Filomat 31:14 (2017), 4331–4340
https://doi.org/10.2298/FIL1714331M

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Caristi-Kirk Type and Boyd&Wong–Browder-Matkowski-Rus Type Fixed Point Results in b-Metric Spaces

Radu Miculescu[a], **Alexandru Mihail**[a]

[a] *University of Bucharest, Faculty of Mathematics and Computer Science, Str. Academiei 14, 010014 Bucharest, Romania*

Abstract. In this paper, based on a lemma giving a sufficient condition for a sequence with elements from a b-metric space to be Cauchy, we obtain Caristi-Kirk type and Boyd&Wong–Browder-Matkowski-Rus type fixed point results in the framework of b-metric spaces. In addition, we extend Theorems 1, 2 and 3 from [M. Bota,V. Ilea, E. Karapinar, O. Mleşniţe, On $\alpha_* - \varphi$-contractive multi-valued operators in b-metric spaces and applications, Applied Mathematics & Information Sciences, 9 (2015), 2611-2620].

1. Introduction

The notion of b-metric space was introduced by I. A. Bakhtin [3] and S. Czerwik [14], [15] in connection with some problems concerning the convergence of measurable functions with respect to measure.

In the last decades a considerable amount of fixed point results in the framework of b-metric spaces were obtained (see, for example, [1], [2], [6], [7], [8], [13], [17], [20], [21], [22], [23], [24], [25] , [27], [28] and the references therein).

In this paper we present a sufficient condition for a sequence with elements from a b-metric space to be Cauchy (see Lemma 2.6). Then, using this result, in Section 3, we obtain Caristi-Kirk type and Boyd&Wong–Browder-Matkowski-Rus type fixed point results in the framework of b-metric spaces (see Theorems 3.1 and 3.3). Section 4 is devoted to the presentation of a class of comparison functions, denoted by Γ^{γ}, satisfying the hypotheses of the above mentioned Boyd&Wong–Browder-Matkowski-Rus type fixed point result. We also point out that Γ^{γ} is larger that Berinde's class Ψ_b. In Section 5, we show how to extend, using Lemma 2.6, the results from [9] concerning α_*-φ-contractive multivalued operators by considering comparison functions φ from Γ^{γ} rather than from Ψ_b.

2. Preliminaries

In this section we recall some basic facts that will be used in the sequel.

Definition 2.1. *Given a nonempty set X and a real number $s \in [1, \infty)$, a function $d : X \times X \to [0, \infty)$ is called b-metric if it satisfies the following properties:*

i) $d(x, y) = 0$ if and only if $x = y$;

2010 *Mathematics Subject Classification*. 54H25, 47H10
Keywords. fixed point theorems, b-metric spaces, comparison functions, α_*-φ-contractive multivalued operators
Received: 01 February 2016; Accepted: 31 August 2016
Communicated by Ljubomir Ćirić
Email addresses: miculesc@yahoo.com (Radu Miculescu), mihail_alex@yahoo.com (Alexandru Mihail)

ii) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
iii) $d(x,y) \leq s(d(x,z) + d(z,y))$ *for all* $x, y, z \in X$.
The pair (X,d) *is called b-metric space (with constant s).*

Besides the classical spaces $l^p(\mathbb{R})$ and $L^p[0,1]$, where $p \in (0,1)$, more examples of b-metric spaces could be found in [2], [4], [8], [14] and [15].

Remark 2.2. *Every metric space is a b-metric space (with constant* 1*). There exist b-metric spaces which are not metric spaces (see, for example, [2], [13] or [21]).*

Definition 2.3. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements from a b-metric space* (X,d) *is called:*
– convergent if there exists $l \in X$ *such that* $\lim_{n\to\infty} d(x_n, l) = 0$;
– Cauchy if $\lim_{m,n\to\infty} d(x_m, x_n) = 0$ *i.e. for every* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that* $d(x_m, x_n) < \varepsilon$ *for all* $m, n \in \mathbb{N}$, $m, n \geq n_\varepsilon$.
The b-metric space (X,d) *is called complete if every Cauchy sequence of elements from* (X,d) *is convergent.*

Remark 2.4. *A b-metric space can be endowed with the topology induced by its convergence.*

Using the method of mathematical induction, one can easily establish the following result:

Lemma 2.5. *For every sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements from a b-metric space* (X,d)*, with constant s, the inequality*

$$d(x_0, x_k) \leq s^n \sum_{i=0}^{k-1} d(x_i, x_{i+1})$$

is valid for every $n \in \mathbb{N}$ *and every* $k \in \{1, 2, 3, \ldots, 2^n - 1, 2^n\}$.

The following lemma is the key ingredient in the proof of Theorems 3.1, 3.3 and 5.4.

