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Distinguishing Number and Distinguishing Index of Certain Graphs

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Abstract. The distinguishing number (index) $D(G)$ ($D'(G)$) of a graph G is the least integer d such that G has an vertex labeling (edge labeling) with d labels that is preserved only by a trivial automorphism. In this paper we compute these two parameters for some specific graphs. Also we study the distinguishing number and the distinguishing index of corona product of two graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph. We use the standard graph notation ([5]). In particular, $Aut(G)$ denotes the automorphism group of G. A labeling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is said to be r-distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. The point of the labels on the vertices is to destroy the symmetries of the graph, that is, to make the automorphism group of the labeled graph trivial. Formally, ϕ is r-distinguishing if for every non-trivial $\sigma \in Aut(G)$, there exists x in $V = V(G)$ such that $\phi(x) \neq \phi(x\sigma)$. We will often refer to a labeling as a coloring, but there is no assumption that adjacent vertices get different colors. Of course the goal is to minimize the number of colors used. Consequently the distinguishing number of a graph G is defined by

$$D(G) = min\{r |\ G \textit{ has a labeling that is } r\textit{-distinguishing}\}.$$

This number has defined by Albertson and Collins [2]. Similar to this definition, Kalinowski and Pilśniak [6] have defined the distinguishing index $D'(G)$ of G which is the least integer d such that G has an edge colouring with d colours that is preserved only by a trivial automorphism. If a graph has no nontrivial automorphisms, its distinguishing number is 1. In other words, $D(G) = 1$ for the asymmetric graphs. The other extreme, $D(G) = |V(G)|$, occurs if and only if $G = K_n$. The distinguishing index of some examples of graphs was exhibited in [6]. For instance, $D(P_n) = D'(P_n) = 2$ for every $n \geqslant 3$, and $D(C_n) = D'(C_n) = 3$ for $n = 3, 4, 5$, $D(C_n) = D'(C_n) = 2$ for $n \geqslant 6$. It is easy to see that the value $|D(G) - D'(G)|$ can be large. For example $D'(K_{p,p}) = 2$ and $D(K_{p,p}) = p + 1$, for $p \geq 4$. The Cartesian product of graphs G and H is a graph denoted $G\Box H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. We denote $G\Box G$ by G^2, and we recursively define the k-th Cartesian power of G as $G^k = G\Box G^{k-1}$ [4]. A graph G is called prime if $G = G_1\Box G_2$ implies that either G_1 or G_2 is K_1. The distinguishing number and index of the Cartesian powers of graphs has been thoroughly

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investigated. It was first proved by Albertson [1] that if G is a connected prime graph, then $D(G^k) = 2$ whenever $k \geq 4$, and if $|V(G)| \geq 5$, then also $D(G^3) = 2$. Next, Klavžar and Zhu [8] showed that for any connected graph G with a prime factor of order at least 3, $D(G^k) = 2$ for $k \geq 3$. Michael and Garth in [9] have determined the distinguishing number of the Cartesian product of complete graphs. Pilśniak studied the Nordhaus-Gaddum bounds for the distinguishing index in [10]. Also the distinguishing number of the hypercube has been investigated in [3]. Similar to definition of $D(G)$ and $D'(G)$, authors in [7] introduced the total distinguishing number of a graph G, $D''(G)$ as the least number d such that G has a total colouring (not necessarily proper) with d colours that is only preserved by the trivial automorphism. They proved that $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.

In this paper, we continue the study of two parameters $D(G), D'(G)$ and proceed as follows.

In the next section, we consider two specific graphs, friendship graphs and book graphs and compute their distinguishing number and index. Also we study the distinguishing number and the distinguishing index of corona product of two graphs in Section 3.

2. The Distinguishing Number and Index of some Graphs

In this section, we consider friendship graphs and book graphs and compute their distinguishing number and their distinguishing index. We begin with friendship graph. The friendship graph F_n ($n \geqslant 2$) can be constructed by joining n copies of the cycle graph C_3 with a common vertex. First we state the following lemma:

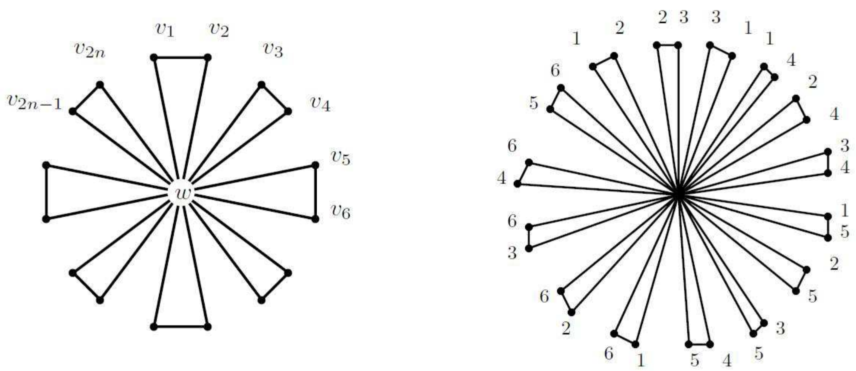

Figure 1: Friendship graph F_n and the vertex labeling of F_{15}, respectively.

