Weighted $\beta\gamma$ - summability of Fuzzy Functions of Order θ

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Abstract. In this paper, the concept of weighted $\beta\gamma$ - summability of fuzzy functions of order θ is introduced and classified into ordinary and absolute sense. Further, a relationship between the weighted $\beta\gamma$ - summability and the corresponding statistical convergence is established. Also, in case of ordinary weighted $\beta\gamma$ - summability, a result analogous to Tauberian theorem for fuzzy functions is obtained.

1. Introduction

Convergence of sequences, in classical or fuzzy sense, means that almost all elements of the sequence have to belong to an arbitrary small neighborhood of the limit. The aim to introduce the concept of statistical convergence is to relax the above condition and to examine the convergence criterion only for majority of elements. The recent trend in mathematics is to investigate this convergence in a more generalized way and link it with summability theory for sequences of real as well as fuzzy numbers.

After introducing the concept of statistical convergence for scalar sequences by Fast [6], several development in this direction i.e. convergence along with the summability methods, for the sequences of real or complex numbers as well as for fuzzy numbers, can be enumerated as follows:

(i) λ - statistical convergence of order β [4], [3], [5].

(ii) lacunary statistical convergence of order β [14], [2].

(iii) weighted statistical convergence [13], [1].

Recently, Aktuglu [1] has given a general type of convergence by defining $\beta\gamma$ - density and in the light of this he has defined statistical convergence. Motivated by Gong et. al. [23], Ojha and Srivastava [19] has introduced weighted $\beta\gamma$ - statistical convergence for fuzzy functions by using the above density and given three types of classifications of it.

In the existing literature of summability method for fuzzy numbers, it is observed that the authors are dealing either with the fuzzy numbers or with the metric values of the fuzzy numbers. The main aim of introducing weighted $\beta\gamma$ - summability is to investigate these both type of definitions and their inter-relations. A generalized version of Tauberian theorem is also established. This summability method not only includes some well known matrix methods but also gives some non-regular matrix methods.

2. Preliminaries

A fuzzy real number $\hat{x} : \mathbb{R} \to [0,1]$ is a fuzzy set which is normal, fuzzy convex, upper semicontinuous and $[\hat{x}]_0 = \{t \in \mathbb{R} : \hat{x}(t) > 0\}$ is compact. Clearly, $\mathbb{R}$ is embedded in $L(R)$, the set of all fuzzy numbers, in the following way:

For each $r \in \mathbb{R}, \bar{r} \in L(R)$ is defined as,

$$\bar{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r. \end{cases}$$

For $0 < \alpha \leq 1$, α - cut of $\hat{x}$ is defined by $[\hat{x}]_{(\alpha)} = \{t \in \mathbb{R} : \hat{x}(t) \geq \alpha\} = [(\hat{x})_\alpha^-, (\hat{x})_\alpha^+]$, a closed and bounded interval of $\mathbb{R}$. Now for any two fuzzy numbers $\hat{x}, \hat{y}$, Matloka [11] has proved that $L(R)$ is a complete metric space under the metric d defined by

$$d(\hat{x}, \hat{y}) = \sup_{0 \leq \alpha \leq 1} \max \left\{ |(\hat{x})_\alpha^- - (\hat{y})_\alpha^-|, |(\hat{x})_\alpha^+ - (\hat{y})_\alpha^+| \right\}.$$

For any $\hat{x}, \hat{y}, \hat{z}, \hat{w} \in L(R)$, the metric d satisfies

(i) $d(c\hat{x}, c\hat{y}) = |c| \, d(\hat{x}, \hat{y})$, $c \in \mathbb{R}$.
(ii) $d(\hat{x} + \hat{z}, \hat{y} + \hat{z}) = d(\hat{x}, \hat{y})$.
(iii) $d(\hat{x} + \hat{z}, \hat{y} + \hat{w}) \leq d(\hat{x}, \hat{y}) + d(\hat{z}, \hat{w})$.

For any two fuzzy numbers $\hat{x}, \hat{y} \in L(R)$, a partial order relation "$\preceq$" is defined as

$$\hat{x} \preceq \hat{y} \iff (\hat{x})_\alpha^- \leq (\hat{y})_\alpha^- \text{ and } (\hat{x})_\alpha^+ \leq (\hat{y})_\alpha^+$$

holds for each $\alpha \in [0,1]$.

Lemma 2.1. *For the given fuzzy numbers $\hat{x}, \hat{y} \in L(R)$, the following statements are equivalent:*

(i) $d(\hat{x}, \hat{y}) \leq \varepsilon$.
(ii) $\hat{x} - \bar{\varepsilon} \preceq \hat{y} \preceq \hat{x} + \bar{\varepsilon}$

where $\varepsilon > 0$.

Definition 2.2 (Statistical convergence). *A sequence $x = (x_k)$ of real or complex numbers is said to be statistically convergent to a number l if for every $\varepsilon > 0$,*

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : |x_k - l| \geq \varepsilon \right\} \right| = 0$$

where the vertical bars $|\cdot|$ indicate the number of elements in the enclosed set.

Many authors have generalized the definition of statistical convergence for the real or complex numbers. The most recent generalization in this direction is the idea of $\beta\gamma$ - statistical convergence which is introduced by Aktuglu [1] as follows:

Definition 2.3. *Let $\beta = (\beta_n)$, $\gamma = (\gamma_n)$ be two sequences of positive numbers s.t.*

(i) $(\beta_n), (\gamma_n)$ *are both non-decreasing*
(ii) $\gamma_n \geq \beta_n$
(iii) $\gamma_n - \beta_n \to \infty$ *as $n \to \infty$.*

Then a sequence (x_k) of real numbers is said to be $\beta\gamma$ - statistically convergent of order θ to L if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{(\gamma_n - \beta_n + 1)^\theta} \left| \left\{ k \in [\beta_n, \gamma_n] : |x_k - L| \geq \varepsilon \right\} \right| = 0.$$

Choosing suitable β and γ, the above definition includes statistical convergence, λ - statistical convergence, lacunary statistical convergence, etc., as particular cases.

