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Disease Extinction and Persistence in a Discrete-time SIS Epidemic Model with Vaccination and Varying Population Size

Rahman Farnoosh[a], Mahmood Parsamanesh[a]

[a] *School of Mathematics,Iran University of Science & Technology, Narmak, Tehran 16844, Iran*

Abstract. A discrete-time SIS epidemic model with vaccination is introduced and formulated by a system of difference equations. Some necessary and sufficient conditions for asymptotic stability of the equilibria are obtained. Furthermore, a sufficient condition is also presented. Next, bifurcations of the model including transcritical bifurcation, period-doubling bifurcation, and the Neimark-Sacker bifurcation are considered. In addition, these issues will be studied for the corresponding model with constant population size. Dynamics of the model are also studied and compared in detail with those found theoretically by using bifurcation diagrams, analysis of eigenvalues of the Jacobian matrix, Lyapunov exponents and solutions of the models in some examples.

1. Introduction

The spread and control of the infectious diseases is an important issue in the study of behavior of a population. Mathematical models and computer simulations have been extensively used in the study of epidemics. These models may be either continues- or discrete-time which are described by differential or difference equations, respectively[1, 2]. Many epidemic models are formulated as differential equations[3–6]. However, there is an increasing interest and popularity in using discrete models[7–9]. Discrete models may be found directly by discrete formulation of the model[10] or from continuous version of the model by using (forward/backward) Euler methods[11] or a non-standard finite difference (NSFD) scheme developed by Mickens[12]. We usually deal with discrete data, therefore difference equations appear as a more natural way to describe the epidemic models. Moreover, numerical solutions of differential equations use discretization and this encourages one to employ difference equations directly. Generally, dynamics of a continues-time model is determined by a threshold $\mathcal{R}_0$ which is called basic reproduction number. When $\mathcal{R}_0 < 1$, the disease extinction occurs and disease-free equilibrium is asymptotically stable, whereas when $\mathcal{R}_0 > 1$, the disease-free equilibrium is unstable and the disease will persist in the population. However, discrete-time epidemic models exhibit more complex dynamical behaviors, such as oscillations, chaos and bifurcations.

As an efficient strategy to control and eliminate infectious diseases, vaccination is included in many epidemic models among other behavioral changes such as, isolation and quarantine, mixing patterns, treatment, and education programs[12–16]. Vaccination may give permanent immunity or temporary

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Email addresses: rfarnoosh@iust.ac.ir (Rahman Farnoosh), m.parsamanesh@uoz.ac.ir (Mahmood Parsamanesh)

protection from disease. In this investigation, a discrete-time model of SIS type is developed for the spread of disease in a population with a vaccination program in effect. The vaccination is temporary and the immunity can be lost as time passes.
The continuous-time version of this model was considered in [14] and conditions for global stability of disease-free equilibrium and endemic equilibrium with respect to a threshold $\mathcal{R}_0$ were found. Indeed, the authors showed that if $\mathcal{R}_0 \leq 1$ then the disease-free equilibrium is globally stable in an invariant set, whereas if $\mathcal{R}_0 > 1$ then the endemic state is globally stable under a sufficient condition. As we will show in the following sections and it will be observed in numerical examples, the transcritical bifurcation, the period-doubling bifurcation, maybe the Neimark-Sacker bifurcation, and also chaotic and oscillatory behavior of solutions appear in the discrete-time epidemic model which are different from the dynamical behaviors in the corresponding continuous-time model.
The epidemic model composed with difference equations is formulated in the next section. In section 3, stability of the equilibria of the model are analyzed and the occurrence of bifurcations is considered. Section 4 is devoted to studying the model when population size is constant. At last, numerical simulations are carried out in section 5 to confirm the results obtained in previous sections. Bifurcation diagrams, eigenvalues and Lyapunov exponents of the model are also studied in some examples. In addition, chaos phenomenon and oscillation of solutions are studied numerically.