Lemma 2.6. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements from a b -metric space* (X,d)*, with constant s, is Cauchy provided that there exists* $\gamma > \log_2 s$ *such that the series* $\sum_{n=1}^{\infty} n^\gamma d(x_n, x_{n+1})$ *is convergent.*

Proof. With the notation $\alpha := \gamma \log_s 2 > 1$, we have $\lim_{n\to\infty} s^{(n+1)(n+2)-\alpha n^2} = 0$ since $\lim_{n\to\infty} (n+1)(n+2) - \alpha n^2 = -\infty$. Therefore the sequence $(s^{(n+1)(n+2)-\alpha n^2})_{n\in\mathbb{N}}$, i.e. $(\frac{s^{(n+1)(n+2)}}{(2^{n^2})^\gamma})_{n\in\mathbb{N}}$, is bounded and since $\sup_{x\in\mathbb{R}} (x+1)(x+2) - \alpha x^2 = 2 + \frac{9}{4(\alpha-1)}$, we have

$$\sup_{n\in\mathbb{N}} \frac{s^{(n+1)(n+2)}}{(2^{n^2})^\gamma} \leq s^{2+\frac{9}{4(\alpha-1)}} := M. \tag{1}$$

We claim that

$$d(x_n, x_{n+m}) \leq M \sum_{i=n}^{\infty} i^\gamma d(x_i, x_{i+1}), \tag{2}$$

for all $m, n \in \mathbb{N}$.

Indeed, with the notation $l = [\sqrt{\log_2(m+1)}]$ (having in mind that $2^{l^2} - 1 \leq m < 2^{(l+1)^2} - 1$), we get

$$d(x_n, x_{n+m}) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_{n+m}) \leq$$

$$\leq sd(x_{n+2^{0^2}-1}, x_{n+2^{(0+1)^2}-1}) + s^2 d(x_{n+2^{1^2}-1}, x_{n+2^{2^2}-1}) + s^2 d(x_{n+2^{2^2}-1}, x_{n+m}) \leq$$

....

$$\leq \sum_{i=0}^{l-1} s^{i+1} d(x_{n+2^{i^2}-1}, x_{n+2^{(i+1)^2}-1}) + s^l d(x_{n+2^{l^2}-1}, x_{n+m}) \overset{\text{Lemma 2.5}}{\leq}$$

$$\leq \sum_{i=0}^{l-1} s^{i+1} s^{(i+1)^2} \left(\sum_{j=2^{i^2}}^{2^{(i+1)^2}-1} d(x_{n+j-1}, x_{n+j}) \right) + s^{l+1} s^{(l+1)^2} \sum_{j=2^{l^2}}^{2^{(l+1)^2}-1} d(x_{n+j-1}, x_{n+j})) =$$

$$= \sum_{i=0}^{l} s^{i+1} s^{(i+1)^2} \left(\sum_{j=2^{i^2}}^{2^{(i+1)^2}-1} d(x_{n+j-1}, x_{n+j}) \right) \leq$$

$$\leq \sum_{i=0}^{\infty} \frac{s^{(i+1)(i+2)}}{(2^{i^2})^\gamma} \left(\sum_{j=2^{i^2}-1}^{2^{(i+1)^2}-1} (j+1)^\gamma d(x_{n+j-1}, x_{n+j}) \right) \overset{(1)}{\leq}$$

$$\leq M \sum_{i=0}^{\infty} (i+1)^\gamma d(x_{n+i}, x_{n+i+1}) \leq M \sum_{i=0}^{\infty} (n+i)^\gamma d(x_{n+i}, x_{n+i+1}) =$$

$$= M \sum_{i=n}^{\infty} i^\gamma d(x_i, x_{i+1}).$$

As the series $\sum\limits_{n=1}^{\infty} n^\gamma d(x_n, x_{n+1})$ is convergent, from (2), we infer that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. □

Using the comparison test, from the above lemma, we obtain the following two results.

Corollary 2.7. *A sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b -metric space (X, d), with constant s, is Cauchy provided that there exist $\gamma > \log_2 s$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that:*

a) the series $\sum\limits_{n=1}^{\infty} a_n d(x_n, x_{n+1})$ is convergent;

b) $\underline{\lim} \frac{a_n}{n^\gamma} > 0$.

Corollary 2.8. *A sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (X, d) is Cauchy provided that there exists $\alpha > 1$ such that the series $\sum\limits_{n=1}^{\infty} \alpha^n d(x_n, x_{n+1})$ is convergent.*

3. Caristi-Kirk Type and Boyd&Wong–Browder-Matkowski-Rus Type Fixed Point Results in b-Metric Spaces

In this section, using Lemma 2.6, we obtain two fixed point theorems in the framework of *b*-metric spaces.

Our first result is a Caristi-Kirk type fixed point result (see [12], [16] and [18]).

Theorem 3.1. *Let (X, d) be a complete b-metric space, $\varphi : X \to [0, \infty)$, $f : X \to X$ and $\alpha > 1$ such that:*

a) f is continuous;

b) $d(x, f(x)) \leq \varphi(x) - \alpha\varphi(f(x))$ for every $x \in X$.