In 2016, Ojha and Srivastava [19] have introduced the concept of weighted $\beta\gamma$ - statistical convergence in case of fuzzy functions with the modification given by Ghoshal [8] for the weighted convergence definition as follows:

Let θ be a real number such that $0 < \theta \leq 1$ and $t = (t_k)$ be a sequence of non-negative real numbers such that $\liminf_k t_k > 0$ and suppose

$$T_{\beta\gamma(n)} = \sum_{k \in [\beta_n, \gamma_n]} t_k, \quad n \in \mathbb{N}.$$

Definition 2.4. *A sequence $\hat{f}_k : [a, b] \to L(R)$ of fuzzy functions is said to be weighted $\beta\gamma$ - pointwise statistically convergent to $\hat{f}$ of order θ if for every $\varepsilon > 0$ and for each $x \in [a, b]$,*

$$\lim_{n \to \infty} \frac{1}{T_{\beta\gamma(n)}^{\theta}} \left| \left\{ k \leq T_{\beta\gamma(n)} : t_k \, d\left(\hat{f}_k(x), \hat{f}(x) \right) \geq \varepsilon \right\}_x \right| = 0$$

where $\{\cdot\}_x$ denotes that the set depends on the point x. Denote the set of all weighted $\beta\gamma$ - pointwise statistically convergent sequence of fuzzy functions of order θ by $SP_{\beta\gamma}^{\theta}(t)$.

Definition 2.5 (Weighted strong Cesaro summability). *[7]*
Let (t_n) be a sequence of non-negative real numbers such that $t_1 > 0$, $T_n = t_1 + t_2 + \cdots + t_n$ and $T_n \to \infty$ as $n \to \infty$. Then a sequence of real numbers (x_n) is said to be weighted strong Cesaro convergent to a real number x if

$$\lim_{n \to \infty} \frac{1}{T_n} \sum_{k=1}^{n} t_k |x_k - x| = 0.$$

Use of various type of summability methods has motivated us to define the notion of weighted $\beta\gamma$ - summability of order θ for fuzzy functions. We have also classified the method into ordinary and absolute sense and obtained some interesting results on it.

3. Absolutely Weighted $\beta\gamma$ - summability of Order θ

Let $t = (t_n)$ be a sequence of positive real numbers such that $t_1 > 0$ and

$$T_{\beta\gamma(n)} = \sum_{k \in [\beta_n, \gamma_n]} t_k \to \infty \tag{1}$$

as $n \to \infty$. Let $\hat{f}, \hat{f}_k : [a, b] \to \mathbb{R}$ be a sequence of fuzzy functions and θ belongs to $(0, 1]$. For each $x \in [a, b]$, consider

$$s_n(x) = \frac{1}{T_{\beta\gamma(n)}^{\theta}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x) \right)$$

Definition 3.1. *A sequence $(\hat{f}_k)$ of fuzzy functions is said to be **absolutely weighted $\beta\gamma$ - summable** to a fuzzy function $\hat{f}$ of order θ if for every $x \in [a, b]$,*

$$s_n(x) \to 0 \text{ as } n \to \infty.$$

We denote this convergence as $N_{\beta\gamma}^{\theta}(t)$. In case of $\theta = 1$, the set $N_{\beta\gamma}^{\theta}(t)$ reduces to $N_{\beta\gamma}(t)$.

Remark 3.2 (Particular cases). *For $t_n = 1$ and*

(i) *$\beta_n = 1, \gamma_n = n$ and $\theta = 1$, the class $N_{\beta\gamma}^\theta(t)$ reduces to the class of Cesaro summability which is defined for fuzzy numbers by Kwon [9].*

(ii) *Let (λ_n) be a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$ for all n. Then choosing $\beta_n = n - \lambda_n + 1, \gamma_n = n$, $N_{\beta\gamma}^\theta(t)$ gives the class of fuzzy numbers which is λ - summable [17], λ -summable of order θ [3] and the class of λ -summable of order θ of fuzzy functions by [18].*

(iii) *$\beta_n = k_{n-1} + 1, \gamma_n = k_n$ where (k_n) is an increasing integer sequence with $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$, then $N_{\beta\gamma}^\theta(t)$ is the class of all lacunary summable sequences which is due to Kwon et al [10] for fuzzy numbers.*

Theorem 3.3. *Let $\hat{f}_k : [a, b] \to L(R)$ and $\hat{g}_k : [a, b] \to L(R)$ be two sequences of fuzzy functions. Then*

(i) *If $(\hat{f}_k), (\hat{g}_k) \in N_{\beta\gamma}^\theta(t)$, then $(\hat{f}_k + \hat{g}_k), (c\hat{f}_k) \in N_{\beta\gamma}^\theta(t)$ for $c \in \mathbb{R}$.*

(ii) *If $0 < \theta \leq \delta \leq 1$, then $N_{\beta\gamma}^\theta(t) \subseteq N_{\beta\gamma}^\delta(t)$. The inclusion is strict for some θ, δ where $\theta < \delta$.*

Proof. (i) This part follows from the following relations:

$$d\left(\hat{f}_k(x) + \hat{g}_k(x), \hat{f}(x) + \hat{g}(x)\right) \leq d\left(\hat{f}_k(x), \hat{f}(x)\right) + d\left(\hat{g}_k(x), \hat{g}(x)\right)$$
$$\text{and } d\left(c\hat{f}_k(x), c\hat{f}(x)\right) = |c| \, d\left(\hat{f}_k(x), \hat{f}(x)\right)$$

for each $x \in [a, b]$. Since (t_k) are positive real numbers, so the result follows.

(ii) Since $T_{\beta\gamma(n)} \to \infty$ as $n \to \infty$, so for some fixed $n_0 \in \mathbb{N}$, $T_{\beta\gamma(n)} \geq 1$ for all $n \geq n_0$ and hence for $\theta \leq \delta$, we have $T_{\beta\gamma(n)}^\theta \leq T_{\beta\gamma(n)}^\delta$ for all $n \geq n_0$. Therefore

$$\frac{1}{T_{\beta\gamma(n)}^\delta} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \leq \frac{1}{T_{\beta\gamma(n)}^\theta} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right)$$

for all $n \geq n_0$. This implies $N_{\beta\gamma}^\theta(t) \subset N_{\beta\gamma}^\delta(t)$. To show the inequality is strict, consider the following example.