Lemma 2.1. *The order of automorphism group of F_n ($n \geqslant 2$) is $|Aut(F_n)| = n!2^n$.*

Proof. Since w is the only vertex of F_n which is not of degree 2 (Figure 1), so w is fixed by all elements of the automorphism group. We can get the automorphism group of F_n by interchanging the base of triangles together and rotating about the center w. Therefore $|Aut(F_n)| = n!2^n$. □

Theorem 2.2. *The distinguishing number of the friendship graph F_n ($n \geq 2$) is*

$$D(F_n) = \lceil \frac{1 + \sqrt{8n+1}}{2} \rceil.$$

Proof. First we shall find a lower bound for $D(F_n)$ and then we present a distinguishing vertex labeling with this number of labels. Let $\{x_i, y_i\}$, $1 \leqslant i \leqslant n$ be the set of two labels that has been assigned to the two vertices of the base of i-th triangle, and $L = \Big\{1, \{x_i, y_i\} \mid 1 \leqslant i \leqslant n,\ x_i, y_i \in \mathbb{N}\Big\}$ is the labeling of F_n such that the label of the central vertex w is 1 and the label of the two vertices on the base of i-th triangle is $\{x_i, y_i\}$. If L is a distinguishing labeling for F_n, then it satisfies the following properties:

(i) For every $i \in \{1, \dots n\}$, $x_i \neq y_i$. Because for every $1 \leq i \leq n$, the map $f_i : V(F_n) \to V(F_n)$ which maps v_{2i-1} and v_{2i} to each other and fixes the rest of vertices of F_n, is an automorphism of F_n.

(ii) For every $i, j \in \{1, \dots n\}$ where $i \neq j$, $\{x_i, y_i\} \neq \{x_j, y_j\}$. Because for every $i, j \in \{1, \dots n\}$ where $i \neq j$, the map $f_{i,j} : V(F_n) \to V(F_n)$ which maps v_{2i-1} and v_{2j} to each other and v_{2i} and v_{2j-1} to each other and fixes the rest of vertices of F_n, is an automorphism of F_n. Also the map $g_{i,j} : V(F_n) \to V(F_n)$ which maps v_{2i-1} and v_{2j-1} to each other and v_{2i} and v_{2j} to each other and fixes the rest of vertices of F_n, is an automorphism of F_n.

So it can be obtained that with labels $\{1, \dots, s\}$ we can make at most $\binom{s}{2}$ numbers of the pairs (x, y) such that they satisfy (i) and (ii). Hence $D(F_n) \geqslant min\{s : \binom{s}{2} \geqslant n\}$. By a simple computation we get $D(F_n) \geqslant \lceil \frac{1+\sqrt{8n+1}}{2} \rceil$. Now we define a distinguishing vertex labeling on F_n with $\lceil \frac{1+\sqrt{8n+1}}{2} \rceil$ labels. Consider the friendship graph in Figure 1. The function that maps v_1 to v_2 and v_2 to v_1 and fixes the rest of vertices, is a non-trivial automorphism. Thus the labels v_1 and v_2 should be different. We assign the vertex v_1 the label 1 and the vertex v_2 the label 2. Similarly, the function that maps v_3 to v_4 and v_4 to v_3 and fixes the rest, is a non-trivial automorphism. Thus the labels v_3, v_4 should be distinct. Let assign the vertex v_3 the label 2 and the vertex v_4 the label 3. We continue this method to label all vertices of friendship graph (see the label of F_{15} in Figure 1). Note that the label of vertex w is 1. Hence this method gives a distinguishing vertex labeling with the minimum number of labels. By the above process, observe that the distinguishing number of F_n, $D(F_n)$, is the n-th term of the sequence $\{D(F_i)\}$ which defines as follows:

$$\{D(F_i)\}_{i\geqslant 1} = \{-, 3, 3, 4, 4, 4, \underbrace{5, \dots, 5}_{4-times}, \underbrace{6, \dots, 6}_{5-times}, \underbrace{7, \dots, 7}_{6-times}, \dots, \underbrace{m, \dots, m}_{(m-1)-times}, \dots\}.$$

In fact,

$$D(F_n) = min\{k : \sum_{i=2}^{k}(i-1) \geqslant n\}.$$

By an easy computation, we see that

$$min\{k : \sum_{i=2}^{k}(i-1) \geqslant n\} = \lceil \frac{1+\sqrt{8n+1}}{2} \rceil.$$

Therefore we have the result. □

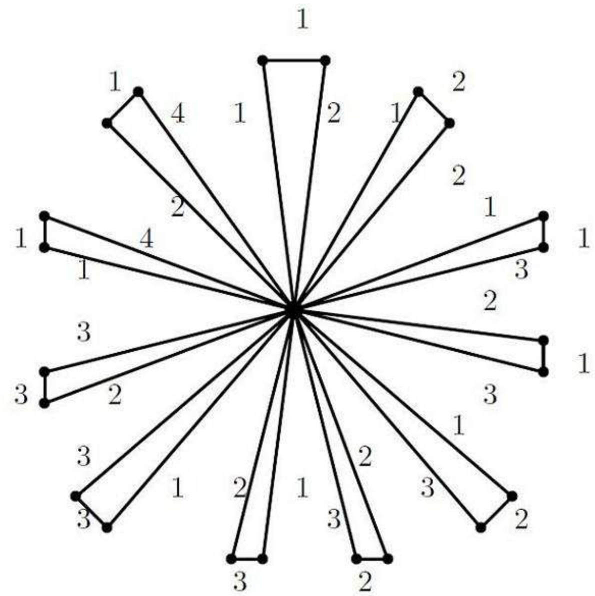

Figure 2: The edge labeling of F_{11}.

Now we shall compute the distinguishing index of the friendship graph, i.e., $D'(F_n)$.

Theorem 2.3. *Let $a_n = 1 + 27n + 3\sqrt{81n^2 + 6n}$. For every $n \geq 2$,*

$$D'(F_n) = \lceil \frac{1}{3}(a_n)^{\frac{1}{3}} + \frac{1}{3(a_n)^{\frac{1}{3}}} + \frac{1}{3} \rceil.$$

Proof. First we show that $D'(F_n) \geqslant min\{k : k^3 - k^2 \geq 2n\}$ and next we present a distinguishing edge labeling such that it obtains this bound.