2. Model Description

According to the disease status, the population is divided to susceptible, infected and vaccinated individuals. The number of individuals in each sub-population at time t is denoted by S_t, I_t and V_t, respectively and $N_t = S_t + I_t + V_t$ denotes the total population size at time t. A constant number of individuals denoted by Λ enters the population per unit of time, in which the new members consist of immigrants and newborn individuals. A fraction q of new members are vaccinated and the rest are susceptible to infection. Vaccination is also done on susceptible individuals with the rate φ and vaccinated individuals lose their immunity with the rate θ. The vaccine assumed completely effective and no vaccinated individual becomes infected. The rate of new infectives at time t is $\beta S_t I_t$ and recovery rete is γ. Natural death rate and disease-caused death rate are μ and α respectively. All parameters are assumed to be non-negative. Moreover, μ and Λ are positive and $\mu, q, \varphi, \beta, \theta$ and γ are less than one. Therefore the system of difference equations of the model has the following form:

$$\begin{aligned} S_{t+1} &= (1-q)\Lambda - \beta S_t I_t + [1-(\mu+\varphi)]S_t + \gamma I_t + \theta V_t, \\ I_{t+1} &= \beta S_t I_t + [1-(\mu+\gamma+\alpha)]I_t, \\ V_{t+1} &= q\Lambda + \varphi S_t + [1-(\mu+\theta)]V_t. \end{aligned} \tag{1}$$

The following assumptions on the parameters ensure that solutions of (1) remain non-negative:

$$\begin{aligned} &0 < \mu + \varphi + \beta < 1, \\ &0 < \mu + \gamma + \alpha < 1, \\ &0 < \mu + \varphi + \theta < 1. \end{aligned} \tag{2}$$

Note that the total population size N_t is not constant over time,

$$N_{t+1} = \Lambda + (1-\mu)N_t - \alpha I_t.$$

But we have

$$N_{t+1} \leq (1-\mu)^{t+1}N(0) + [1-(1-\mu)^{t+1}]\frac{\Lambda}{\mu}. \tag{3}$$

This implies that $N_t \leq N(0) + \frac{\Lambda}{\mu}$ for all $t > 0$, and moreover, $N(\infty) \leq \frac{\Lambda}{\mu}$. Therefore total population is always bounded.

3. Analysis of Stability

3.1. Equilibria and Asymptotic Stability

The equilibria of the model (1) are solutions of the following system;

$$\begin{aligned}(\mu+\varphi)\bar{S} &= (1-q)\Lambda - \beta\bar{S}\bar{I} + \gamma\bar{I} + \theta\bar{V},\\ (\mu+\gamma+\alpha)\bar{I} &= \beta\bar{S}\bar{I},\\ (\mu+\theta)\bar{V} &= q\Lambda + \varphi\bar{S}.\end{aligned} \tag{4}$$

We obtain two equilibria: a disease-free equilibrium $E_0 = (I_0, S_0, V_0)^T$ where $I_0 = 0$, $S_0 = \frac{\Lambda(\mu(1-q)+\theta)}{\mu(\mu+\theta+\varphi)}$ and $V_0 = \frac{\Lambda(\mu q+\varphi)}{\mu(\mu+\theta+\varphi)}$, and an endemic equilibrium $E^* = (I^*, S^*, V^*)^T$ where $I^* = \frac{\beta\Lambda((1-q)\mu+\theta)-\mu(\mu+\gamma+\alpha)(\mu+\theta+\varphi)}{\beta(\mu+\alpha)(\mu+\theta)}$, $S^* = \frac{\mu+\gamma+\alpha}{\beta}$ and $V^* = \frac{q\Lambda+\varphi(\mu+\gamma+\alpha)/\beta}{\mu+\theta}$. Positive I^* exists when

$$\mathcal{R}_0 = \frac{\beta\Lambda((1-q)\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\theta+\varphi)} > 1. \tag{5}$$

The expression $\mathcal{R}_0$ is referred to as the basic reproduction number for this model. On the other hand it is obvious that disease-free equilibrium E_0 always exists for all values of $\mathcal{R}_0$. Thus we have the following:

Theorem 3.1. *If $\mathcal{R}_0 \leq 1$ then there exists only disease-free equilibrium E_0 for the model (1) and the model has only two equilibria E_0 and E^* obtained from (4), if $\mathcal{R}_0 > 1$.*

The Jacobian matrix of model (1) at $\bar{X} = (\bar{I}, \bar{S}, \bar{V})^T$ has the following form:

$$J(X) = \begin{bmatrix} 1-(\mu+\gamma+\alpha)+\beta\bar{S} & \beta\bar{I} & 0\\ -\beta\bar{S}+\gamma & 1-(\mu+\varphi)-\beta\bar{I} & \theta\\ 0 & \varphi & 1-(\mu+\theta)\end{bmatrix}. \tag{6}$$