Example 3.4. *Construct the sequence of fuzzy functions $(\hat{f}_k)$ as:*

$$\hat{f}_k(x) = \begin{cases} \bar{0}, & k \text{ is a square number} \\ \hat{f}(x), & \text{otherwise} \end{cases}$$

for each $x \in [a, b]$ and $\hat{f} : [a, b] \to L(R)$ be a bounded fuzzy function. Let $t_k = 1$ for all k. Then we have

$$s_n(x) = \frac{1}{(\gamma_n - \beta_n + 1)^\theta} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right)$$

$$= \frac{\left[\sqrt{(\gamma_n - \beta_n + 1)}\right]}{(\gamma_n - \beta_n + 1)^\theta} d(\hat{f}(x), \bar{0})$$

where [y] is the greatest integer function less than or equal to y. Now

$$\frac{\sqrt{(\gamma_n - \beta_n + 1)} - 1}{(\gamma_n - \beta_n + 1)^\theta} d(\hat{f}(x), \bar{0}) \leq s_n(x) \leq \frac{\sqrt{(\gamma_n - \beta_n + 1)}}{(\gamma_n - \beta_n + 1)^\theta} d(\hat{f}(x), \bar{0})$$

So from the above inequality, it is clear that $s_n(x) \to 0$ $(n \to \infty)$ for choosing $\theta > 1/2$ and $s_n(x)$ is divergent for $\theta < 1/2$ as $\hat{f}(x)$ is bounded. So by choosing $\theta < 1/2 < \delta$, we can conclude that $(\hat{f}_k) \in N_{\beta\gamma}^\delta(t)$ but $(\hat{f}_k) \notin N_{\beta\gamma}^\theta(t)$.

This completes the proof. $\square$

Theorem 3.5. *The set $N_{\beta\gamma}(t) \cap B^F[a,b]$ is a closed subset of $B^F[a,b]$ where $B^F[a,b]$ is the set of all bounded fuzzy functions on $[a,b]$.*

Proof. Let $(\hat{f}^i) = (\hat{f}^i_k) \in N_{\beta\gamma}(t) \cap B^F[a,b]$ such that for each $k \in \mathbb{N}$ and $x \in [a,b]$, $\hat{f}^i_k(x) \to \hat{f}_k(x)$ as $i \to \infty$. Write $\hat{f} = (\hat{f}_k)$. We have to show $\hat{f} \in N_{\beta\gamma}(t) \cap B^F[a,b]$.

Since $(\hat{f}^i) = (\hat{f}^i_k) \in N_{\beta\gamma}(t)$, so there exists fuzzy functions $\hat{g}_i : [a,b] \to L(R)$ and an integer n_0 such that for all $n \geq n_0$ and for all $x \in [a,b]$,

$$\frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) < \frac{\varepsilon}{3}$$

Further as $(\hat{f}^i_k)$ converges to $\hat{f}_k(x)$, so it is Cauchy i.e. for $i, j \geq n_1$,

$$d\left(\hat{f}^i_k(x), \hat{f}^j_k(x)\right) < \frac{\varepsilon}{3}$$

holds. So for all $i, j, n \geq \max\{n_0, n_1\}$, we have

$$d\left(\hat{g}_i(x), \hat{g}_j(x)\right) \leq d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) + d\left(\hat{f}^i_k(x), \hat{f}^j_k(x)\right) + d\left(\hat{f}^j_k(x), \hat{g}_j(x)\right)$$

i.e. $\dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{g}_i(x), \hat{g}_j(x)\right) \leq \dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \left[d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) + \dfrac{\varepsilon}{3} \right.$

$$\left. + d\left(\hat{f}^j_k(x), \hat{g}_j(x)\right) \right]$$

i.e. $d\left(\hat{g}_i(x), \hat{g}_j(x)\right) \leq \dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) + \dfrac{\varepsilon}{3}$

$$+ \frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}^j_k(x), \hat{g}_j(x)\right)$$

$$< \varepsilon .$$

Thus $(\hat{g}_i(x))$ is a Cauchy sequence in $L(R)$ and $L(R)$ is complete, so $(\hat{g}_i(x))$ is convergent, say $\hat{g}_i(x) \to \hat{g}(x)$ for each $x \in [a,b]$ as $i \to \infty$.

So there exists an integer n_2 such that for all $i, n \geq n_2$ and $\forall \, x \in [a,b]$,

$$d\left(\hat{g}_i(x), \hat{g}(x)\right) < \frac{\varepsilon}{4}$$

$$d\left(\hat{f}^i_k(x), \hat{f}_k(x)\right) < \frac{\varepsilon}{4} \text{ (by assumption)}$$

and $\dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) < \dfrac{\varepsilon}{2}$ (by assumption)

holds. Then for all $i, n \geq n_2$ and for all $x \in [a,b]$,

$$d\left(\hat{f}_k(x), \hat{g}(x)\right) \leq d\left(\hat{f}^i_k(x), \hat{f}_k(x)\right) + d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) + d\left(\hat{g}_i(x), \hat{g}(x)\right)$$

$$< \frac{\varepsilon}{2} + d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right)$$

i.e. $\dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{g}(x)\right) \leq \dfrac{\varepsilon}{2} + \dfrac{1}{T_{\beta\gamma(n)}} \displaystyle\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}^i_k(x), \hat{g}_i(x)\right) < \varepsilon .$

So we can conclude now, $(\hat{f}_k) \in N_{\beta\gamma}(t) \cap B^F[a,b]$ i.e. $N_{\beta\gamma}(t) \cap B^F[a,b]$ is a closed subset of $B^F[a,b]$. $\quad\square$