Let $\{x_i, y_i, z_i\}$, $1 \leqslant i \leqslant n$ be the label of three sides of a triangle in the friendship graph such that z_i is the label which it is assigned to the base and x_i, y_i are the labels of two sides, and let $L' = \Big\{(x_i, y_i, z_i) \mid 1 \leqslant i \leqslant n,\ x_i, y_i \in \mathbb{N}\Big\}$ be the labeling of F_n. If L' is a distinguishing labeling for F_n, then it satisfies the following properties:

(i) For all $j = 1, \ldots, n$, $(x_j, y_j, z_j) \neq (y_j, x_j, z_j)$. Because for every $i \in \{1, \ldots n\}$, the map $f_i : V(F_n) \to V(F_n)$ which maps v_{2i-1} and v_{2i} to each other and fixes the rest of the vertices of F_n, is an automorphism of F_n.

(ii) For every $j \neq i$, $(x_i, y_i, z_i) \neq (x_j, y_j, z_j)$ and $(y_i, x_i, z_i) \neq (x_j, y_j, z_j)$. Because for every $i, j \in \{1, \ldots n\}$ where $i \neq j$, the map $f_{i,j} : V(F_n) \to V(F_n)$ which maps v_{2i-1} and v_{2j} to each other and v_{2i} and v_{2j-1} to each other and fixes the rest of vertices of F_n, is an automorphism of F_n. Also the map $g_{i,j} : V(F_n) \to V(F_n)$ such that it maps v_{2i-1} and v_{2j-1} to each other and v_{2i} and v_{2j} to each other and fixes the rest of vertices of F_n, is an automorphism of F_n.

So it can be obtained that with labels $\{1, \ldots, s\}$ we can make at most $\binom{s}{2}s$ numbers of the 3-ary's (x, y, z) such that they satisfy (i) and (ii) (there are $\binom{s}{2}$ choices for x_i and y_i and s choices for z_i). Hence $D'(F_n) \geqslant min\{s : \binom{s}{2}s \geq n\}$ and it can be calculated that $D'(F_n) \geqslant \lceil \frac{1}{3}(a_n)^{\frac{1}{3}} + \frac{1}{3(a_n)^{\frac{1}{3}}} + \frac{1}{3} \rceil$.

Now we define a distinguishing edge labeling on F_n with $min\{k : k^3 - k^2 \geq 2n\}$ labels. Similar to the vertex labeling of F_n, in the edge labeling of F_n, the labels of two sides of every triangle should be distinct, otherwise, we have a non-trivial automorphism, which preserves the labeling. We assign the first triangle, the 3-ary $(1,2,1)$ and the second the 3-ary $(1,2,2)$. Now we assign the third triangle, the 3-ary $(1,3,1)$ and the forth triangle the 3-ary $(2,3,1)$. Continuing this method we can obtain a distinguishing labeling for the graph (see the Figure 2 for the labeling of F_{11}). It is easy to see that the distinguishing index of F_n, $D'(F_n)$, is the n-th term of the sequence $\{D'(F_i)\}$ which defines as follows:

$$\{D'(F_i)\}_{i\geqslant 1} = \{-,2,\underbrace{3,\ldots,3}_{7-times},\underbrace{4,\ldots,4}_{15-times},\underbrace{5,\ldots,5}_{26-times},\underbrace{6,\ldots,6}_{40-times},\ldots,\underbrace{m,\ldots,m}_{2(m-1)+3\binom{m-1}{2}-times},\ldots\}.$$

In fact,

$$D'(F_n) = min\{k : \sum_{i=2}^{k}\left(2(i-1)+3\binom{i-1}{2}\right) \geqslant n\}.$$

By an easy computation, we see that

$$min\{k : \sum_{i=2}^{k}\left(2(i-1)+3\binom{i-1}{2}\right) \geqslant n\} = min\{k : k^3 - k^2 \geq 2n\} =$$

$$\lceil \frac{1}{3}(1+27n+3\sqrt{81n^2+6n})^{1/3} + \frac{1}{3(1+27n+3\sqrt{81n^2+6n})^{1/3}} + \frac{1}{3}\rceil.$$

So, our method for edge labeling of F_n which as shown in Figure 2 lead to use the minimum number of labels. Therefore we have the result. □

The n-book graph ($n \geqslant 2$) (Figure 3) is defined as the Cartesian product $K_{1,n}\square P_2$. We call every C_4 in the book graph B_n, a page of B_n. All pages in B_n have a common side v_1v_2. If we change the labels of vertices of parallel side of v_1v_2 (for example the labels of v_3 and v_4 in Figure 3), then we call this new page as the inverse of the first page. We shall compute the distinguishing number and index of B_n. The following result gives the order of automorphism group B_n.

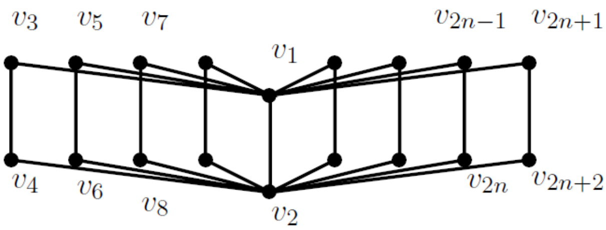

Figure 3: Book graph B_n.

