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Complete Moment Convergence for Weighted Sums of Extended Negatively Dependent Random Variables

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Abstract. In this paper, the complete moment convergence for the weighted sums of extended negatively dependent (END, in short) random variables is investigated. Some general conditions to prove the complete moment convergence are provided. The results obtained in the paper generalize and improve the corresponding ones for some dependent sequences.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins [8] as follows: A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

Hsu and Robbins [8] proved that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summands is finite. Erdös [5] proved the converse. The result of Hsu-Robbins-Erdös is a fundamental theorem in probability theory and has been generalized and extended in several directions by many authors. One can refer to Spitzer [17], Baum and Katz [2], Gut [7], Qiu et al. [12], Sung [20], and so forth.

Chow [4] introduced a more general concept of the complete convergence. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{{b_n}^{-1}|Zn| - \varepsilon\}_+^q < \infty \text{ for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence.

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Recently Wang et al. [21] obtained the following result on complete convergence for weighted sums of extended negatively dependent (END, in short) random variables.

Theorem 1.1. *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed END random variables with $EX_1 = 0$ and $E|X_1|^p < \infty$ for some $p > 1/\alpha$ and $1/2 < \alpha \leq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^q = O(n) \quad \text{for some } q > p\,. \tag{1}$$

Then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0. \tag{2}$$

Noting that complete moment convergence is more general than complete convergence, we aim to generalize the complete convergence for weighted sums of END random variables with identical distribution to the case of complete moment convergence with non-identical distribution. Now let us recall the concept of END random variables.

Definition 1.2. *Random variables $Y_1, Y_2, ...$ are said to be extended negatively dependent (END, in short) if there exists a constant $M > 0$ such that, for each $n \geq 2$,*

$$P(Y_1 \leq y_1, ..., Y_n \leq y_n) \leq M \prod_{i=1}^{n} P(Y_i \leq y_i)$$

and

$$P(Y_1 > y_1, ..., Y_n > y_n) \leq M \prod_{i=1}^{n} P(Y_i > y_i)$$

hold for every sequence $\{y_1, ..., y_n\}$ of real numbers.

The concept of END random variables was introduced by Liu [10]. When $M = 1$, the notion of END random variables reduces to the well-known notion of negatively orthant dependent (NOD, in short) random variables, which was firstly introduced by Ghosh [6]; some properties and limit results can be found in Alam and Saxena [1], Block et al. [3], Joag-Dev and Proschan [9], Wu and Zhu [23], Shen [15], and so on. As is mentioned in Liu [10], the END structure is substantially more comprehensive than the NOD structure in that it can reflect not only a negative dependence structure but also a positive one, to some extent. Liu [10] pointed out that the END random variables can be taken as negatively or positively dependent and provided some interesting examples to support this idea. Joag-Dev and Proschan [9] also pointed out that negatively associated (NA, in short) random variables must be NOD and NOD is not necessarily NA, thus NA random variables are END. A great number of articles for NA random variables have appeared in the literature. But very few papers are written for END random variables. For example, for END random variables with heavy tails Liu [10] obtained the precise large deviations and Liu [11] studied sufficient and necessary conditions for moderate deviations, Shen [14] established some probability inequalities and moment inequalities for END random variables, Qiu et al. [13] and Wu and Guan [22] studied complete convergence for weighted sums and arrays of rowwise END random variables, and so on.

The following concept of stochastic domination will play an important role throughout the paper.

Definition 1.3. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that*

$$P(|X_n| > x) \leq CP(|X| > x) \tag{3}$$

for all $x \geq 0$ and $n \geq 1$.

Our main results are as follows.

