then $(y_n, y) \in E(G^{-1})$ for all $n \in \mathbb{N}$.

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in $X \times X$.

Proof. Let $(x_0, y_0) \in X\times X$ be such that $(x_0, F(x_0, y_0)) \in E(G^{-1})$ and $(y_0, F(y_0, x_0)) \in E(G)$. Then $((x_0, y_0), T(x_0, y_0)) \in E(G')$. Notice that (2) is equivalent to

$$\psi\Big(\frac{\eta(T_F(x, y), T_F(u, v))}{2}\Big) \le \psi\Big(\frac{\eta((x, y), (u, v))}{2}\Big) - \phi\Big(\frac{\eta((x, y), (u, v))}{2}\Big).$$

By letting

$$d_2((x, y), (u, v)) = \frac{\eta((x, y), (u, v))}{2} = \frac{d(x, u) + d(y, v)}{2},$$

we have

$$\psi(d_2(T_F(x, y), T_F(u, v))) \le \psi(d_2((x, y), (u, v))) - \phi(d_2((x, y), (u, v)))$$

and d_2 is a complete metric on $X \times X$.
Next, let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in $X\times X$ such that $(x_n, y_n) \to (x, y)$ in $(X\times X, d_2)$ and $((x_n, y_n), (x_{n+1}, y_{n+1})) \in E(G)$. It implies that

$$x_n \to x, (x_n, x_{n+1}) \in E(G) \text{ and } y_n \to y, (y_n, y_{n+1}) \in E(G^{-1}).$$

By assumption (2), we have $(x_n, x) \in E(G)$ and $(y_n, y) \in E(G)$. Therefore, $((x_n, y_n), (x, y)) \in E(G')$. Thus, $X \times X$ has Property A. By Lemma 4.4, we have that T_F is graph preserving. Therefore, all conditions of Corollary 3.9 are satisfied. Hence T_F has a fixed point i.e., there exists $(x, y) \in X$ such that $(x, y) = T_F(x, y) = (F(x, y), F(y, x))$. Therefore, F has a coupled fixed point. □

The following corollary is a consequence of Theorem 4.5.

Corollary 4.6. *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and there exist $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose that there exist $\psi, \phi \in \Psi$ for which F and g satisfy*

$$\phi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \leq \phi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \tag{4.2}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$. Suppose that either

(1) F is continuous or ;

(2) X has the following property:

(a) if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$

(b) if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$

Then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in $X \times X$.

Proof. Let $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x, y) | x \leq y\}$. We can directly check that all conditions of Theorem 4.5 are satisfied. Therefore, F has a coupled fixed point. □

Remark 4.7. *Corollary 4.6 extends Theorem 2.1 in [8] and improves Theorem 1 in [23].*

Example 4.8. *Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, G be a directed graph such that $V(G) = \mathbb{Z}$ and $E(G) = \{(x, y) : x, y \in \mathbb{R}$ such that $x < y\}$. Define $F : X \times X \to X$ by $F(x, y) = \frac{x-4y}{8}$. Then F is edge preserving. Let $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{16}$. Then we have*

$$\begin{aligned}
\psi\left(\frac{d(F(x,y),f(u,v)) + d(F(y,x),F(v,u))}{2}\right) &= \frac{1}{4}\left(\left|\frac{x-4y}{8} - \frac{u-4v}{8}\right| + \left|\frac{y-4x}{8} - \frac{v-4u}{8}\right|\right) \\
&= \frac{1}{4}\left(\left|\frac{x-u}{8} + \frac{4v-4y}{8}\right| + \left|\frac{y-v}{8} + \frac{4u-4x}{8}\right|\right) \\
&\leq \frac{1}{16}\left(\frac{|x-u| + |y-v|}{2}\right) + \frac{1}{4}\left(\frac{|x-u| + |v-y|}{2}\right) \\
&\leq \frac{1}{2}\left(\frac{|x-u| + |y-v|}{2}\right) - \frac{1}{16}\left(\frac{|x-u| + |v-y|}{2}\right) \\
&= \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right),
\end{aligned}$$

for all $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$. So F satisfies the condition (2) of Theorem 4.5. Now choose $(x_0, y_0) = (-8, 6)$, $(x_0, F(x_0, y_0)) = (-8, -4) \in E(G)$ and $(y_0, F(y_0, x_0)) = (6, -4) \in E(G^{-1})$. Hence, F satsifies all conditions of Theorem 4.5 and we see that $(0, 0)$ is a coupled fixed point of F.