Eigenvalues of the Jacobian matrix J at an equilibrium determine the dynamic of the nonlinear system. If the eigenvalues satisfy $|\lambda_i| < 1$, that is, the spectral radius is less than one, $\rho(J) < 1$, then $\lim_{t\to\infty} J^t = 0$. Letting $Y_t = X_t - \bar{X}$ we have $Y_t = (J^t(\bar{X}))Y_0 \to 0$ as $t \to \infty$ and this concludes local asymptotic stability. The Jacobian matrix at the disease-free equilibrium can also be used to obtain the basic reproduction number $\mathcal{R}_0$ for the model[17].
The Jacobian matrix at disease-free equilibrium E_0 is

$$J_0 = J(E_0) = \begin{bmatrix} 1-(\mu+\gamma+\alpha)+\beta S_0 & 0 & 0\\ -\beta S_0+\gamma & 1-(\mu+\varphi) & \theta\\ 0 & \varphi & 1-(\mu+\theta)\end{bmatrix}. \tag{7}$$

Eigenvalues of J_0 are $\lambda_1 = 1-(\mu+\gamma+\alpha)+\beta S_0$ and those for the following sub-matrix

$$\begin{bmatrix} 1-(\mu+\varphi) & \theta\\ \varphi & 1-(\mu+\theta)\end{bmatrix}, \tag{8}$$

that are $\lambda_2 = 1-\mu$ and $\lambda_3 = 1-(\mu+\varphi+\theta)$. For two last eigenvalues we have $|\lambda_{2,3}| < 1$ by assumptions (2). But $|\lambda_1| < 1$ implies $\frac{\beta S_0}{\mu+\gamma+\alpha} < 1$, i.e. $\mathcal{R}_0 < 1$.
For endemic equilibrium E^* the Jacobian matrix is

$$J^* = J(E^*) = \begin{bmatrix} 1 & \beta I^* & 0\\ -(\mu+\alpha) & 1-(\mu+\varphi+\beta I^*) & \theta\\ 0 & \varphi & 1-(\mu+\theta)\end{bmatrix}. \tag{9}$$

The characteristic equation of matrix J^* is

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{10}$$

where

$$\begin{aligned} a_1 &= -tr(J^*) = -3 - b_1, \\ a_2 &= -\tfrac{1}{2}[tr(J^{*2}) - tr^2(J^*)] = 3 + 2b_1 + b_2, \\ a_3 &= -det(J^*) = -1 - b_1 - b_2 + b_3, \end{aligned} \tag{11}$$

with

$$\begin{aligned} b_1 &= -[(\mu + \varphi + \beta I^*) + (\mu + \theta)] < 0, \\ b_2 &= \beta I^*(\mu + \alpha) - \theta\varphi + (\mu + \varphi + \beta I^*)(\mu + \theta) = \beta I^*(2\mu + \alpha + \theta) + \mu(\mu + \varphi + \theta) > 0, \\ b_3 &= (\mu + \alpha)(\mu + \theta)\beta I^* > 0. \end{aligned} \tag{12}$$

Theorem 3.2. *Under assumptions of the model (1),*

(I) Disease-free equilibrium E_0 is asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 \geq 1$;

(II) Endemic equilibrium E^* is asymptotically stable if

(1) $1 - a_1 + a_2 - a_3 > 0$,

(2) $1 - a_3^2 > \mid a_2 - a_1 a_3 \mid$,

and is unstable otherwise;

(III) If in addition $\mathcal{R}_0 < 1 + \frac{(\mu+\alpha)(\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}$, then the condition (2) in part (II) is sufficient for asymptotic stability of E^*.

Proof. Preceding discussion proves (I). For (II) we use the Jury conditions. According to the Jury conditions, the roots $\lambda_i, i = 1, 2, 3$, of the characteristic equation (10) satisfy $\mid \lambda_i \mid < 1$ if and only if the following three conditions hold[18]:

(i) $P(1) = 1 + a_1 + a_2 + a_3 > 0$,

(ii) $-P(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(iii) $1 - a_3^2 > \mid a_2 - a_1 a_3 \mid$.