Theorem 2.4. *For every $n \geq 2$, $|Aut(B_n)| = 2n!$.*

Proof. All vertices of B_n, except two vertices v_1 and v_2 have degree 2 (Figure 3). So the two vertices v_1 and v_2 are mapped into each other under the elements of automorphism group. In fact each automorphism maps the pages to each other. Note that as soon as the first page is mapped to the inverse of a page, the rest of pages are mapped to inverse of themselves or to inverse of another page. Therefore $|Aut(B_n)| = 2n!$. □

Theorem 2.5. *The distinguishing number of B_n ($n \geq 2$) is $D(B_n) = \lceil\sqrt{n}\rceil$.*

Proof. First we show that $D(B_n) \geqslant \lceil\sqrt{n}\rceil$ and next we present a distinguishing edge labeling such that it obtains this bound. Let x_i, y_i be the labels of upper and lower vertices of i-th page respectively (except for v_1, v_2 in Figure 3) and let $L = \{(1,2),(x_i,y_i) \mid 1 \leqslant i \leqslant n,\ x_i,y_i \in \mathbb{N}\}$ be the labeling of B_n such that $(1,2)$ is the two labels that has been assigned to the vertices v_1 and v_2. If L is a distinguishing labeling for B_n, then it has the following property:

(i) For all $i, j \in \{1, \ldots, n\}$ where $j \neq i$, $(x_i, y_i) \neq (x_j, y_j)$,

because for every $i, j \in \{1, \ldots n\}$, the map $f_i : V(B_n) \to V(B_n)$ which maps v_{2i-1} and v_{2j-1} to each other and maps v_{2i} and v_{2j} to each other and fixes the rest of vertices of B_n, is an automorphism of B_n.

So it can be obtained that with labels $\{1, \ldots, s\}$ we can make at most $2\binom{s}{2} + s$ numbers of the pairs (x, y) such that they satisfy (i) (there are $2\binom{s}{2}$ choices for pairs (x, y) such that $x \neq y$ and (x, y) satisfies (i) and since the pairs (i, i) for every $i \in \{1, \ldots, s\}$ are not counted in these $2\binom{s}{2}$ choices, so we add s to $2\binom{s}{2}$). Hence $D(B_n) \geqslant min\{s : 2\binom{s}{2} + s \geq n\}$ and it can be calculated that $D(B_n) \geqslant \lceil \sqrt{n} \rceil$.

To present the vertex labeling of B_n, note that the labels of v_1 and v_2 can be the same or distinct. Since we would like to use least number of labels, observe that for this purpose, two labels that have been given to v_1 and v_2 should be different, because if the label of vertices v_1 and v_2 is the same, then we should also check $x_i \neq y_i$ for each i ($1 \leqslant i \leqslant n$). We assign the first page of B_n, the pair $(1, 1)$ and the second, the pair $(2, 1)$. Now we assign the third page of B_n, the pair $(2, 2)$ and the forth page, the pair $(1, 2)$. Now we use the new label 3 for the labeling next five pages. We assign these five pages the labels $(3, 1)$, $(3, 2)$, $(3, 3)$, $(1, 3)$, and $(2, 3)$, respectively. Note that if in the process of labeling, the same labels have been given to v_1 and v_2, then commute them with two labels of the next page. Our method for labeling the vertices of book graph have been shown for B_{10} in Figure 4. By this process, observe that the distinguishing number of B_n, $D(B_n)$, is the n-th term of the sequence $\{D(B_i)\}$ which defines as follows:

$$\{D(B_i)\}_{i\geqslant 1} = \{-, 2, 2, 2, 3, 3, 3, 3, 3, \underbrace{4, \ldots, 4}_{7-times}, \underbrace{5, \ldots, 5}_{9-times}, \ldots, \underbrace{m, \ldots, m}_{(2m-1)-times}, \ldots\}.$$

In fact,

$$D(B_n) = min\{k : \sum_{i=1}^{k} (2i - 1) \geqslant n\}.$$

By an easy computation, we see that

$$min\{k : \sum_{i=1}^{k} (2i - 1) \geqslant n\} = \lceil \sqrt{n} \rceil.$$

Therefore we have the result. □

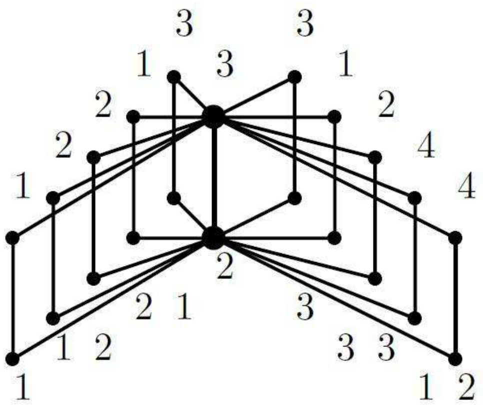

Figure 4: The vertex labeling of B_{10}.

Remark 2.6. *The distinguishing index of Cartesian product of star $K_{1,n}$ with path P_m for $m \geqslant 2$ and $n \geqslant 2$ is $D'(K_{1,n}\square P_m) = \lceil \sqrt[2m-1]{n}\rceil$, unless $m = 2$ and $n = r^3$ for some integer r. In the latter case $D'(K_{1,n}\square P_2) = \sqrt[3]{n} + 1$. ([4]). Since $B_n = K_{1,n}\square P_2$, using this equality we obtain the distinguishing index of book graph B_n.*

3. Distinguishing Number (Index) of Corona of Two Graphs

In this section, we shall study the distinguishing number and the distinguishing index of corona product of two graphs. The corona product $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H.