Theorem 1.4. *Let $p > 1/\alpha$ and $1/2 < \alpha \le 1$. Let $\{X_n, n \ge 1\}$ be a sequence of mean zero END random variables, which is stochastically dominated by a random variable X with $E|X|^p < \infty$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying (1) for some $q > \max\{p, \frac{\alpha p-1}{\alpha-1/2}\}$. Then*

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}^{+} < \infty, \quad \forall \varepsilon > 0. \tag{4}$$

If further assume that $E|X|^p \log^q(1+|X|) < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}^{+} < \infty, \quad \forall \varepsilon > 0. \tag{5}$$

Remark 1.5. *Comparing Theorem 1.1 with Theorem 1.4, we have the following generalizations:*
(i) Complete convergence in Theorem 1.1 is extended to complete moment convergence in Theorem 1.4;
(ii) The condition of identical distribution in Theorem 1.1 is weakened by stochastic domination in Theorem 1.4.

Remark 1.6. *Theorem 1.4 only discusses the case $p > \frac{1}{\alpha} \ge 1$. Actually, the case $p = 1$ is also very interesting and it is still an open problem whether (2) holds for the partial sums of an END sequence when $p = 1$. However, if we add some strong condition on moment, we can get the following result.*

Theorem 1.7. *Let $\{X_n, n \ge 1\}$ be a sequence of mean zero END random variables, which is stochastically dominated by a random variable X with $E|X| \log(1+|X|) < \infty$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying (1) for some $q > 1$. Then*

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n\right\}^{+} < \infty, \quad \forall \varepsilon > 0. \tag{6}$$

If further assume that $E|X| \log^3(1+|X|) < \infty$, then

$$\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n\right\}^{+} < \infty, \quad \forall \varepsilon > 0. \tag{7}$$

Throughout the paper, let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Let $I(A)$ be the indicator function of the set A. The symbol C denotes generic positive constants, whose value may vary from one place to another. Denote $\log x = \ln \max\{x, e\}$.

2. Proofs of the Main Results

To prove the main results of the paper, we need the following lemmas. The first one was obtained by Liu [10].

Lemma 2.1. *Let $X_1, X_2, ..., X_n$ be END random variables. Assume that $f_1, f_2, ..., f_n$ are Borel functions all of which are monotone increasing (or all are monotone decreasing). Then $f_1(X_1), f_2(X_2), ..., f_n(X_n)$ are END random variables.*

The second one is a basic property for stochastic domination. For the proof, one can refer to Shen et al. [16].

Lemma 2.2. *Assume that $\{X_n, n \geq 1\}$ is an array of random variables stochastically dominated by a random variable X. Then for all $\alpha > 0$ and $b > 0$, there exists positive constants C_1 and C_2 such that*

$$E|X_n|^\alpha I(|X_n \leq b|) \leq C_1 \left[E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\right], \tag{8}$$

and

$$E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b). \tag{9}$$

Consequently, $E|X_n|^\alpha \leq CE|X|^\alpha$.

The third one is very important to prove complete moment convergence. For the proof, one can refer to Sung [19].

Lemma 2.3. *Let $\{Y_i, 1 \leq i \leq n\}$ and $\{Z_i, 1 \leq i \leq n\}$ be sequences of random variables. Then for any $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$E\left(\left|\sum_{i=1}^{n}(Y_i + Z_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}} E\left|\sum_{i=1}^{n} Y_i\right|^q + E\left|\sum_{i=1}^{n} Z_i\right|$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(Y_i + Z_i)\right| - \varepsilon a\right)^+ \leq \left(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\right)\frac{1}{a^{q-1}} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Y_i\right|^q\right) + E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} Z_i\right|\right).$$

The next one was obtained by Shen [14].

Lemma 2.4. *For any $r > 1$, if $\{X_n, n \geq 1\}$ is a sequence of END random variables with $EX_n = 0$ for every $n \geq 1$, then there exists is a positive constant C_r depending only on r such that for all $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq C_r \sum_{i=1}^{n} E|X_i|^r$$

holds when $1 < r < 2$ and

$$E\left|\sum_{i=1}^{n} X_i\right|^r \leq C_r \left\{\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{r/2}\right\}$$

holds when $r \geq 2$.