We have $P(1) = 1 + a_1 + a_2 + a_3 = b_3$ and therefore (i) obviously is satisfied.
Now, if in addition $\mathcal{R}_0 < 1 + \frac{(\mu+\alpha)(\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}$ which holds if and only if $\beta I^* < 1$, then

$$\begin{aligned} -P(-1) &= 8 + 4b_1 + 2b_2 - b_3 \\ &= 2(2 - \mu)(2 - (\mu + \varphi + \theta)) - \beta I^*(2 - (\mu + \alpha))(2 - (\mu + \theta)) \\ &> 2(2 - \mu)(2 - (\mu + \varphi + \theta)) - (2 - (\mu + \alpha))(2 - (\mu + \theta)) > 0, \end{aligned} \tag{13}$$

because $(2 - (\mu + \alpha)) < (2 - \mu)$ and $2(2 - (\mu + \varphi + \theta)) > 2 > (2 - (\mu + \theta))$. Hence, in this case condition (ii) also holds and only condition (iii) concludes asymptotic stability of E^*. This completes the proof of part (III). □

We shall now consider a sufficient condition under which the model is stable. Let $X(t + 1) = G(X(t))$ describes the model (1). We already showed in (3) that the total population size is always bounded. Thus, $I_t \leq N(0) + \frac{\Lambda}{\mu}$ and therefore

$$\begin{aligned} S_{t+1} &\leq (1 - q)\Lambda + [1 - (\mu + \varphi)]S_t + \gamma I_t + \theta V_t, \\ I_{t+1} &\leq \beta(N(0) + \tfrac{\Lambda}{\mu})S_t + [1 - (\mu + \gamma + \alpha)]I_t, \\ V_{t+1} &= q\Lambda + \varphi S_t + [1 - (\mu + \theta)]V_t. \end{aligned} \tag{14}$$

Now let $X(t+1) \leq F(X(t))$ describes (14). The Jacobian matrix related to F is

$$\tilde{J} = \begin{bmatrix} 1-(\mu+\gamma+\alpha) & \beta(N(0)+\frac{\Lambda}{\mu}) & 0 \\ \gamma & 1-(\mu+\varphi) & \theta \\ 0 & \varphi & 1-(\mu+\theta) \end{bmatrix}. \tag{15}$$

If one take $Y_t = X_t - \bar{X}$ then $Y_t \leq \tilde{J}Y_{t-1}$. Because $\tilde{J}$ is a matrix with non-negative components, we find that $Y_t \leq \tilde{J}^t Y_0$. Now if $\lim_{t\to\infty} \tilde{J}^t = 0$ then $Y_t \to 0$ as $t \to \infty$. Some simple sufficient but not necessary conditions for local asymptotic stability of the equilibria use matrix norms; If all of the absolute column sums are less than one, then the absolute value of all eigenvalues of the Jacobian matrix are less than one. For matrix $\tilde{J}$,

$$\|\tilde{J}\|_1 = max\{1-(\mu+\alpha), 1-\mu+\beta(N(0)+\frac{\Lambda}{\mu}), 1-\mu\}. \tag{16}$$

From $\rho(\tilde{J}) \leq \|\tilde{J}\|$, we conclude that sufficient condition to spectral radius be less than one is $\mu > \beta(N(0)+\frac{\Lambda}{\mu})$, because obviously other inequalities hold. Therefore the following was proven:

Theorem 3.3. *Under assumptions of the model (1), a sufficient condition for local asymptotic stability of the model is that* $\Lambda < \mu(\frac{\mu}{\beta} - N(0))$.

Remark 3.4. *If the model considered without vaccination, the compartment V is removed, we have a model of SIS type and its dynamics is obtained by letting q, φ and θ be zero in the counterpart model with vaccination. The basic reproduction number* $\mathcal{R}_0$ *in (5) is reduced by increasing the values of q and φ. This indicates the direct effect of vaccination in control and eliminate the disease.*

3.2. Bifurcations

When the Jacobian matrix has eigenvalues with the modules equal to one, bifurcations may appear in non-hyperbolic equilibrium. Transcritical (fold) bifurcation or period-doubling (flip) bifurcation takes place if for a real eigenvalue we have $\lambda_i = 1$ or $\lambda_i = -1$, respectively. Whereas, the Neimark-Sacker bifurcation occurs when the Jacobian matrix has a pair of conjugate complex eigenvalues with the modules one. This type of bifurcation corresponds to the Hopf bifurcation in continuous-time systems[19]. Indeed, the third-degree polynomial (10) has either three real roots if $\Delta \leq 0$ or one real root and a pair of conjugate complex roots if $\Delta > 0$[8, 20]. Here, $\Delta = B^2 - 4AC$ and

$$\begin{aligned} A &= a_1^2 - 3a_2, \\ B &= a_1a_2 - 9a_3, \\ C &= a_2^2 - 3a_1a_3. \end{aligned} \tag{17}$$