Theorem 3.1. *For every $n \geq 4$, $D(P_n \circ K_1) = (D'(P_n \circ K_1)) = 2$.*

Proof. Since $|Aut(P_n \circ K_1)| = 2$, so $P_n \circ K_1$ has only one non-trivial automorphism. Therefore $D(P_n \circ K_1) = D'(P_n \circ K_1) = 2$. □

Before presenting the main result for $D(G \circ H)$, we explain the relationship between the automorphism group of the graph $G \circ H$ with the automorphism groups of two connected graphs G and H such that $G \neq K_1$. Note that there is no vertex in the copies of H which has the same degree as a vertex in G. Because if there exists a vertex w in one of the copies of H and a vertex v in G such that $deg_{G\circ H}v = deg_{G\circ H}w$, then $deg_G(v) + |V(H)| = deg_H(w) + 1$. So we have $deg_H(w) + 1 > |V(H)|$, which is a contradiction. By this note, we state and prove the following theorem:

Theorem 3.2. *For every two connected graphs G and H such that $G \neq K_1$, we have $|Aut(G\circ H)| = |Aut(G)||Aut(H)|$.*

Proof. Let the vertex set of G be $\{v_1, \ldots, v_{|V(G)|}\}$ and the vertex set of i-th copy of H, $H^{(i)}$, be $\{w_1^{(i)}, \ldots, w_{|V(H)|}^{(i)}\}$. Since there is no vertex in copies of H which has the same degree as a vertex in G, for every $f \in Aut(G \circ H)$, we have $f|_H \in Aut(H)$ and $f|_G \in Aut(G)$. In addition, for $i, j \in \{1, \ldots, |V(G)|\}$ we have

$$f(v_i) = v_j \Longleftrightarrow f(H^{(i)}) = H^{(j)}.$$

Conversely, let $\varphi \in Aut(G)$ and $\phi \in Aut(H)$ such that $\varphi(v_i) = v_{j_i}$, where $i, j_i \in \{1, \ldots, |V(G)|\}$. Now we define the following automorphism h of $G \circ H$:

$$\begin{cases} h : G \circ H \to G \circ H \\ v_i \mapsto \varphi(v_i) = v_{j_i} & i, j_i \in \{1, \ldots, |V(G)|\}, \\ w_k^{(i)} \mapsto (\phi(w_k))^{(j_i)} & k \in \{1, \ldots, |V(H)|\}. \end{cases}$$

Therefore $|Aut(G \circ H)| = |Aut(G)||Aut(H)|$. □

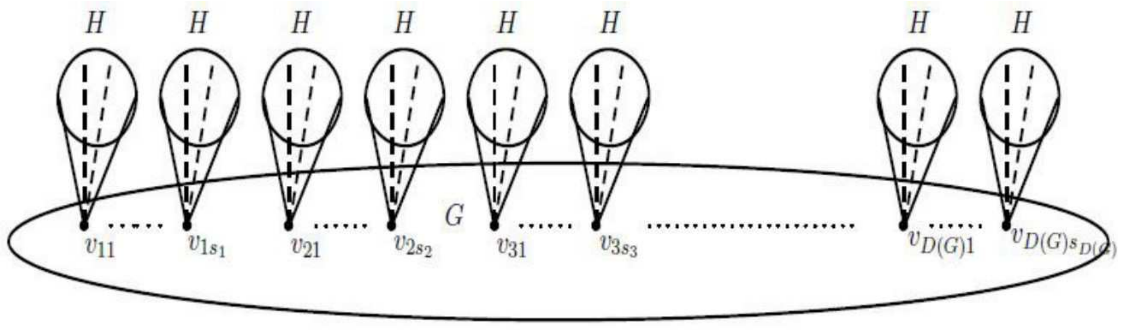

Figure 5: The partition of the vertices of $G \circ H$ by its labels.

By the elements of the automorphism group of $G \circ H$ (Theorem 3.2), we have the following result.

Theorem 3.3. *Let G and H be two connected graphs and $G \neq K_1$.*

(i) If $D(G) = 1$, then $D(G \circ H) = D(H)$.
(ii) $D(G \circ H) = 1$ if and only if $D(G) = D(H) = 1$.

Theorem 3.4. *Let G and H be two connected graphs such that $G \neq K_1$. If $D(G) \leqslant D(H)$, then $D(H) = D(G \circ H)$*

Proof. If $D(G) = 1$, then we have the result by Theorem 3.3, so we suppose that $D(G) \neq 1$. If we label $G \circ H$ with less than $D(H)$ labels in a distinguishing way, then we can find a non-identity automorphism of H such as f, such that it preserves the labeling of H. Expanding f to $G \circ H$ such that f acts as the identity function on G, we obtain a non-identity automorphism of $G \circ H$ preserving the labeling of $G \circ H$, which is contradiction. So we have $D(H) \leqslant D(G \circ H)$. Now we show the inequality $D(H) \geqslant D(G \circ H)$. By the definition of distinguishing vertex labeling, the vertex set $V(G)$ is partitioned to at most $D(H)$-classes (because $D(G) \leqslant D(H)$), say, $[1], [2], ..., [D(H)]$. The vertices of the class $[i]$ denoted by $v_{i1}, \ldots, v_{is_i}$ in Figure 5 where s_i is the size of the $[i]$-class and $i = 1, \ldots, D(H)$. We label the vertices in the class $[i]$ and s_i-copies of H to obtain a distinguishing vertex labeling of $G \circ H$ as follows: we label all vertices in the class $[i]$ with the label i, and vertices in the s_i copies of H with $D(H)$ labels in a distinguishing way where $i = 1, \ldots, D(H)$. By Theorem 3.2 this labeling is a distinguishing labeling of $G \circ H$, and so $D(H) \geqslant D(G \circ H)$. □

Theorem 3.5. *Let G and H be two connected graphs such that $D(G) > D(H)$. Then*

$$D(H) \leqslant D(G \circ H) \leqslant D(H) + \lfloor \frac{-(1 + D(H)) + \sqrt{(D(H) - 1)^2 + 4D(G)}}{2} \rfloor.$$

Proof. The proof of the left inequality is the same as the first part of the proof of Theorem 3.4. For the second inequality, by the definition of distinguishing vertex labeling, the vertex set $V(G)$ will be partitioned to $D(G)$-classes, say, $[1], [2], ..., [D(G)]$. The vertices of the class $[i]$ denoted by $v_{i1}, \ldots, v_{is_i}$ ($i = 1, \ldots, D(G)$) in Figure 5. We label the vertices in the class $[i]$ and s_i-copies of H to obtain a distinguishing vertex labeling of $G \circ H$ as follows:

Step 1) Labeling the vertices in the classes $[i]$ and the vertices of s_i-copies of H for $1 \leq i \leq D(H)$:
We label all vertices in the class $[i]$ with the label i, and vertices in the s_i copies of H with $D(H)$ labels in a distinguishing way.