By Lemma 2.4 and the same argument as that of Theorem 2.3.1 in Stout [18], the following lemma holds.

Lemma 2.5. *For any $r > 1$, if $\{X_n, n \geq 1\}$ is a sequence of END random variables with $EX_n = 0$ for every $n \geq 1$, then there exists is a positive constant C_r depending only on r such that for all $n \geq 1$,*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^r\right) \leq C_r (\log n)^r \sum_{i=1}^{n} E|X_i|^r$$

holds when $1 < r < 2$ and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^r\right) \leq C_r (\log n)^r \left\{\sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} E|X_i|^2\right)^{r/2}\right\}$$

holds when $r \geq 2$.

Based on the above lemmas, we can present the proofs for the main results.

Proof. *(Proof of Theorem 1.4)* First, we will give the proof of (4). Without loss of generality, we assume $a_{ni} \geq 0$ for all $1 \leq i \leq n, n \geq 1$. Define

$$Y_i = -n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \leq n^\alpha) + n^\alpha I(X_i > n^\alpha)$$

and

$$Y_i' = (X_i + n^\alpha) I(X_i < -n^\alpha) + (X_i - n^\alpha) I(X_i > n^\alpha).$$

By Lemma 2.1, it is easy to see that $\{Y_i, 1 \leq i \leq n\}$ and $\{a_{ni}Y_i, 1 \leq i \leq n\}$ are END for each $n \geq 1$. Applying Lemma 2.3 with $a = n^\alpha$, we can obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^\alpha\right\}^+ \\
\leq\ & \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\left\{Cn^{-\alpha(q-1)}E\left|\sum_{i=1}^{n} a_{ni}(Y_i - EY_i)\right|^q + E\left|\sum_{i=1}^{n} a_{ni}(Y_i{}' - EY_i')\right|\right\} \qquad (10)
\end{aligned}$$

We consider the following two cases.

(i) If $q > 2$, by (10), it is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\left|\sum_{i=1}^{n} a_{ni}X_i\right| - \varepsilon n^\alpha\right\}^+ \\
\leq\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n}|a_{ni}|^q E|Y_i - EY_i|^q + \left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2}\right\} + \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{i=1}^{n}|a_{ni}|E|Y_i'| \\
\leq\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n}|a_{ni}|^q E|Y_i|^q + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2} + \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{i=1}^{n}|a_{ni}|E|Y_i'| \\
\doteq\ & I_1 + I_2 + I_3. \qquad (11)
\end{aligned}$$

For I_1, we have by Lemma 2.2, (1) and $E|X|^p < \infty$ that

$$\begin{aligned}
I_1 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n}|a_{ni}|^q E\left|-n^\alpha I(X_i < -n^\alpha) + X_i I(|X_i| \leq n^\alpha) + n^\alpha I(X_i > n^\alpha)\right|^q \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n}|a_{ni}|^q \left(E|X_i|^q I(|X_i| \leq n^\alpha) + n^{\alpha q}P(|X_i| > n^\alpha)\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n}|a_{ni}|^q \left(E|X|^q I(|X| \leq n^\alpha) + n^{\alpha q}P(|X| > n^\alpha)\right) \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}E|X|^q I(|X| \leq n^\alpha) + C\sum_{n=1}^{\infty} n^{\alpha p-1}P(|X| > n^\alpha) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{m=1}^{n}E|X|^q I((m-1)^\alpha < |X| \leq m^\alpha) + C\sum_{n=1}^{\infty} n^{\alpha p-1}\sum_{m=n}^{\infty}P(m^\alpha < |X| \leq (m+1)^\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}E|X|^q I(0<|X|\le 1) + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\sum_{m=2}^{n} E|X|^q I((m-1)^\alpha<|X|\le m^\alpha) \\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-1}\sum_{m=n}^{\infty} P(m^\alpha<|X|\le (m+1)^\alpha) \\
&\doteq I_{11}+I_{12}+I_{13}.
\end{aligned}
$$