Remark 3.6. *The above theorem is not true in case of $\theta \neq 1$. To prove this, consider the following example. Let $t_n = 1$ for all n. Take the following sequences of fuzzy functions.*

$$f_k^{(n)}(x) = \begin{cases} \bar{0}, & k \leq n \text{ and } k \text{ is a square number} \\ \hat{f}(x), & \text{otherwise} \end{cases}$$

$$f_k(x) = \begin{cases} \bar{0}, & k \text{ is a square number} \\ \hat{f}(x), & \text{otherwise} \end{cases}$$

where $\hat{f}(x)$ is bounded on $[a, b]$. It is clear that $f_k^{(n)}(x) \to f_k(x)$ as $n \to \infty$ and for each $x \in [a, b]$. Also $\left(f_k^{(n)}(x)\right) \in N_{\beta\gamma}^\theta(t)$ to the function $f_k(x)$ for any choice $(\beta_n), (\gamma_n)$ and θ. But from the Example 3.4, it is clear that $\left(f_k(x)\right) \notin N_{\beta\gamma}^\theta(t)$ for $\theta < 1/2$.

3.1. Inclusion Relations Between Weighted $\beta\gamma$ - Pointwise Statistical Convergence and Absolutely Weighted $\beta\gamma$ - summability

We know that summability methods eg. Cesaro, λ - statistical, lacunary summability, etc., imply the corresponding type of statistical convergence in real or in fuzzy number systems. So it is interesting to know the behavior of the weighted $\beta\gamma$ - summability for the case of fuzzy functions of order θ with the corresponding type of weighted $\beta\gamma$ - statistical convergence of fuzzy functions of order θ. To investigate the inclusions by our definitions, we assume

$$0 < \theta_1 \leq \theta_2 \leq 1 \quad \text{and} \quad \liminf_{n \to \infty} t_n > 0$$

Now we prove the following theorems.

Theorem 3.7. *Let $\liminf\limits_{n \to \infty} \frac{\gamma_n}{T_{\beta\gamma(n)}} > 1$ and $0 < \beta_n \leq 1$. Then if a sequence of fuzzy functions is absolutely weighted $\beta\gamma$ - summable of order θ_1, it is also weighted $\beta\gamma$ - pointwise statistically convergent of order θ_2 i.e. $N_{\beta\gamma}^{\theta_1}(t) \subset SP_{\beta\gamma}^{\theta_2}(t)$.*

Proof. Since $\liminf\limits_{n \to \infty} \frac{\gamma_n}{T_{\beta\gamma(n)}} > 1$, so $\gamma_n > T_{\beta\gamma(n)}$ for all n. Also since $0 < \beta_n \leq 1$, so

$$[1, T_{\beta\gamma(n)}] \subseteq [\beta_n, \gamma_n].$$

Let $(\hat{f}_k) = (\hat{f}_k(x)) \in N_{\beta\gamma}^{\theta_1}(t)$ and $\varepsilon > 0$. Then for each $x \in [a, b]$

$$
\begin{aligned}
\sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) &\geq \sum_{\substack{k \in [\beta_n, \gamma_n] \\ t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon}} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \\
&= \varepsilon \left| \left\{ k \in [\beta_n, \gamma_n] : t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon \right\} \right| \\
&\geq \varepsilon \left| \left\{ k \leq T_{\beta\gamma(n)} : t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon \right\} \right|.
\end{aligned}
$$

Diving by $T_{\beta\gamma(n)}^{\theta_1}$ and applying $\theta_1 \leq \theta_2$, we get

$$\frac{1}{T_{\beta\gamma(n)}^{\theta_1}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \frac{\varepsilon}{T_{\beta\gamma(n)}^{\theta_2}} \left| \left\{ k \leq T_{\beta\gamma(n)} : t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon \right\} \right|.$$

Now taking $n \to \infty$ in the above inequality we get $(\hat{f}_k) \in SP_{\beta\gamma}^{\theta_2}(t)$ i.e. $N_{\beta\gamma}^{\theta_1}(t) \subset SP_{\beta\gamma}^{\theta_2}(t)$. To prove the strictness of the above inclusion, consider the example.

Example 3.8. *Let $\beta_n = 1$, $\gamma_n = n^2$ and $t_n = \frac{1}{5}$. Then $T_{\beta\gamma(n)} = \frac{n^2}{5}$ and hence $\frac{\gamma_n}{T_{\beta\gamma(n)}} = 5 > 1$.*
Let $(\hat{f}_k) : [a, b] \to L(R)$ be a sequence of fuzzy functions defined as

$$\hat{f}_k(x) = \begin{cases} \begin{cases} \frac{t}{xk} + 1, & -xk \leq t \leq 0 \\ 1 - \frac{t}{xk}, & 0 \leq t \leq xk \end{cases}, & \text{when } k \text{ is a square number} \\ \bar{0}, & \text{otherwise} \end{cases}$$

and so

$$t_k\, d\!\left(\hat{f}_k(x), \bar{0}\right) = \begin{cases} \frac{kx}{5}, & k \text{ is a square number} \\ 0, & \text{otherwise.} \end{cases}$$

Take $\theta_1 = \frac{1}{4}$ and $\theta_2 = 1$. Then for any pre-assigned positive $\varepsilon > 0$,

$$\frac{1}{T_{\beta\gamma(n)}^{\theta_2}} \left| \left\{ k \leq \frac{n^2}{5} : t_k\, d\!\left(\hat{f}_k(x), \bar{0}\right) \geq \varepsilon \right\} \right| = \frac{\left[\sqrt{n^2/5}\right]}{n^2/5}$$

$$\leq \frac{1}{\sqrt{n^2/5}} = \frac{\sqrt{5}}{n} \to 0 \text{ as } n \to \infty.$$