Step 2) Labeling the vertices in the classes $[i]$ and the vertices of s_i-copies of H, for $D(H)+1 \leq i \leq 2D(H)+2$:
Now for the labeling of the vertices of classes $[i]$, $(D(H) + 1 \leq i \leq 2D(H) + 1)$ we use the label $i - D(H)$, and for s_i copies of H, we add the number one to the label of each vertex of H in the prior step, i.e., if the label of a vertex of H is l $(1 \leqslant l \leqslant D(H))$ in distinguishing labeling of H with $D(H)$ labels, then we replace it by $l + 1$. For the class $[2D(H) + 2]$ we use the label $D(H) + 1$ for the vertices in this class and we label $s_{2D(H)+2}$ copies of H with $D(H)$ labels in a distinguishing way.

Step 3) Labeling the vertices in the classes $[i]$ and the vertices of s_i-copies of H, for $2D(H) + 3 \leq i \leq 3D(H) + 6$:
Now for the labeling of the vertices of classes $[i]$, $(2D(H)+3 \leq i \leq 3D(H)+4)$ we use the label $i-(2D(H)+2)$, and for s_i copies of H, we add the number two to the label of each vertex of H in the firs step, i.e., if the label of a vertex of H is l $(1 \leqslant l \leqslant D(H))$ in distinguishing labeling of H with $D(H)$ labels, then we replace it by $l + 2$. For the class $[3D(H) + 5]$ we use the label $D(H) + 2$ for the vertices in this class and label $s_{3D(H)+5}$ copies of H with $D(H)$ labels in a distinguishing way. For the class $[3D(H) + 6]$ we use the label $D(H) + 2$ for the vertices in this class and we label $s_{3D(H)+6}$ copies of H with $D(H) + 1$ labels in a distinguishing way (i.e., if the label of a vertex of H is l $(1 \leqslant l \leqslant D(H))$ in distinguishing labeling of H with $D(H)$ labels, then we replace it by $l + 1$).

Continuing this method and by Theorem 3.2 it can be observed that this method makes a distinguishing labeling with $D(H) + min\{k : \left(\sum_{i=0}^{k}(D(H) + 2i)\right) \geqslant D(G)\}$ labels. By an easy computation we get

$$min\{k : \left(\sum_{i=0}^{k}(D(H) + 2i)\right) \geqslant D(G)\} = \lfloor \frac{-(1 + D(H)) + \sqrt{(D(H) - 1)^2 + 4D(G)}}{2} \rfloor.$$

So we have the result. □

Theorem 3.6. *Let H be a connected graph, then* $D(H) \leqslant D(K_1 \circ H) \leqslant D(H) + 1$.

Proof. First we prove $D(H) \leqslant D(K_1 \circ H)$. Suppose to the contrary that $D(H) > D(K_1 \circ H)$, so if we label $K_1 \circ H$ in a distinguishing way with $D(K_1 \circ H)$ labels and transfer this labeling to H, then there exists a non-identity automorphism of H such as f, that it preserves the labeling. Expanding f to $K_1 \circ H$ such that f acts as the identity on K_1, we have a non-identity automorphism of $K_1 \circ H$ that it preserves the labeling, which is a contradiction. Now we shall show that $D(K_1 \circ H) \leqslant D(H) + 1$. For this purpose, we define a distinguishing labeling of $K_1 \circ H$ with $D(H) + 1$ labels. First we label H with $D(H)$ labels in a distinguishing way and next assign a new label to the only vertex of K_1. This labeling is a distinguishing labeling for $K_1 \circ H$, because if f is an automorphism of $K_1 \circ H$ preserving the labeling, then $f(K_1) = K_1$ and $f|_H \in Aut(H)$. Since we labeled H in a distinguishing way, $f|_H$ is the identity automorphism. Therefore f is the identity automorphism on $K_1 \circ H$. Therefore, the result follows. □

Here we study the distinguishing index of corona of two graphs. First we compute the distinguishing index of some special cases and exclude them subsequently. The special cases are as follows:

$$D'(K_1 \circ K_1) = 1,\ \ D'(K_1 \circ K_2) = 3,\ \ D'(K_2 \circ K_1) = 2,\ \ D'(K_2 \circ K_2) = 2.$$

Theorem 3.7. *Let G and H be two connected graphs such that* $G \neq K_1$ *and* $D'(H) \geqslant 2$, *then* $D'(G \circ H) \leqslant max\{D'(G), \lceil \sqrt{D'(H)} \rceil\}$.