It follows by $q>p$ that $\alpha p-\alpha q-1<-1$, hence $I_{11}<\infty$, and

$$
\begin{aligned}
I_{12}+I_{13} &\le C\sum_{m=2}^{\infty} E|X|^q I((m-1)^\alpha<|X|\le m^\alpha)\sum_{n=m}^{\infty} n^{\alpha p-\alpha q-1} + C\sum_{m=1}^{\infty} P(m^\alpha<|X|\le (m+1)^\alpha)\sum_{n=1}^{m} n^{\alpha p-1} \\
&\le C\sum_{m=2}^{\infty} m^{\alpha p-\alpha q}E|X|^q I((m-1)^\alpha<|X|\le m^\alpha) + C\sum_{m=1}^{\infty} m^{\alpha p} EI(m^\alpha<|X|\le (m+1)^\alpha) \\
&\le CE|X|^p<\infty. \qquad (12)
\end{aligned}
$$

For the second part of (11), if $p\ge 2$, we take $q>\frac{\alpha p-1}{\alpha-1/2}$ that

$$
\begin{aligned}
I_2 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n}|a_{ni}|^2 (E|X_i|^2 I(|X_i|\le n^\alpha)+n^{2\alpha}P(|X_i|>n^\alpha))\right\}^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n}|a_{ni}|^2 (E|X|^2 I(|X|\le n^\alpha)+E|X|^2 I(|X|>n^\alpha))\right\}^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+q/2}<\infty. \qquad (13)
\end{aligned}
$$

If $1<p<2$, it follows by $q>2$ that $\alpha p-2+q/2-\frac{\alpha pq}{2}<-1$, hence

$$
\begin{aligned}
I_2 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n}|a_{ni}|^2 (E|X_i|^2 I(|X_i|\le n^\alpha)+n^{2\alpha}P(|X_i|>n^\alpha))\right\}^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left\{\sum_{i=1}^{n}|a_{ni}|^2 n^{\alpha(2-p)}(E|X|^p I(|X|\le n^\alpha)+E|X|^p I(|X|>n^\alpha))\right\}^{q/2} \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\left(\sum_{i=1}^{n}|a_{ni}|^2 n^{\alpha(2-p)}E|X|^p\right)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2+q/2-\frac{\alpha pq}{2}}<\infty. \qquad (14)
\end{aligned}
$$

By C_r inequality, we have

$$
\sum_{i=1}^{n}|a_{ni}| = \left[\left(\sum_{i=1}^{n}|a_{ni}|\right)^q\right]^{1/q} \le \left(n^{q-1}\sum_{i=1}^{n}|a_{ni}|^q\right)^{1/q} \le Cn.
$$

For the third part of (11), by the definition of Y_i', we have

$$\begin{aligned}
I_3 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{i=1}^{n}|a_{ni}|E|Y_i'| \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>n^{\alpha}) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}E|X|I(|X|>n^{\alpha}) \\
&= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\sum_{m=n}^{\infty}E|X|I(m^{\alpha}<|X|\le(m+1)^{\alpha}) \\
&= C\sum_{m=1}^{\infty} E|X|I(m^{\alpha}<|X|\le(m+1)^{\alpha})\sum_{n=1}^{m}n^{\alpha p-\alpha-1} \\
&\le C\sum_{m=1}^{\infty} m^{\alpha p-\alpha}E|X|I(m^{\alpha}<|X|\le(m+1)^{\alpha}) \\
&\le CE|X|^p<\infty. \qquad (15)
\end{aligned}$$

(ii) If $1<q\le 2$, by (10), we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\left\{\left|\sum_{i=1}^{n}a_{ni}X_i\right|-\varepsilon n^{\alpha}\right\}^+ \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\sum_{i=1}^{n}|a_{ni}|^q\left(E|X_i|^qI(|X_i|\le n^{\alpha})+n^{\alpha q}P(|X_i|>n^{\alpha})\right)+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>n^{\alpha}) \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}E|X_i|^qI(|X_i|\le n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha p-1}P(|X_i|>n^{\alpha})+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}E|X_i|I(|X_i|>n^{\alpha}) \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}E|X|^qI(|X|\le n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha p-1}P(|X|>n^{\alpha})+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}E|X|I(|X|>n^{\alpha}) \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}E|X|^qI(|X|\le n^{\alpha})+C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}E|X|I(|X|>n^{\alpha}) \\
\doteq\ & I_4+I_5. \qquad (16)
\end{aligned}$$

The rest of the proof is similar to those of I_1 and I_3, so we omit the details.