So, $(\hat{f}_k) \in SP_{\beta\gamma}^{\theta_2}(t)$. But

$$\frac{1}{T_{\beta\gamma(n)}^{\theta_1}} \sum_{k \in [\beta_n, \gamma_n]} t_k\, d\!\left(\hat{f}_k(x), \bar{0}\right) = \frac{1}{(n^2/5)^{1/4}} \sum_{\substack{k \in [1, n^2] \\ k = p^2 \text{ for } p \in \mathbb{N}}} \frac{kx}{5}$$

$$= \frac{x}{5^{3/4}} \frac{1}{\sqrt{n}} [1 + 4 + 9 + \cdots + n^2] \to \infty \text{ as } n \to \infty$$

i.e. $(\hat{f}_k) \notin N_{\beta\gamma}^{\theta_1}(t)$.
$\square$

Theorem 3.9. *If (t_n) be a bounded sequence of real numbers such that $\limsup\limits_{n \to \infty} \frac{\gamma_n}{T_{\beta\gamma(n)}^{\theta_2}} < \infty$, then $SP_{\beta\gamma}^{\theta_1}(t) \cap B^F[a, b] \subset N_{\beta\gamma}^{\theta_2}(t)$*

i.e. if a sequence of bounded fuzzy functions which is weighted $\beta\gamma$ - statistically convergent of order θ_1, then it is absolutely weighted $\beta\gamma$ - summable of order θ_2.

Proof. Let $\left(\hat{f}_k(x)\right) \in SP_{\beta\gamma}^{\theta_1}(t) \cap B^F[a, b]$ and since $\limsup\limits_{n \to \infty} \frac{\gamma_n}{T_{\beta\gamma(n)}^{\theta_2}} < \infty$, so for some reals $M_1, M_2 > 0$ such that

$$\frac{\gamma_n}{T_{\beta\gamma(n)}^{\theta_2}} < M_1$$

and $t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right) < M_2$.

Let $\varepsilon > 0$ be given. For every $x \in [a, b]$ consider the set

$$A_n = \left\{ k \leq T_{\beta\gamma(n)} : t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon \right\}.$$

Then we have

$$\sum_{k \in [\beta_n, \gamma_n]} t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right) = \sum_{k \in [\beta_n, \gamma_n] \cap A_n} t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right) + \sum_{k \in [\beta_n, \gamma_n] \cap A_n^c} t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right)$$

$$\leq M_2 \left| \left\{ k \leq T_{\beta\gamma(n)} : t_k\, d\!\left(\hat{f}_k(x), \hat{f}(x)\right) \geq \varepsilon \right\} \right| + (\gamma_n + 1)\varepsilon$$

i.e. $\dfrac{1}{T^{\theta_2}_{\beta\gamma(n)}} \displaystyle\sum_{k\in[\beta_n,\gamma_n]} t_k \, d\big(\hat{f}_k(x),\hat{f}(x)\big) \;\leq\; \dfrac{M_2}{T^{\theta_1}_{\beta\gamma(n)}} \Big| \big\{ k \leq T_{\beta\gamma(n)} : t_k \, d\big(\hat{f}_k(x),\hat{f}(x)\big) \geq \varepsilon \big\} \Big| + \Big(M_1 + \dfrac{1}{T^{\theta_2}_{\beta\gamma(n)}}\Big)\,\varepsilon\;.$

Since $\varepsilon > 0$ is arbitrary, so the result follows.
To show that the inclusion is strict, consider the following example.

Example 3.10. *Let $\beta_n = 1$, $\gamma_n = n^2$ and $t_n = 1 + \frac{1}{n}$. Then $\liminf\limits_{n\to\infty} t_n = 1 > 0$ and $t_n \leq 2$ for all n. Also*
$T_{\beta\gamma(n)} = n^2 + (1 + \frac{1}{2} + \cdots + \frac{1}{n^2})$. *Let $\theta_2 = 1$.*

$$\limsup_{n\to\infty} \frac{\gamma_n}{T^{\theta_2}_{\beta\gamma(n)}} = \limsup_{n\to\infty} \frac{n^2}{n^2 + (1 + \frac{1}{2} + \cdots + \frac{1}{n^2})} \leq 1$$

Define a sequence of fuzzy functions $(\hat{f}_k)$ as

$$\hat{f}_k(x) = \begin{cases} \begin{cases} 1 + \frac{tk}{x}, & -\frac{x}{k} \leq t \leq 0 \\ 1 - \frac{tk}{x}, & 0 \leq t \leq \frac{x}{k} \end{cases}, & k = p^3 \text{ for some } p \in \mathbb{N} \\ \bar{0}, & \text{otherwise} \end{cases}$$

and so

$$t_k \, d\big(\hat{f}_k(x),\bar{0}\big) = \begin{cases} \frac{x}{k} + \frac{x}{k^2}, & k = p^3 \text{ for some } p \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore we have

$$\frac{1}{T^{\theta_2}_{\beta\gamma(n)}} \sum_{k\in[\beta_n,\gamma_n]} t_k \, d\big(\hat{f}_k(x),\hat{f}(x)\big) \;=\; \frac{1}{n^2 + (1 + \frac{1}{2} + \cdots + \frac{1}{n^2})} \sum_{\substack{k\in[1,n^2] \\ k=p^3}} \Big(\frac{x}{k} + \frac{x}{k^2}\Big)$$

$$\leq\; \frac{x}{n^2}\Big(1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{[n^{2/3}]}\Big) + \frac{x}{n^2}\Big(1 + \frac{1}{64} + \cdots + \frac{1}{[n^{4/3}]}\Big) \to 0$$

as $n \to \infty$. So, $(\hat{f}_k) \in N^{\theta_2}_{\beta\gamma}(t)$. Now

$$\frac{1}{T^{\theta_1}_{\beta\gamma(n)}} \Big| \big\{ k \leq T_{\beta\gamma(n)} : t_k \, d\big(\hat{f}_k(x),\bar{0}\big) \geq \varepsilon \big\} \Big| \;=\; \frac{[T^{1/3}_{\beta\gamma(n)}]}{T^{\theta_1}_{\beta\gamma(n)}}$$

$$\geq\; \frac{T^{1/3}_{\beta\gamma(n)} - 1}{T^{\theta_1}_{\beta\gamma(n)}}\;.$$

Choosing $\theta_1 < 1/3$, it is easy to see that R.H.S tends to ∞ as $n \to \infty$. So $(\hat{f}_k) \notin SP^{\theta_1}_{\beta\gamma}(t)$.