Then, for every $x_0 \in X$, the sequence $(f^{[n]}(x_0))_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of f.

Proof. With the notation $x_n := f^{[n]}(x_0)$, according to b), we have $d(x_n, x_{n+1}) \leq \varphi(x_n) - \alpha\varphi(x_{n+1})$, so

$$\alpha^n d(x_n, x_{n+1}) \leq \alpha^n \varphi(x_n) - \alpha^{n+1}\varphi(x_{n+1}),$$

for every $n \in \mathbb{N}$ and consequently the series $\sum\limits_{n=1}^{\infty} \alpha^n d(x_n, x_{n+1})$ is convergent, its partial sums being between 0 and $\alpha\varphi(x_1)$. According to Corollary 2.8, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and if we denote by u its limit, then passing to limit as $n \to \infty$ in the relation $x_{n+1} = f(x_n)$, based on a), we infer that $f(u) = u$, i.e. u is a fixed point of f. □

Remark 3.2. *The above result gives us a sufficient condition for f to be a weakly Picard operator.*

Our second result is a Boyd&Wong–Brower-Matkowski-Rus type fixed point result (see [10], [11], [19] and [26]).

Theorem 3.3. *Let (X, d) be a complete b-metric space, with constant s, $\gamma > \log_2 s$, $\varphi : [0, \infty) \to [0, \infty)$ and $f : X \to X$ such that:*

a) $\varphi(r) < r$ for every $r > 0$;

b) the series $\sum\limits_{n=1}^{\infty} n^{\gamma}\varphi^{[n]}(r)$ is convergent for every $r > 0$;

c) f is a φ-contraction, i.e. $d(f(x), f(y)) \leq \varphi(d(x, y))$ for all $x, y \in X$.

Then f has a unique fixed point u and the sequence $(f^{[n]}(x_0))_{n\in\mathbb{N}}$ is convergent to u for every $x_0 \in X$.

Proof. With the notation $x_n := f^{[n]}(x_0)$, taking into account c), we have

$$n^{\gamma} d(x_n, x_{n+1}) \leq n^{\gamma}\varphi^{[n]}(d(x_1, x_0)), \tag{3}$$

for every $n \in \mathbb{N}$.

Based on (3), b) and the comparison test, we conclude that the series $\sum\limits_{n=1}^{\infty} n^{\gamma} d(x_n, x_{n+1})$ is convergent and Lemma 2.6 assures us that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent. If we denote by u its limit, then passing to limit as $n \to \infty$ in the relation $x_{n+1} = f(x_n)$, since f is continuous (see c)), we infer that $f(u) = u$, i.e. u is a fixed point of f.

Now let us prove that u is unique.

Indeed, if there exists $v \in X \setminus \{u\}$ having the property that $f(v) = v$, then we get the following contradiction: $d(u, v) = d(f(u), f(v)) \leq \varphi(d(u, v)) \overset{a)}{<} d(u, v)$. □

Remark 3.4. *We have*

$$d(x_n, u) \leq sd(x_n, x_{n+m}) + sd(x_{n+m}, u) \overset{(2)\ \textit{from the proof of Lemma 2.6}}{\leq} sM\sum_{i=n}^{\infty} i^{\gamma} d(x_i, x_{i+1}) + sd(x_{n+m}, u),$$

for all $m, n \in \mathbb{N}$. By passing to limit as $m \to \infty$ in the above inequality we get the following estimation of the speed of convergence for $(f^{[n]}(x_0))_{n\in\mathbb{N}}$:

$$d(x_n, u) \leq sM\sum_{i=n}^{\infty} i^{\gamma} d(x_i, x_{i+1}),$$

for every $n \in \mathbb{N}$.

Remark 3.5. *The above result gives us a sufficient condition for f to be a Picard operator.*

4. Some Classes of Comparison Functions Satisfying Conditions a) and b) from the Hypotheses of Theorem 3.3

In this section we introduce and study the class Γ^{γ} of x^{γ}-summable comparison functions - which is larger that Berinde's class Ψ_b - and whose elements satisfy conditions a) and b) from the hypotheses of Theorem 3.3.

Definition 4.1. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if:*
i) φ is increasing;
ii) $\lim\limits_{n\to\infty} \varphi^{[n]}(r) = 0$ for every $r \in [0, \infty)$.

Remark 4.2. *Every comparison function φ has the property that $\varphi(r) < r$ for every $r \in (0, \infty)$.*

Definition 4.3. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called a x^{γ}-summable comparison function, where $\gamma > 0$, if:*
i) φ is increasing;
ii) the series $\sum\limits_{n=1}^{\infty} n^{\gamma} \varphi^{[n]}(r)$ is convergent for every $r \in [0, \infty)$.
We denote the family of x^{γ}-summable comparison functions by Γ^{γ}.