According to (11)-(16), (4) holds. Next, we present the proof of (5). Similar to the proof of (4) and using the second inequality of Lemma 2.3, we also get that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|-\varepsilon n^{\alpha}\right\}^+ \\
\le\ & \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\left\{Cn^{-\alpha(q-1)}E\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}(Y_i-EY_i)\right|^q+E\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}(Y_i'-EY_i')\right|\right\} \\
=\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}E\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}(Y_i-EY_i)\right|^q+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\max_{1\le k\le n}\left|\sum_{i=1}^{k}a_{ni}(Y_i'-EY_i')\right|.
\end{aligned}$$

We consider the following two cases.

Case 1: if $q > 2$, then we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} E\left\{\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| - \varepsilon n^{\alpha}\right\}^{+} \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left\{\sum_{i=1}^{n}|a_{ni}|^q E|Y_i|^q + \left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2}\right\} \\
&+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\log n\sum_{i=1}^{n}|a_{ni}|E|Y_i'| \\
=\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\sum_{i=1}^{n}|a_{ni}|^q E|Y_i|^q + C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2} \\
&+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\log n\sum_{i=1}^{n}|a_{ni}|E|Y_i'| \\
\doteq\ & I_6 + I_7 + I_8.
\end{aligned}
\tag{17}
$$

For I_6, we have by Lemma 2.2, (1) and $E|X|^p\log^q(1+|X|) < \infty$ that

$$
\begin{aligned}
I_6 \ \le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\sum_{i=1}^{n}|a_{ni}|^q\left(E|X_i|^q I(|X_i|\le n^{\alpha}) + n^{\alpha q}P(|X_i|>n^{\alpha})\right) \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\log^q nE|X|^q I(|X|\le n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-1}\log^q nP(|X|>n^{\alpha}) \\
=\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-1}\log^q n\sum_{m=1}^{n} E|X|^q I((m-1)^{\alpha}<|X|\le m^{\alpha}) \\
&+C\sum_{n=1}^{\infty} n^{\alpha p-1}\log^q n\sum_{m=n}^{\infty} P(m^{\alpha}<|X|\le (m+1)^{\alpha}) \\
\le\ & C\sum_{m=1}^{\infty} m^{\alpha p-\alpha q}\log^q mE|X|^q I((m-1)^{\alpha}<|X|\le m^{\alpha}) \\
&+C\sum_{m=1}^{\infty} m^{\alpha p}\log^q mEI(m^{\alpha}<|X|\le (m+1)^{\alpha}) \\
\le\ & CE|X|^p\log^q(1+|X|)\ <\ \infty.
\end{aligned}
\tag{18}
$$

For the second part of (17), if $p \ge 2$, we have by $q > \frac{\alpha p-1}{\alpha-1/2}$ and Lemma 2.2 that

$$
\begin{aligned}
I_7 \ =\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2 E|Y_i|^2\right)^{q/2} \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left\{\sum_{i=1}^{n}|a_{ni}|^2\left(E|X_i|^2 I(|X_i|\le n^{\alpha}) + n^{2\alpha}P(|X_i|>n^{\alpha})\right)\right\}^{q/2} \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left\{\sum_{i=1}^{n}|a_{ni}|^2\left(E|X|^2 I(|X|\le n^{\alpha}) + E|X|^2 I(|X|>n^{\alpha})\right)\right\}^{q/2} \\
\le\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2+q/2}\log^q n < \infty.
\end{aligned}
\tag{19}
$$