$\square$

Remark 3.11. *From the above two theorems it is clear that neither of the classes $SP^{\theta}_{\beta\gamma}(t)$ and $N^{\theta}_{\beta\gamma}(t)$ is contained in the other.*

4. Ordinary Weighted $\beta\gamma$ - summability of Order θ

Now we define a more general class of summability which is defined as follows:

Definition 4.1. *A sequence $(\hat{f}_k(x))$ of fuzzy functions is said to be* **ordinary weighted** $\beta\gamma$ **- summable** *to a fuzzy function $\hat{f}$ of order θ if for every $x \in [a, b]$,*

$$\hat{S}_{\beta\gamma(n)}(x) = \frac{1}{T^\theta_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, \hat{f}_k(x) \xrightarrow{d} \hat{f}(x) \ \text{as } n \to \infty$$

i.e. $d\left(\hat{S}_{\beta\gamma(n)}(x), \hat{f}(x)\right) \to 0$ as $n \to \infty$ for each $x \in [a, b]$. We denote this convergence as $\bar{N}^\theta_{\beta\gamma}(t)$. For $\theta = 1$, the class $\bar{N}^\theta_{\beta\gamma}(t)$ reduces to $\bar{N}_{\beta\gamma}(t)$.

Remark 4.2. *As particular cases, we get*

1. *For $t_n = 1, \beta_n = 1, \gamma_n = n$, $\bar{N}_{\beta\gamma}(t)$ is the Cesaro summability (non-absolute type) discussed in [21].*
2. *For $\beta_n = 1, \gamma_n = n$, $\bar{N}_{\beta\gamma}(t)$ reduces to Riesz mean as introduced in [22].*

Theorem 4.3. *Let $\hat{f}_k : [a, b] \to L(R)$ and $\hat{g}_k : [a, b] \to L(R)$ be two sequences of fuzzy functions. If $(\hat{f}_k), (\hat{g}_k) \in \bar{N}_{\beta\gamma}(t)$, then $(\hat{f}_k + \hat{g}_k), (c\hat{f}_k) \in \bar{N}_{\beta\gamma}(t)$ for $c \in \mathbb{R}$.*

Proof. Proof is easy. So we omit it. □

Theorem 4.4. *If a sequence of fuzzy functions is absolutely weighted $\beta\gamma$ - summable, then it is ordinary weighted $\beta\gamma$ - summable i.e. $N_{\beta\gamma}(t) \subset \bar{N}_{\beta\gamma}(t)$.*

Proof. Let $\left(\hat{f}_k(x)\right)$ be a sequence of fuzzy functions such that $\left(\hat{f}_k(x)\right) \in N_{\beta\gamma}(t)$ for a fuzzy function $\hat{f}(x)$. Now

$$
\begin{aligned}
d(\hat{S}_{\beta\gamma(n)}(x), \hat{f}(x)) &= d\left(\frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, \hat{f}_k(x), \hat{f}(x)\right) \\
&= d\left(\frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, \hat{f}_k(x), \ \frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, \hat{f}(x)\right) \\
&\leq \frac{1}{T_{\beta\gamma(n)}} \sum_{k \in [\beta_n, \gamma_n]} t_k \, d\left(\hat{f}_k(x), \hat{f}(x)\right).
\end{aligned}
$$

Taking $n \to \infty$ in both sides, we get the inclusion.
To show the inclusion is strict, we give the following example.

Example 4.5. *Let $t_n = 1, \beta_n = 1, \gamma_n = n$ and the sequence of fuzzy functions be defined for any $x \in [a, b]$ as*

$$\hat{f}_k(x) = \bar{a}_k \ \text{ where } a_k = (-1)^{k+1}$$

Then we have

$$
\begin{aligned}
\hat{S}_{\beta\gamma(n)}(x) &= \frac{1}{n} \sum_{k=1}^{n} \overline{(-1)^{k+1}} \\
&= \begin{cases} \bar{0}, & \text{if } n \text{ is even} \\ \frac{1}{n}\bar{1}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

i.e. $\hat{S}_{\beta\gamma(n)}(x) \ \to \ \bar{0}$ as $n \to \infty$.

So, $\left(\hat{f}_k(x)\right) \in \bar{N}_{\beta\gamma}(t)$. But

$$\frac{1}{n} \sum_{k \in [1,n]} d\left(\hat{f}_k(x), \bar{0}\right) = \frac{1}{n} \sum_{k \in [1,n]} 1 = 1$$

$$\text{i.e. } \left(\hat{f}_k(x)\right) \notin N_{\beta\gamma}(t).$$

This completes the proof. □

In order to get the generalized version of Tauberian theorem, given by [16], we note the following definition.

Definition 4.6. *[20] A sequence of fuzzy numbers* $(\hat{u}_n)$ *is said to be slowly decreasing if for every* $\varepsilon > 0$*, there exists* $n_0 = n_0(\varepsilon)$ *and* $\lambda = \lambda(\varepsilon) > 1$ *such that for all* $n > n_0$

$$\hat{u}_k \succeq \hat{u}_n - \bar{\varepsilon} \text{ whenever } n < k \le [\lambda n]$$

or equivalently, if for every $\varepsilon > 0$*, there exists* $n_0 = n_0(\varepsilon)$ *and* $\lambda = \lambda(\varepsilon) < 1$ *such that for all* $n > n_0$

$$\hat{u}_n \succeq \hat{u}_k - \bar{\varepsilon} \text{ whenever } [\lambda n] < k \le n.$$

Now on the basis of the above definition, we prove our results.
First we assume that (γ_n) be a real sequence taking integer values only and we use the notation $([\lambda \gamma]) = ([\lambda \gamma_n])$.

Lemma 4.7. *For the sequence* $(T_{\beta\gamma}(n))$*, defined in (1), the following conditions*

$$\liminf_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)}} > 1 \text{ for every } \lambda > 1 \tag{2}$$

$$\text{and } \liminf_{n\to\infty} \frac{T_{\beta\gamma(n)}}{T_{\beta([\lambda\gamma])(n)}} > 1 \text{ for every } 0 < \lambda < 1 \tag{3}$$

are equivalent.