Proof. We define a distinguishing edge labeling for $G \circ H$ with $max\{D'(G), \lceil \sqrt{D'(H)} \rceil\}$ labels. First we label G with the labels $\{1, \ldots, D'(G)\}$ in a distinguishing way. Now we present a labeling for a copy of H and all middle edges that are incident to this copy of H and G, and next we transfer this labeling to all copies of H and their middle edges. For this we partition the edge set of H with respect to a distinguishing edge labeling of H with the label set $\{1, \ldots, D'(H)\}$. So we have $D'(H)$ classes of edges such that $[i]$-class $(1 \leqslant i \leqslant D'(H))$ contains all the edges of H which they have the label i in the distinguishing edge labeling of H. It is clear that there are vertices of H that are incident to the edges in different classes, such as $[i]$ and $[j]$ with $i \geqslant j$. In this case the middle edges incident to such vertex are considered as the middle edges of the $[i]$-class. The new labeling of H and all its middle edges are as follows:

Step 1) We label all edges in class [1] with the label 1. Next we label all its middle edges that are incident to a vertex in [1]-class, with the label 1. We label all edges in class [2] with the label 1. Next we label all its middle edges that are incident to a vertex in [2]-class, with the label 2.

Step 2) We label all edges in class [3] with the label 2. Next we label all its middle edges that are incident to a vertex in [3]-class with the label 1. We label all edges in class [4] with the label 2. Next we label all its middle edges that are incident to a vertex in [4]-class with the label 2.

Step 3) We label all edges in class [5] with the label 1. Next we label all its middle edges that are incident to a vertex in [5]-class with the label 3. We label all edges in class [6] with the label 2. Next we label all its middle edges that are incident to a vertex in [6]-class with the label 3. We label all edges in class [7] with the label 3. Next we label all its middle edges that are incident to a vertex in [7]-class with the label 3.

Step 4) We label all edges in class [8] with the label 3. Next we label all its middle edges that are incident to a vertex in [8]-class with the label 1. We label all edges in class [9] with the label 3. Next we label all its middle edges that are incident to a vertex in [9]-class with the label 2.

Continuing this method, in the next step we label all edges in class [10] with the label 4 and next we label all its middle edges that are incident to a vertex in [10]-class with the label 1, we obtain a labeling for $G \circ H$ that is distinguishing. Because if f is an automorphism of $G \circ H$ preserving the labeling, then the restriction of f to G is the identity automorphism of G. On the other hand, for each non-identity automorphism of H, there exists an edge in a class that is mapped to an edge in another class. So by considering our labeling of H and all its middle edges we obtain that the restriction of f to H is the identity automorphism of H. Therefore f is the identity automorphism of $G \circ H$. Since we used $min\{k : \sum_{i=1}^{k}(2i - 1) \geqslant D'(H)\}$ labels for the labeling of copies of H (and since this number is equal with $\lceil \sqrt{D'(H)}\rceil$) and used $D'(G)$ labels for G, so we have the result. □

Theorem 3.8. *Let G and H be two connected graphs of orders $n \geqslant 3$ and $m \geqslant 3$, respectively. If $D'(G) = D'(H) = 1$ then $D'(G \circ H) = 1$.*

Proof. Since the orders of G and H are greater than two and $D'(G) = D'(H) = 1$, so $|Aut(G)| = |Aut(H)| = 1$. By Theorem 3.2, $|Aut(G \circ H)| = 1$, and so $D'(G \circ H) = 1$. □

Theorem 3.9. *Let H be a connected graph of order $n \geqslant 3$. Then $D'(K_1 \circ H) \leqslant D'(H) + 1$.*

Proof. We label the edges of H with the labels $\{1, \dots, D'(H)\}$ in a distinguishing way and next label all its middle edges with the new label 0. If f is an automorphism of $K_1 \circ H$ preserving the labeling, then $f(K_1) = K_1$ and $f|_H$ is an automorphism of H. Since we labeled H in a distinguishing way, so this labeling is a distinguishing labeling for $K_1 \circ H$. Hence $D'(K_1 \circ H) \leqslant D'(H) + 1$. □

Theorem 3.10. *Let G be a connected graph such that $G \neq K_1$. Then $D'(G \circ K_2) \leqslant max\{D'(G), 2\}$.*

Proof. If we label G with $D'(G)$ labels in a distinguishing way and label all copies of K_2 with the label 1 and next assign the two middle edges of each copy of K_2, the labels 1 and 2, then we have a distinguishing labeling of $G \neq K_1$ with $max\{D'(G), 2\}$ labels. □

Theorem 3.11. *Let G and H be two connected graphs such that $G \neq K_1$ and $H \neq K_2$.*

(i) If $|Aut(G)| = 1$, then $D'(G \circ H) \leqslant min\{D'(H), |V(H)|\}$.
(ii) If $|V(G)| \leqslant |V(H)| + 1$ and $D'(H) = 1$, then $D'(G \circ H) \leqslant 2$.

Proof. (i) If $|Aut(G)| = 1$, then every element of the automorphism group of $G \circ H$ treats as the identity on G. If $|V(H)| < D'(H)$ then we assign the edges between G and $H^{(i)}$, the labels $1, 2, \dots, |V(H)|$ for $1 \leqslant i \leqslant |V(G)|$ and assign the remaining edges the label 1. If $|V(H)| \geqslant D'(H)$, then we label each copy of H with $D'(H)$ labels in a distinguishing way and assign the remaining edges the label 1. In both cases we made a distinguishing edge labeling, and so the result follows.

(ii) Let the vertex set of G be $\{v_1, \dots, v_{|V(G)|}\}$ and the vertex set of i-th copy of H be $\{w_1^{(i)}, \dots, w_{|V(H)|}^{(i)}\}$. Let e_{ik} be the edge from v_i to $w_k^{(i)}$. If $|Aut(G)| \geqslant 2$, then there exists a non-trivial automorphism φ of $G \circ H$ and $r, s \in \{1, \dots, |V(G)|\}$, $r \neq s$ such that $\varphi(v_r) = v_s$. So e_{rk} is mapped to $e_{sk'}$ under φ where $k, k' \in \{1, \dots, |V(H)|\}$. Now we assign $e_{i1}, \dots, e_{i(i-1)}$ the label 2 for $2 \leqslant i \leqslant |V(G)|$, and assign the remaining edges the label 1. Clearly, this labeling is distinguishing (see Theorem 3.2), and so $D'(G \circ H) \leqslant 2$. □

Corollary 3.12. *Let G and H be two connected graphs such that $G \neq K_1$ and $H \neq K_2$.*

(i) If $D(G) = 1$, then $D'(G \circ H) \leqslant min\{D'(H), |V(H)|\}$.
(ii) If $|V(G)| \leqslant |V(H)| + 1$ and $D'(H) = 1$, then $D'(G \circ H) \leqslant 2$.