If $1 < p < 2$, it follows by $q > 2$ that $\alpha p - 2 + q/2 - \frac{\alpha pq}{2} < -1$, hence

$$\begin{aligned} I_7 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n(\sum_{i=1}^{n}|a_{ni}|^2E|Y_i|^2)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left\{\sum_{i=1}^{n}|a_{ni}|^2(E|X_i|^2I(|X_i|\leq n^{\alpha})+n^{2\alpha}P(|X_i|>n^{\alpha}))\right\}^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left\{\sum_{i=1}^{n}|a_{ni}|^2n^{\alpha(2-p)}(E|X|^pI(|X|\leq n^{\alpha})+E|X|^pI(|X|>n^{\alpha}))\right\}^{q/2} \\ &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\left(\sum_{i=1}^{n}|a_{ni}|^2n^{\alpha(2-p)}E|X|^p\right)^{q/2} \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-2+q/2-\frac{\alpha pq}{2}}\log^q n<\infty. \end{aligned} \tag{20}$$

For the third part of (17), by the definition of Y'_i, we have

$$\begin{aligned} I_8 &= C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\log n\sum_{i=1}^{n}|a_{ni}|E|Y'_i| \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\log n\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>n^{\alpha}) \\ &\leq C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1}\log nE|X|I(|X|>n^{\alpha}) \\ &\leq CE|X|^p\log(1+|X|)<\infty. \end{aligned} \tag{21}$$

Case 2: if $1 < q \leq 2$, we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\left\{\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}a_{ni}X_i\right|-\varepsilon n^{\alpha}\right\}^{+} \\ \leq\ & C\sum_{n=1}^{\infty} n^{\alpha p-\alpha q-2}\log^q n\sum_{i=1}^{n}|a_{ni}|^qE|Y_i|^q+\sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\log n\sum_{i=1}^{n}|a_{ni}|E|Y'_i|. \end{aligned} \tag{22}$$

The rest of the proof is similar to those of I_6 and I_8, so we omit the details.

According to (17)-(22), (5) holds. The proof is completed. □

Proof. *(Proof of Remark 1.6 (i))*

The complete moment convergence implies complete convergence, which can be verified by the following statement:

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}E\left\{\left|\sum_{i=1}^{n}a_{ni}X_i\right|-\varepsilon n^{\alpha}\right\}^{+} \\ &= \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2}\int_0^{\infty}P\left(\left|\sum_{i=1}^{n}a_{ni}X_i\right|>\varepsilon n^{\alpha}+x\right)dx \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} \int_0^{\varepsilon n^\alpha} P\left(\left|\sum_{i=1}^n a_{ni} X_i\right| > \varepsilon n^\alpha + x\right) dx \\
&\geq \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{i=1}^n a_{ni} X_i\right| > 2\varepsilon n^\alpha\right).
\end{aligned}
$$

The proof is completed. □

Proof. (Proof of Theorem 1.7) First, we give the proof of (6). Without loss of generality, we assume $a_{ni} \geq 0$ for all $1 \leq i \leq n, n \geq 1$. Define