Proof. Let (2) holds.
Set $m = (m_n) = [\lambda\gamma_n]$. Then $\frac{m}{\lambda} = \frac{[\lambda\gamma_n]}{\lambda} \le \gamma_n$ and thus

$$T_{\beta([m/\lambda])(n)} \le T_{\beta\gamma(n)}$$

and so
$$\frac{T_{\beta\gamma(n)}}{T_{\beta([\lambda\gamma_n])(n)}} \ge \frac{T_{\beta([m/\lambda])(n)}}{T_{\beta m(n)}}$$

As the denominators are same and thus

$$\liminf_{n\to\infty} \frac{T_{\beta\gamma(n)}}{T_{\beta([\lambda\gamma_n])(n)}} \ge \liminf_{n\to\infty} \frac{T_{\beta([m/\lambda])(n)}}{T_{\beta m(n)}} > 1$$

since $1/\lambda > 1$.
Now assume (3) holds.
Choose λ_1 be such that $1 < \lambda_1 < \lambda$. Set $\lambda_1 \le \lambda - 1/\gamma_n$ for large n. Then

$$\gamma_n \le \frac{[\lambda\gamma_n] - 1}{\lambda_1} < \frac{[\lambda\gamma_n]}{\lambda_1} = \frac{m}{\lambda_1}.$$

For such n, we have

$$T_{\beta\gamma(n)} \le T_{\beta([m/\lambda_1])(n)}$$

i.e.
$$\frac{T_{\beta([\lambda\gamma_n])(n)}}{T_{\beta\gamma(n)}} \ge \frac{T_{\beta m(n)}}{T_{\beta([m/\lambda_1])(n)}}$$

As the numerators are same, so we get

$$\liminf_{n\to\infty} \frac{T_{\beta([\lambda\gamma_n])(n)}}{T_{\beta\gamma(n)}} \ge \liminf_{n\to\infty} \frac{T_{\beta m(n)}}{T_{\beta([m/\lambda_1])(n)}} > 1$$

This completes the proof. □

Lemma 4.8. *If $\left(T_{\beta\gamma}(n)\right)$ satisfies (2) or (3), then the following conditions*

$$\limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} \quad < \quad \infty \text{ for every } \lambda > 1 \tag{4}$$

$$and \limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \quad < \quad \infty \text{ for every } 0 < \lambda < 1 \tag{5}$$

holds accordingly.

Proof. Let (2) holds. So for $\lambda > 1$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} &= \limsup_{n\to\infty} \frac{1}{1 - \frac{T_{\beta\gamma(n)}}{T_{\beta([\lambda\gamma])(n)}}} \\
&= \left\{ 1 - \limsup_{n\to\infty} \frac{T_{\beta\gamma(n)}}{T_{\beta([\lambda\gamma])(n)}} \right\}^{-1} \\
&= \left\{ 1 - \frac{1}{\liminf\limits_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)}}} \right\}^{-1} < \infty
\end{aligned}$$

by condition (2). Similarly by using (3), we can prove (5). $\quad\square$

Now on the basis of the above lemmas, we shall try to prove the following theorem.

Theorem 4.9. *Assume $(\hat{f}_k(x))$ be a sequence of fuzzy functions which is slowly decreasing and (2) is satisfied. Moreover let $(\hat{f}_k(x))$ is $\bar{N}_{\beta\gamma}(t)$ summable to a fuzzy function $\hat{f}(x)$, $x \in [a,b]$. Then the subsequence $\left(\hat{f}_{\gamma_k}(x)\right)$ of fuzzy functions converges to $\hat{f}(x)$ for all $x \in [a,b]$.*

Proof. Let $(\hat{f}_k(x))$ is $\bar{N}_{\beta\gamma}(t)$ summable to a fuzzy function $\hat{f}(x)$ and is slowly decreasing. Now consider these two cases.

Case 1 : Let $\lambda > 1$. Then for each n, $T_{\beta([\lambda\gamma])(n)} > T_{\beta\gamma(n)}$. So for each $x \in [a,b]$, we have

$$\begin{aligned}
\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) + \hat{S}_{\beta\gamma(n)}(x) &= \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} \frac{1}{T_{\beta([\lambda\gamma])(n)}} \sum_{k\in\left[\beta_n,[\lambda\gamma_n]\right]} t_k \hat{f}_k(x) \\
&\quad + \frac{1}{T_{\beta\gamma(n)}} \sum_{k\in[\beta_n,\gamma_n]} t_k \hat{f}_k(x) \\
&= \left[\frac{1}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} + \frac{1}{T_{\beta\gamma(n)}} \right] \sum_{k\in[\beta_n,\gamma_n]} t_k \hat{f}_k(x) \\
&\quad + \frac{1}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} \sum_{k\in\left[\gamma_n+1,[\lambda\gamma_n]\right]} t_k \hat{f}_k(x) \\
&= \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} \hat{S}_{\beta\gamma(n)}(x) \\
&\quad + \frac{1}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma}(n)} \sum_{k\in\left[\gamma_n+1,[\lambda\gamma_n]\right]} t_k \hat{f}_k(x). \tag{6}
\end{aligned}$$

It is given that $\lim\limits_{n\to\infty} \hat{S}_{\beta\gamma(n)}(x) = \hat{f}(x)$. Thus for each $x \in [a,b]$,

$$\hat{f}(x) - \frac{\bar{\varepsilon}}{3} \leq \hat{S}_{\beta\gamma(n)}(x) \leq \hat{f}(x) + \frac{\bar{\varepsilon}}{3}. \tag{7}$$