Remark 4.4. *Every $\varphi \in \Gamma^{\gamma}$, where $\gamma > 0$, is a comparison function, so it satisfies conditions a) and b) from the hypotheses of Theorem 3.3. Hence we are interested in finding sufficient conditions for a comparison function φ to belong to Γ^{γ}.*

Definition 4.5. *Given $\alpha > 1$, we denote by Γ_{α} the family of all comparison functions φ for which there exist $a > 0$ and $\varepsilon > 0$ such that $\varphi(x) \le x - ax^{\alpha}$ for every $x \in [0, \varepsilon]$.*

Lemma 4.6. *Let us consider $\alpha > 1$, $\varepsilon > 0$ and $a > 0$ such that $x - ax^{\alpha} \ge 0$ for every $x \in [0, \varepsilon]$. Then the sequence $(x_n)_{n\in\mathbb{N}}$, given by $x_0 \in [0, \varepsilon]$ and $x_{n+1} = x_n - ax_n^{\alpha}$ for every $n \in \mathbb{N}$, has the following property:*

$$\lim_{n\to\infty} \frac{x_n}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} = (\frac{1}{a(\alpha-1)})^{\frac{1}{\alpha-1}}.$$

Proof. It is clear that $(x_n)_{n\in\mathbb{N}}$ is decreasing and $\lim\limits_{n\to\infty} x_n = 0$, so $(x_n^{1-\alpha})_{n\in\mathbb{N}}$ is increasing and $\lim\limits_{n\to\infty} x_n^{1-\alpha} = \infty$.
As

$$\lim_{n\to\infty} \frac{(n+1)-n}{x_{n+1}^{1-\alpha} - x_n^{1-\alpha}} = \lim_{n\to\infty} \frac{1}{x_n^{1-\alpha}((\frac{x_{n+1}}{x_n})^{1-\alpha} - 1)} =$$

$$= \lim_{n\to\infty} \frac{1}{x_n^{1-\alpha}((1-ax_n^{\alpha-1})^{1-\alpha} - 1)} = \lim_{n\to\infty} \frac{1}{-a\frac{(1-ax_n^{\alpha-1})^{1-\alpha}-1}{-ax_n^{\alpha-1}}} = \frac{1}{a(\alpha-1)},$$

in virtue of Stolz-Cesaro lemma we obtain that $\lim\limits_{n\to\infty} \frac{n}{x_n^{1-\alpha}} = \frac{1}{a(\alpha-1)}$, i.e.
$\lim\limits_{n\to\infty} \frac{x_n}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} = (\frac{1}{a(\alpha-1)})^{\frac{1}{\alpha-1}}$. □

Lemma 4.7. *Given $\alpha > 1$, for every $\varphi \in \Gamma_{\alpha}$ and every $r \ge 0$, we have*

$$\overline{\lim} \frac{\varphi^{[n]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} \in [0, \infty).$$

Proof. Taking into account ii) from the definition of a comparison function, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{[n_0]}(r) \in [0,\varepsilon]$. Since $\varphi \in \Gamma_\alpha$, we infer that $\varphi^{[n+n_0]}(r) \leq x_n$ for every $n \in \mathbb{N}$, where $(x_n)_{n\in\mathbb{N}}$ is given by $x_0 = \varphi^{[n_0]}(r) \in [0,\varepsilon]$ and $x_{n+1} = x_n - ax_n^\alpha$ for every $n \in \mathbb{N}$. Consequently

$$\overline{\lim}\frac{\varphi^{[n+n_0]}(r)}{(\frac{1}{n+n_0})^{\frac{1}{\alpha-1}}} = \overline{\lim}\frac{\varphi^{[n+n_0]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}}\frac{1}{(\frac{n}{n+n_0})^{\frac{1}{\alpha-1}}} \leq$$

$$\leq \lim_{n\to\infty}\frac{x_n}{(\frac{1}{n})^{\frac{1}{\alpha-1}}}(\frac{n+n_0}{n})^{\frac{1}{\alpha-1}} \overset{\text{Lemma 4.6}}{=} (\frac{1}{a(\alpha-1)})^{\frac{1}{\alpha-1}},$$

so $\overline{\lim}\frac{\varphi^{[n]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} \in [0,\infty)$. $\square$

Our next result provides a sufficient condition for the validity of the inclusion $\Gamma_\alpha \subseteq \Gamma^\gamma$.

Proposition 4.8. *For every $\alpha \in (1,2)$ and $\gamma \in (0,\frac{2-\alpha}{\alpha-1})$, we have $\Gamma_\alpha \subseteq \Gamma^\gamma$.*

Proof. It suffices to prove that the series $\sum\limits_{n=1}^{\infty} n^\gamma\varphi^{[n]}(r)$ is convergent for every $r \geq 0$.