$$Y_i = -nI(X_i < -n) + X_i I(|X_i| \leq n) + nI(X_i > n)$$

and

$$Y_i' = (X_i + n)I(X_i < -n) + (X_i - n)I(X_i > n).$$

Taking $q = 2$ and $a = n$ in Lemma 2.3, we have by Lemma 2.2 and Lemma 2.4 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E\left\{\left|\sum_{i=1}^n a_{ni} X_i\right| - \varepsilon n\right\}^+ \\
&\leq \sum_{n=1}^{\infty} n^{-2}\left\{Cn^{-1} E\left|\sum_{i=1}^n a_{ni}(Y_i - EY_i)\right|^2 + E\left|\sum_{i=1}^n a_{ni}(Y_i{}' - EY_i')\right|\right\} \\
&= C\sum_{n=1}^{\infty} n^{-3} E\left|\sum_{i=1}^n a_{ni}(Y_i - EY_i)\right|^2 + \sum_{n=1}^{\infty} n^{-2} E\left|\sum_{i=1}^n a_{ni}(Y_i{}' - EY_i')\right| \\
&\leq C\sum_{n=1}^{\infty} n^{-3} \sum_{i=1}^n |a_{ni}|^2 [EX_i{}^2 I(|X_i| \leq n) + n^2 P(|X_i| > n)] + \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^n |a_{ni}| E|X_i| I(|X_i| > n) \\
&\leq C\sum_{n=1}^{\infty} n^{-2} EX^2 I(|X| \leq n) + C\sum_{n=1}^{\infty} n^{-1} E|X| I(|X| > n) + C\sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E|X| I(m < |X| \leq (m+1)) \\
&= C\sum_{m=1}^{\infty} m^2 P((m-1) < |X| \leq m) \sum_{n=m}^{\infty} n^{-2} + C\sum_{m=1}^{\infty} E|X| I(m < |X| \leq (m+1)) \sum_{n=1}^{m} n^{-1} \\
&\leq C\sum_{m=1}^{\infty} mP((m-1) < |X| \leq m) + C\sum_{m=1}^{\infty} E|X| I(m < |X| \leq (m+1)) \log m \\
&\leq CE|X| + CE|X| \log(1 + |X|) < \infty,
\end{aligned}
$$

which implies (6).

Next, we will prove (7). Taking $q = 2$ and $a = n$ in Lemma 2.3, we have by Lemma 2.2 and Lemma 2.5 that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-2} E\left\{\max_{1\leq k\leq n}\left|\sum_{i=1}^k a_{ni} X_i\right| - \varepsilon n\right\}^+ \\
&\leq \sum_{n=1}^{\infty} n^{-2}\left\{Cn^{-1} E\max_{1\leq k\leq n}\left|\sum_{i=1}^k a_{ni}(Y_i - EY_i)\right|^2 + E\max_{1\leq k\leq n}\left|\sum_{i=1}^k a_{ni}(Y_i{}' - EY_i')\right|\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{-3} E\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}(Y_i - EY_i)\right|^2 + \sum_{n=1}^{\infty} n^{-2} E\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}(Y_i{}' - EY_i')\right| \\
&\le C\sum_{n=1}^{\infty} n^{-3}\log^2 n\sum_{i=1}^{n}|a_{ni}|^2[EX_i{}^2 I(|X_i|\le n) + n^2 P(|X_i|>n)] + \sum_{n=1}^{\infty} n^{-2}\log n\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>n) \\
&\le C\sum_{n=1}^{\infty} n^{-2}\log^2 nEX^2 I(|X|\le n) + C\sum_{n=1}^{\infty} n^{-1}\log^2 nE|X|I(|X|>n) + \sum_{n=1}^{\infty} n^{-1}\log nE|X|I(|X|>n) \\
&\le C\sum_{n=1}^{\infty} n^{-2}\log^2 n\sum_{m=1}^{n} m^2 P((m-1)<|X|\le m) + C\sum_{n=1}^{\infty} n^{-1}\log^2 n\sum_{m=n}^{\infty} E|X|I(m<|X|\le(m+1)) \\
&\quad + \sum_{n=1}^{\infty} n^{-1}\log n\sum_{m=n}^{\infty} E|X|I(m<|X|\le(m+1)) \\
&\le C\sum_{m=1}^{\infty} m\log^2 mP((m-1)<|X|\le m) + C\sum_{m=1}^{\infty} E|X|\log^3 mI(m<|X|\le(m+1)) \\
&\quad + C\sum_{m=1}^{\infty} E|X|\log^2 mI(m<|X|\le(m+1)) \\
&\le CE|X|\log^3(1+|X|)<\infty,
\end{aligned}
$$

which implies (7). The proof is completed. □

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