Proof. (i) We note that for every graph G, $D(G) = 1$ if and only if $|Aut(G)| = 1$. So we have the result by Theorem 3.11 (i).

(ii) It is easy to see that for every G, $D(G) \geqslant 2$ if and only if $|Aut(G)| \geqslant 2$. So we have the result by Theorem 3.11 (ii). □

Now, we shall present an upper bound for $D'(G \circ H)$ with $D'(H) = 1$ without any condition on $|V(G)|$. For this purpose we need two following parameters:

$$x'_r = \begin{cases} 1 & r = 1 \\ m - 1 & r = 2 \\ \sum_{i_{r-2}=r-1}^{m} \dots \sum_{i_2=i_3}^{m} \sum_{i_1=i_2}^{m} (m - i_1) & r \geqslant 3, \end{cases}$$

$$y'_r = \begin{cases} 1 & r = 1 \\ m & r = 2 \\ \sum_{i=0}^{r-1} \binom{r-1}{i} x_{i+1} & r \geqslant 3. \end{cases}$$

In fact, x'_r is the number of copies of H in $G \circ H$, that their middle edges (edges between H and G), have been labeled with r labels such that these r labels are used in each copy at least one time. Also y'_r is the number of copies of H that their middle edges, the edges between H and G, have been labeled with the labels $1, \dots, r$ such that the label r is used in each copy at least one time.

Theorem 3.13. *Let G and H be the two connected graphs of orders n and m, respectively such that $G \neq K_1$ and $H \neq K_2$ and $D'(H) = 1$. If $D(G) \geqslant 2$ then $D'(G \circ H) \leqslant min\{D'(G), min\{k : \sum_{r=1}^{k} y_r \geqslant n\}\}$.*

Proof. Similar to the proof of part (ii) of Theorem 3.11, let the vertex set of G be $\{v_1, \dots, v_{|V(G)|}\}$, the vertex set of i-th copy of H be $\{w_1^{(i)}, \dots, w_{|V(H)|}^{(i)}\}$ and e_{ik} be the edge from v_i to $w_k^{(i)}$. We present an edge labeling of $G \circ H$ that is continuation of used edge labeling in the proof of part (ii) of Theorem 3.11. We have the following steps:

Step 1) We assign the edges $e_{11}, \dots, e_{1m}$ the label 1. Set $x'_1 = 1$ and $y'_1 = 1$.

Step 2) We assign the edges $e_{i1}, \dots, e_{i(i-1)}$ the label 2 and $e_{ii}, \dots, e_{im}$ the label 1, for $2 \leqslant i \leqslant m$. Set $x'_2 = m - 1$.

Step 3) Label The edges $e_{(m+1)1}, \dots, e_{(m+1)m}$ with the label 2.

So we used the label 2 for labeling the edges between G and m copies of H. Set $y'_2 = m$.

Step 4) Label $e_{(m+2)1}, \dots, e_{(m+2)m}$ with the label 3. Next we do the same work as in Step 2 with the label 1, 3 and 2, 3. So we labeled the edges $e_{i1}, \dots, e_{im}$ for $m + 3 \leqslant i \leqslant 3m$.

Step 5) In this step we use the labels 1, 2, 3 for the labeling of middle edges. We assign the first three edges e_{i1}, e_{i2}, e_{i3} the labels 1, 2, 3 for $3m + 1 \leqslant i \leqslant 3m + x'_3$, where $x'_3 = \sum_{j=2}^{m}(m - j)$. For labeling $e_{i4}, \dots, e_{im}$ we use the label 1, 2, 3 such that $(L_1^{(i)}, L_2^{(i)}, L_3^{(i)})$ are distinct for each $3m + 1 \leqslant i \leqslant 3m + x'_3$, where $L_j^{(i)}$ is the number of the label j in edges $e_{i1}, \dots, e_{im}$. It can be seen that this number is $x'_3 = \sum_{j=2}^{m}(m - j)$.

So we used the label 3 for labeling the edges between G and $1 + 2(m - 1) + x'_3$ copies of H. Set $y'_3 = 1 + 2(m - 1) + x'_3$.

By continuing this method we get:

$$x'_r = \begin{cases} 1 & r = 1 \\ m - 1 & r = 2 \\ \sum_{i_{r-2}=r-1}^{m} \dots \sum_{i_2=i_3}^{m} \sum_{i_1=i_2}^{m} (m - i_1) & r \geqslant 3, \end{cases}$$

$$y'_r = \begin{cases} 1 & r = 1 \\ m & r = 2 \\ \sum_{i=0}^{r-1} \binom{r-1}{i} x_{i+1} & r \geqslant 3. \end{cases}$$

With this method we have labeled (distinguishing) all edges between the vertices of G and the vertices of copies of H. We use the label 1 for the rest of edges. Therefore $D'(G \circ H) \leqslant min\{k : \sum_{r=1}^{k} y_r \geqslant n\}$.

On the other hand if we label G in a distinguishing way with $D'(G)$ labels and assign the remaining edges the label 1, then we obtain a distinguishing labeling of $G \circ H$ with $D'(G)$ labels, because $D'(H) = 1$ and $H \neq K_2$. Therefore by the above paragraph we have the result. □

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