In virtue of Lemma 4.7, there exists $n_0 \in \mathbb{N}$ such that $\frac{\varphi^{[n]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} \leq C := \overline{\lim}\frac{\varphi^{[n]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} + 1 \in \mathbb{R}$, i.e. $n^\gamma\varphi^{[n]}(r) = n^\gamma(\frac{1}{n})^{\frac{1}{\alpha-1}}\frac{\varphi^{[n]}(r)}{(\frac{1}{n})^{\frac{1}{\alpha-1}}} \leq C\frac{1}{n^{\frac{1}{\alpha-1}-\gamma}}$ for every $n \in \mathbb{N}$, $n \geq n_0$. Since the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{1}{\alpha-1}-\gamma}}$ is convergent (as $\frac{1}{\alpha-1}-\gamma > 1$), based on the comparison test, we conclude that the series $\sum\limits_{n=1}^{\infty} n^\gamma\varphi^{[n]}(r)$ is convergent. $\square$

Now we provide some other sufficient conditions for a comparison function φ to belong to Γ^γ.

Let us suppose that for the comparison function φ there exist the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that:
i) $a_n \in (0,1)$ and $b_n \in (0,\infty)$ for every $n \in \mathbb{N}$;
ii) $\varphi^{[n+1]}(r) \leq a_n\varphi^{[n]}(r) + b_n$ for every $n \in \mathbb{N}$ and every $r \geq 0$.
Then

$$\varphi^{[n+1]}(r) \leq a_na_{n-1}...a_2a_1r + a_na_{n-1}...a_2b_1 + a_na_{n-1}...a_3b_2 + ... + a_na_{n-1}b_{n-2} + a_nb_{n-1} + b_n,$$

so, with the convention that $b_0 = r$ and $\prod\limits_{j=n+1}^{n} a_j = 1$, we have

$$n^\gamma\varphi^{[n+1]}(r) \leq n^\gamma\sum_{k=0}^{n} b_k\prod_{j=k+1}^{n} a_j,$$

for every $n \in \mathbb{N}$, $r \geq 0$ and $\gamma > 0$.

Consequently, *a sufficient condition for φ to belong to Γ^γ is the convergence of the series $\sum\limits_{n=0}^{\infty}(n^\gamma\sum\limits_{k=0}^{n} b_k\prod\limits_{j=k+1}^{n} a_j)$, i.e. of the series $\sum\limits_{k=0}^{\infty}(b_k\sum\limits_{n=k}^{\infty} n^\gamma\prod\limits_{j=k+1}^{n} a_j)$.*

Now we are going to take a closer look on this sufficient condition in two particular cases.

The first particular case is the one for which the sequence $(a_n)_{n\in\mathbb{N}}$ is constant (so there exists $a \in (0,1)$ such that $a_n = a$ for every $n \in \mathbb{N}$).

We claim that, in this case, the series $\sum_{k=0}^{\infty}(b_k\sum_{n=k}^{\infty}n^{\gamma}a^{n-k})$ is convergent if and only if the series $\sum_{k=0}^{\infty}k^{\gamma}b_k$ is convergent.

The implication "$\Rightarrow$" is obvious as $k^{\gamma}\leq\sum_{n=k}^{\infty}n^{\gamma}a^{n-k}$ for every $k\in\mathbb{N}$.

For the implication "$\Leftarrow$" let us note that $\sum_{k=0}^{\infty}(b_k\sum_{n=k}^{\infty}n^{\gamma}a^{n-k})$ can be rewritten as $\sum_{k=0}^{\infty}(b_k\sum_{j=0}^{\infty}(k+j)^{\gamma}a^{j})$ and that there exists $C_{\gamma}\in\mathbb{R}$ such that $(k+j)^{\gamma}\leq C_{\gamma}(k^{\gamma}+j^{\gamma})$ for every $j,k\in\mathbb{N}$. As the series $\sum_{k=0}^{\infty}k^{\gamma}b_k$ is convergent, in virtue of the comparison test, we infer that the series $\sum_{k=0}^{\infty}(b_k\sum_{j=0}^{\infty}C_{\gamma}k^{\gamma}a^{j})$ and $\sum_{k=0}^{\infty}(b_k\sum_{j=0}^{\infty}C_{\gamma}j^{\gamma}a^{j})$ are convergent (as, with the notation $C_{a,\gamma}:=\sum_{j=0}^{\infty}j^{\gamma}a^{j}$, we have $b_k\sum_{j=0}^{\infty}C_{\gamma}k^{\gamma}a^{j}\leq b_kk^{\gamma}\frac{C_{\gamma}}{1-a}$ and $b_k\sum_{j=0}^{\infty}C_{\gamma}j^{\gamma}a^{j}\leq b_kk^{\gamma}C_{\gamma}C_{a,\gamma}$ for every $k\in\mathbb{N}$). Consequently $\sum_{k=0}^{\infty}(b_k\sum_{j=0}^{\infty}C_{\gamma}(k^{\gamma}+j^{\gamma})a^{j})$ is convergent, and, based on the comparison test, we conclude that $\sum_{k=0}^{\infty}(b_k\sum_{j=0}^{\infty}(k+j)^{\gamma}a^{j})$, i.e. $\sum_{k=0}^{\infty}(b_k\sum_{n=k}^{\infty}n^{\gamma}a^{n-k})$, is convergent. So we proved the following:

Proposition 4.9. *A comparison function φ for which there exist $a\in(0,1)$ and $b_n\in(0,\infty)$, $n\in\mathbb{N}$, such that $\varphi^{[n+1]}(r)\leq a\varphi^{[n]}(r)+b_n$ for every $n\in\mathbb{N}$ and every $r\geq 0$, belongs to Γ^{γ}, where $\gamma>0$, provided that the series $\sum_{n=0}^{\infty}n^{\gamma}b_n$ is convergent.*

The second particular case is the one for which there exists an increasing sequence $(c_n)_{n\in\mathbb{N}}$ converging to ∞ such that $a_n=\frac{c_{n-1}}{c_n}$ for every $n\in\mathbb{N}$.

A sufficient condition for the convergence of the series $\sum_{k=0}^{\infty}(b_k\sum_{n=k}^{\infty}n^{\gamma}\frac{c_k}{c_n})$, i.e. $\sum_{k=0}^{\infty}(b_kc_k\sum_{n=k}^{\infty}\frac{n^{\gamma}}{c_n})$, is the convergence of the series $\sum_{n=0}^{\infty}\frac{n^{\gamma}}{c_n}$ and $\sum_{k=0}^{\infty}b_kc_k$. This happens, for example, if $c_n=n^{\varepsilon+1+\gamma}$ for every $n\in\mathbb{N}$, where $\varepsilon>0$ and the series $\sum_{k=0}^{\infty}b_kk^{\varepsilon+1+\gamma}$ is convergent. So we proved the following:

Proposition 4.10. *A comparison function φ for which there exist $\varepsilon>0$ and $b_n\in(0,\infty)$, $n\in\mathbb{N}$, such that $\varphi^{[n+1]}(r)\leq(\frac{n-1}{n})^{\varepsilon+1+\gamma}\varphi^{[n]}(r)+b_n$ for every $n\in\mathbb{N}$ and every $r\geq 0$, belongs to Γ^{γ}, where $\gamma>0$, provided that the series $\sum_{n=0}^{\infty}b_nn^{\varepsilon+1+\gamma}$ is convergent.*

Let us recall (see [5]) the following:

Definition 4.11. *For a given $b>1$, by Ψ_b we understand the class of functions $\varphi:[0,\infty)\to[0,\infty)$ such that:*

i) φ is increasing;

ii) there exist $a\in(0,1)$ and a convergent series $\sum_{n=0}^{\infty}b_n$, where $b_n\in(0,\infty)$ for every $n\in\mathbb{N}$, such that $b^{n+1}\varphi^{[n+1]}(r)\leq ab^{n}\varphi^{[n]}(r)+b_n$ for every $n\in\mathbb{N}$ and every $r\geq 0$.

Proposition 4.12. $\Psi_b\subseteq\Gamma^{\gamma}$ *for every $\gamma>1$.*

Proof. If $\varphi\in\Psi_b$, then, with the notation $x_n=\frac{b_n}{b^{n+1}}$ for every $n\in\mathbb{N}$, we have $\varphi^{[n+1]}(r)\leq\frac{a}{b}\varphi^{[n]}(r)+x_n$ for every $n\in\mathbb{N}$ and the series $\sum_{n=0}^{\infty}n^{\gamma}x_n$, i.e. $\sum_{n=0}^{\infty}\frac{n^{\gamma}}{b^{n+1}}b_n$, is convergent for every $\gamma>0$. Then, according to Proposition 4.9, we conclude that $\varphi\in\Gamma^{\gamma}$. □

An example. We present an example of comparison function φ which belongs to Γ^{γ} for every $\gamma \in (0,2)$, but not to Ψ_b, for every $b > 1$.

Let us consider the comparison function $\varphi : [0,\infty) \to [0,\infty)$, given by

$$\varphi(x) = \begin{cases} x - x^{\frac{4}{3}}, & x \in \left[0, \dfrac{27}{64}\right] \\ \dfrac{27}{256}, & x \in \left(\dfrac{27}{64}, \infty\right) \end{cases}.$$

Claim 4.13. *$\varphi \in \Gamma^{\gamma}$ for every $\gamma \in (0,2)$.*

Justification of claim 4.13. Note that $\varphi \in \Gamma_{\frac{4}{3}}$ (just take $a = 1$ and $\varepsilon = \frac{27}{64}$ in Definition 4.5), so, according to Proposition 4.8, $\varphi \in \Gamma^{\gamma}$ for every $\gamma \in (0,2)$.