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References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equationsintegrales, Fund. Math., 3 (1922) 133–181.
[2] B. E. Rhoades, Some theorems on weakly contractive maps. Nonlinear Anal. 47 (2001) 2683–2693.
[3] M. Edelstein, An extension of Banach's contraction principle. Proc. Amer. Math. Soc. 12 (1961) 7–10.
[4] A. C. M. Ran, M. C. B. Reurings, a fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. **132**(5), 1435-1443 (2003)
[5] I. Beg, AR. Butt, Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces, Math. Commun. **15**, 65-75 (2010)
[6] I. Beg, A. Latif, Common fixed point and coincidence point of generalized contractions in ordered metric spaces, Fixed point theory and Applications, (2012)
[7] I. Beg, AR. Butt, Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces, Nonlinear Anal. **71**, 3699-3704 (2009)
[8] T. G. Bhaskar, V. Laskhmikantham, Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal TMA. **65**(7), 1379-1393 (2006)
[9] H. Alikhani, D. Gopal, MA. Miandaragh, Sh. Rezapour, N. Shahzad, Some Endpoint Results for β-Generalized Weak Contractive Multifunctions, The Scientific World Journal 2013 (2013), Article ID 948472, 7 pages. http://dx.doi.org/10.1155/2013/948472
[10] D. Gopal, C. Vetro, M. Abbas, D. K. Patel, Some coincidence and periodic points results in a metric space endowed with a graph and applications. Banach Journal of Mathematical Analysis, 9(3) (2015) 128–139.
[11] S. Nadler, Multi-valued contraction mappings. Pacific J Math. **20**(2), 475-488 (1969)
[12] A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, Nonlinear Anal. TMA 72 (2010) 132-134.
[13] S. Moradi, F. Khojasteh, Endpoints of multi-valued generalized weak contraction mappings, Nonlinear Anal. TMA 74 (2011) 2170-2174.
[14] D. Wardowski, Endpoints and fixed points of a set-valued contractions in cone metric spaces, Nonlinear Anal. TMA (2008) doi:10.1016/j.na.2008.10.089.
[15] N. Hussain, A. Amini-Harandi,Y. J. Cho, Approximate endpoints for set-valued contractions in metric spaces, Fixed Point Theory Appl. (2010)
[16] C. Chifu, G. Petrus, M. Bota, Fixed points and strict fixed points for multivalued contractions of Reich type on metric spaces endowed with a graph. Fixed Point Theory and Applications 2013 2013:203.
[17] J. Jachymski, The contraction principle for mappings on a metric with a graph, Proc. Amer. Math. Soc. **1**(136), 1359-1373 (2008).
[18] J. Harjani, K. Sadarangani, Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Analysis. 72 (2010) 1188–1197.
[19] T. Dinevari, M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, J. Math. Anal. Appl. **405**, 507-517 (2013)
[20] A. Sultana, V. Vetrivel, Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications. J. Math. Anal. Appl. 417 (2014) 336–344.
[21] C. Chifu, G. Petrusel, New results on coupled fixed point theory in metric spaces endowed with a directed graph. Fixed Point Theory Appl. 2014, (2014) Article ID 151
[22] B. Samet, E. Karapinar, H. Aydi, V. Rajic, Discussion on some coupled fixed point theorems. Fixed Point Theory Appl. **2013**, 50 (2013).
[23] V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces. Nonlinear Analysis: Theory, Methods and Applications. 74 **18** (2011) 7347–7355.
[24] V. Berinde, M. Pacurar, The role of the Pompeiu-Hausdorff metric in fixed point theory, Creative Math. inf. 22(2) (2013), 143–150.