Also for each $x \in [a, b]$, we have

$$\limsup_{n\to\infty} d\Big(\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])}(n) - T_{\beta\gamma}(n)}\hat{S}_{\beta([\lambda\gamma])(n)}(x), \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x)\Big)$$

$$= \limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}} d\Big(\hat{S}_{\beta([\lambda\gamma])(n)}(x), \hat{S}_{\beta\gamma(n)}(x)\Big)$$

$$\leq \limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])}(n)}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\Big(d\Big(\hat{S}_{\beta([\lambda\gamma])(n)}(x), \hat{f}(x)\Big) + d\Big(\hat{S}_{\beta\gamma(n)}(x), \hat{f}(x)\Big)\Big)$$

$$\leq \Big(\limsup_{n\to\infty} \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma}(n)}\Big)\Big(\lim_{n\to\infty} d\Big(\hat{S}_{\beta([\lambda\gamma])(n)}(x), \hat{f}(x)\Big) + \lim_{n\to\infty} d\Big(\hat{S}_{\beta\gamma(n)}(x), \hat{f}(x)\Big)\Big) = 0\,.$$

since $(\hat{f}_k(x)) \in \bar{N}_{\beta\gamma}(t)$ and (4) holds. So for $\varepsilon > 0$ and for large n and for $x \in [a, b]$

$$\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x) - \frac{\bar{\varepsilon}}{3} \leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta([\lambda\gamma])(n)}(x)$$

$$\leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])}(n) - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x) + \frac{\bar{\varepsilon}}{3}. \tag{8}$$

and thus using (7) and (8), we get

$$\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x) + \frac{2\bar{\varepsilon}}{3} + \hat{f}(x) \geq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])}(n) - T_{\beta\gamma(n)}}\hat{S}_{\beta([\lambda\gamma])(n)}(x) + \hat{S}_{\beta\gamma(n)}(x)$$

$$= \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x)$$

$$+ \frac{1}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\sum_{k\in\left[\gamma_n+1,[\lambda\gamma_n]\right]} t_k\hat{f}_k(x)$$

$$\text{(from (6))}$$

$$\geq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta([\lambda\gamma])(n)} - T_{\beta\gamma(n)}}\hat{S}_{\beta\gamma(n)}(x) + \hat{f}_{\gamma_n}(x) - \frac{\bar{\varepsilon}}{3}$$

follows from the slowly decreasing property of $(\hat{f}_n(x))$ in the interval $\big(\gamma_n, [\lambda\gamma_n]\big]$. Thus we get

$$\hat{f}(x) + \bar{\varepsilon} \geq \hat{f}_{\gamma_n}(x). \tag{9}$$

Case 2 : Let $0 < \lambda < 1$. So for each n, $T_{\beta([\lambda\gamma])(n)} < T_{\beta\gamma(n)}$.

$$\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\hat{S}_{\beta([\lambda\gamma])(n)}(x) + \frac{1}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\sum_{k\in\left[[\lambda\gamma_n]+1,\gamma_n\right]} t_k\,\hat{f}_k(x)$$

$$= \frac{1}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\Big[\sum_{k\in\left[\beta_n,[\lambda\gamma_n]\right]} t_k\,\hat{f}_k(x) + \sum_{k\in[[\lambda\gamma_n]+1,\gamma_n]} t_k\,\hat{f}_k(x)\Big]$$

$$= \frac{1}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\sum_{k\in[\beta_n,\gamma_n]} t_k\,\hat{f}_k(x)$$

$$= \frac{T_{\beta\gamma(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\hat{S}_{\beta\gamma(n)}(x)$$

$$= \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}}\hat{S}_{\beta\gamma(n)}(x) + \hat{S}_{\beta\gamma(n)}(x). \tag{10}$$

Also proceeding as before and with the help of (5), we get

$$\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) - \frac{\bar{\varepsilon}}{3} \leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta\gamma(n)}(x)$$

$$\leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) + \frac{\bar{\varepsilon}}{3} . \tag{11}$$

Thus for each $x \in [a, b]$

$$\frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) + \hat{f}(x) - \frac{2\bar{\varepsilon}}{3}$$

$$\leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta\gamma(n)}(x) + \hat{S}_{\beta\gamma(n)}(x)$$

(from (7) and (11))

$$= \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) + \frac{1}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \sum_{k \in \left[[\lambda\gamma_n]+1,\gamma_n\right]} t_k \hat{f}_k(x)$$

(follows from (10))

$$\leq \frac{T_{\beta([\lambda\gamma])(n)}}{T_{\beta\gamma(n)} - T_{\beta([\lambda\gamma])(n)}} \hat{S}_{\beta([\lambda\gamma])(n)}(x) + \hat{f}_{\lambda_n}(x) + \frac{\bar{\varepsilon}}{3}$$

follows from the slowly-decreasing property of $\left(\hat{f}_n(x)\right)$ in $\left([\lambda\gamma_n], \gamma_n\right]$. Thus we have

$$\hat{f}(x) - \bar{\varepsilon} \leq \hat{f}_{\gamma_n}(x) . \tag{12}$$

Combining (9) and (12), we get

$$\hat{f}(x) - \bar{\varepsilon} \leq \hat{f}_{\gamma_n}(x) \leq \hat{f}(x) + \bar{\varepsilon}, \quad x \in [a, b] .$$

So, $\left(\hat{f}_{\gamma_k}(x)\right)$ converges to $\hat{f}(x)$ for all $x \in [a, b]$ as $k \to \infty$. □

Remark 4.10. *In case of $\gamma_n = n$ and $\beta_n = 1$, we get the Tauberian theorem as in [16] in case of fuzzy numbers.*

Conclusion

Here, in this paper, we have given a detailed discussion on weighted $\beta\gamma$ - summability of fuzzy functions for absolute and ordinary case. It is shown that the two classes $SP^\theta_{\beta\gamma}(t)$ and $N^\theta_{\beta\gamma}(t)$ form an incomparable class of sequences of fuzzy functions which is a different outcome compared to the results for other summability methods and the corresponding statistical convergence in the preceding articles in this direction.
Also, in the last theorem, we have extended the interval $[1, n]$ used in the earlier study of Tauberian theorem to any arbitrary interval $[\beta_n, \gamma_n]$ which is definitely a generalization of the previous versions of Tauberian theorem in the existing literature.

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