Claim 4.14. *There is no $b > 1$ such that $\varphi \in \Psi_b$.*

Justification of claim 4.14. If this is not the case, there exist $a \in (0,1)$ and a convergent series $\sum\limits_{n=0}^{\infty} b_n$, where $b_n \in (0,\infty)$ for every $n \in \mathbb{N}$, such that $b^{n+1}\varphi^{[n+1]}(r) \leq ab^n\varphi^{[n]}(r) + b_n$ for every $n \in \mathbb{N}$ and every $r \geq 0$. Hence, for a fixed $r_0 \in (0, \frac{27}{64})$, we have $b^{n+1}\varphi^{[n+1]}(r_0) \leq ab^n\varphi^{[n]}(r_0) + b_n$, and therefore $\frac{b^{n+1}}{n^3}[(\frac{n}{n+1})^3(n+1)^3\varphi^{[n+1]}(r_0) - \frac{a}{b}n^3\varphi^{[n]}(r_0)] \leq b_n$ for every $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} n^3\varphi^{[n]}(r_0) = 27$ (see Lemma 4.6), by passing to limit as $n \to \infty$ in the previous inequality, we get that $\lim\limits_{n\to\infty} b_n = \infty$. This is a contradiction since the series $\sum\limits_{n=0}^{\infty} b_n$ is convergent.

5. A Fixed Point Theorem for $\alpha_* - \varphi$-Contractive Multivalued Operators in b-Metric Spaces

In this section, inspired by the ideas from [9], we present a fixed point theorem for α_*-φ-contractive multivalued operator in b-metric spaces.

First of all let us recall some notions.

Definition 5.1. *For $T : X \to \mathcal{P}(X) := \{Y \mid Y \subseteq X\}$ and $\alpha : X \times X \to [0,\infty)$, where (X,d) is a b-metric space, we say that T is α_*-admissible if, for all $x, y \in X$, the following implication is valid: $\alpha(x,y) \geq 1 \Rightarrow \alpha_*(T(x),T(y)) := \inf\{\alpha(u,v) \mid u \in T(x), v \in T(y)\} \geq 1$.*

Definition 5.2. *For $T : X \to \mathcal{P}_{cl}(X) := \{Y \in \mathcal{P}(X) \mid Y \text{ is closed}\}$, $\alpha : X \times X \to [0,\infty)$ and $\varphi \in \Gamma^{\gamma}$, where (X,d) is a b-metric space and $\gamma > 1$, we say that T is an $\alpha_* - \varphi$-contractive multivalued operator of type (b) if $\alpha_*(T(x),T(y))h(T(x),T(y)) \leq \varphi(d(x,y))$ for all $x, y \in X$, where h stands for the Hausdorff-Pompeiu metric.*

Definition 5.3. *For $T : X \to \mathcal{P}_{cl}(X)$, where (X,d) is a b-metric space, we say that T is a multivalued weakly Picard operator if for every $x \in X$ and every $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X such that:*

a) $x_0 = x$ and $x_1 = y$;

b) $x_{n+1} \in T(x_n)$ for every $n \in \mathbb{N}$;

c) $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Now we can state our result.

Theorem 5.4. *Let us consider $T : X \to \mathcal{P}_{cl}(X)$, $\alpha : X \times X \to [0, \infty)$ and $\varphi \in \Gamma^{\gamma}$, where (X, d) is a complete b-metric space with constant $s > 1$ and $\gamma > \log_2 s$, such that:*

a) T is an $\alpha_* - \varphi$-contractive multivalued operator of type (b);

b) T is α_*-admissible;

c) there exists $x_0 \in X$ and $x_1 \in T(x_0)$ satisfying the inequality $\alpha(x_0, x_1) \geq 1$;

d) for every convergent sequence $(y_n)_{n\in\mathbb{N}}$ of elements from X having limit y, the following implication is valid: $(\alpha(y_n, y_{n+1}) \geq 1$ for every $n \in \mathbb{N}) \Rightarrow (\alpha(y_n, y) \geq 1$ for every $n \in \mathbb{N})$.

Then T has a fixed point.

Proof. The same line of arguments given in the proof of Theorem 1 from [9] gives us a sequence $(x_n)_{n\in\mathbb{N}}$ of elements from X, with $x_0 \neq x_1$, such that:

i) $x_{n+1} \in T(x_n)$ for every $n \in \mathbb{N}$;

ii) $\alpha(x_n, x_{n+1}) \geq 1$ for every $n \in \mathbb{N}$;

iii) $d(x_n, x_{n+1}) \leq \varphi^{[n]}(d(x_0, x_1))$ for every $n \in \mathbb{N}$.

As $\varphi \in \Gamma^{\gamma}$, the series $\sum_{n=1}^{\infty} n^{\gamma} \varphi^{[n]}(d(x_0, x_1))$ is convergent, so, taking into account the comparison test and iii), we came to the conclusion that $\sum_{n=1}^{\infty} n^{\gamma} d(x_n, x_{n+1})$ is convergent. Consequently, according to Lemma 2.6, $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

The same arguments as the ones used in the proof of Theorem 1 from [9] assure us that the limit of $(x_n)_{n\in\mathbb{N}}$ is a fixed point of T. □

Remark 5.5. *If hypothesis c) is replace by the following condition: $\alpha(x_0, x_1) \geq 1$ for every $x_0 \in X$ and every $x_1 \in T(x_0)$, then the conclusion of the above result is that T is a multivalued weakly Picard operator.*

Remark 5.6. *Using the same technique, one can generalize Theorems 2 and 3 from [9].*

References

[1] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered *b*-metric spaces, Math. Slovaca, **64** (2014), 941-960.

[2] H. Aydi, M.F. Bota, E. Karapinar and S. Mitrović, A fixed point theorem for set-valued quasi-contractions in *b*-metric spaces, Fixed Point Theory Appl., 2012, 2012:88.

[3] I. A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal., Unianowsk Gos. Ped. Inst., **30** (1989), 26-37.

[4] V. Berinde, Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory, 1993, 3-9.

[5] V. Berinde, Sequences of operators and fixed points in quasimetric spaces, Stud. Univ. "Babeş-Bolyai", Math., **16** (1996), 23-27.

[6] M. Boriceanu, M. Bota and A. Petruşel, Multivalued fractals in *b*-metric spaces, Cent. Eur. J. Math., **8** (2010), 367-377.

[7] M. Boriceanu, A. Petruşel and A.I. Rus, Fixed point theorems for some multivalued generalized contraction in *b*-metric spaces, Int. J. Math. Stat., **6** (2010), 65-76.

[8] M. Bota, A. Molnár and C. Varga, On Ekeland's variational principle in *b*-metric spaces, Fixed Point Theory, **12** (2011), 21-28.

[9] M. Bota,V. Ilea, E. Karapinar, O. Mleşniţe, On $\alpha_* - \varphi$-contractive multi-valued operators in *b*-metric spaces and applications, Applied Mathematics & Information Sciences, **9** (2015), 2611-2620.

[10] D.W. Boyd and J.S. Wong, On nonlinear contractions, Proc. Amer Math. Soc., **20** (1969), 458-464.

[11] F.E. Browder, On the convergence of succesive approximations for nonlinear functional equations, Nederl. Akad. Wet., Proc., Ser. A 71, **30** (1968), 27-35.

[12] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Tran. Amer. Math. Soc., **215** (1976), 241-251.

[13] C. Chifu and G. Petruşel, Fixed points for multivalued contractions in *b*-metric spaces with applications to fractals, Taiwanese J. Math., **18** (2014), 1365-1375.

[14] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostraviensis, **1** (1993), 5-11.

[15] S. Czerwik, Nonlinear set-valued contraction mappings in *b*-metric spaces, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263-276.

[16] J. Jachymski, Caristi's fixed point theorem and selections of set-valued contractions, J. Math. Anal. Appl., **227** (1998), 55-67.

[17] M.A. Khamsi and N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal., **73** (2010), 3123-3129.

[18] W. A. Kirk, Caristi's fixed point theorem and metric convexity. Colloq. Math., **36** (1976), 81-86.

[19] J. Matkowski, Integrable solutions of functional equations, Dissertations Math. (Rozprawy), 127 (1976).

[20] S. K. Mohanta, Some fixed point theorems using *wt*-distance in *b*-metric spaces, Fasc. Math., **54** (2015), 125-140.

[21] H.N. Nashine and Z. Kadelburg, Cyclic generalized φ-contractions in *b*-metric spaces and an application to integral equations, Filomat, **28** (2014), 2047-2057.

[22] M.O. Olatinwo, A fixed point theorem for multi-valued weakly Picard operators in b-metric spaces, Demonstratio Math., **42** (2009), 599-606.
[23] M. Păcurar, Sequences of almost contractions and fixed points in b-metric spaces, An. Univ. Vest Timiş. Ser. Mat.-Inform., **48** (2010), 125-137.
[24] J.R. Roshan, N. Hussain, S. Sedghi and N. Shobkolaei, Suzuki-type fixed point results in b-metric spaces, Math. Sci. (Springer), **9** (2015), 153–160.
[25] B.D. Rouhani and S. Moradi, Common fixed point of multivalued generalized φ-weak contractive mappings, Fixed Point Theory Appl., 2010, Article ID 708984.
[26] I. A. Rus, Generalized φ-contractions, Rev. Anal. Numér. Théor. Approximation, Math., **47** (1982), 175-178.
[27] S. Shukla, Partial b-metric spaces and fixed point theorems, Mediterr. J. Math., **11** (2014), 703-711.
[28] H. Yingtaweesittikul, Suzuki type fixed point for generalized multi-valued mappings in b-metric spaces, Fixed Point Theory and Applications, 2013, 2013:215.