The next theorem considers bifurcations of the model:

Theorem 3.5. *For model (1)*

(I) Transcritical bifurcation occurs if $\mathcal{R}_0 = 1$;

(II) Period-doubling bifurcation takes place if $\mathcal{R}_0 = 1 + \left(\frac{(\mu+\alpha)(\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}\right)\left(\frac{2(2-\mu)(2-(\mu+\varphi+\theta))}{(2-(\mu+\alpha))(2-(\mu+\theta))}\right)$;

(III) The Niemark-Sacker bifurcation occurs if and only if

(1) $1 - a_1 + a_2 - a_3 > 0$,

(2) $|\, a_3 \,| < 1$,

(3) $a_2 - a_1a_3 = 1 - a_3^2$.

Furthermore, if $\mathcal{R}_0 < 1 + \frac{(\mu+\alpha)(\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}$, *the Niemark-Sacker bifurcation doesn't occur.*

Proof. As discussed before, the eigenvalues of the Jacobian matrix at equilibrium E_0 were obtained explicitly and we found that its three simple eigenvalues are real and thus the Niemark-Sacker bifurcation doesn't take place for E_0. Nevertheless, we saw that for E_0 we have $|\lambda_{2,3}|<1$ and $\lambda_1 = 1-(\mu+\gamma+\alpha)+\beta S_0$ where $S_0 = \left(\frac{\mu+\gamma+\alpha}{\beta}\right)\mathcal{R}_0$. Therefore transcritical bifurcation occurs when $\lambda_1 = 1$ i.e. $\mathcal{R}_0 = 1$ and period-doubling bifurcation takes place when $\lambda_1 = -1$ which implies $\mathcal{R}_0 = 1 - \frac{2}{\mu+\gamma+\alpha}$. But, this can not be accepted because $0 < \mu+\gamma+\alpha < 1$ and hence $\mathcal{R}_0 < 0$.
For equilibrium E^*, transcritical and period-doubling bifurcation may occur either when all three eigenvalues are real (i.e., $\Delta \leq 0$) and one of them has module one, or when there are two complex and one real eigenvalues (i.e., $\Delta > 0$) from which two complex eigenvalues lie inside the unit circle and the real eigenvalue is of the module one. However, the Niemark-Sacker bifurcation may appear when $\Delta > 0$ and two complex eigenvalues lie on the unit circle while the real one lies inside the unit circle.
Note that $\lambda = 1$ means that for the characteristic equation (10) we must have $P(1) = 0$. Thus transcritical bifurcation occurs when $P(1) = b_3 = 0$. Hence, $(\mu+\alpha)(\mu+\theta)\beta I^* = 0$ and therefore $\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)(\mathcal{R}_0-1) = 0$ and so $\mathcal{R}_0 = 1$. As the same way, for period-doubling bifurcation we must have $P(-1) = 0$. This yields $8+4b_1+2b_2-b_3 = 0$ and from (13) we have

$$\beta I^* = \frac{2(2-\mu)(2-(\mu+\varphi+\theta))}{(2-(\mu+\alpha))(2-(\mu+\theta))}, \tag{18}$$

which concludes the desired result.
For matrix J^* with the characteristic equation (10), a pair of complex conjugate eigenvalues lie on the unit circle and another eigenvalue lies inside the unit circle if and only if the following conditions hold[21]:

(i) $P(1) = 1+a_1+a_2+a_3 > 0$,

(ii) $-P(-1) = 1-a_1+a_2-a_3 > 0$,

(iii) $|a_3| < 1$,

(iv) $a_2 - a_1a_3 = 1-a_3^2$.

Condition (i) holds as in proof of Theorem 3.2. Now suppose that $\mathcal{R}_0 < 1+\frac{(\mu+\alpha)(\mu+\theta)}{\mu(\mu+\gamma+\alpha)(\mu+\varphi+\theta)}$ which is equivalent to $\beta I^* < 1$. Thus as in proof of part (3) in Theorem 3.2, condition (ii) holds. Besides,

$$\begin{aligned} -b_1-b_2 &= 2\mu+\varphi+\theta+\beta I^*-(2\mu+\alpha+\theta)\beta I^*-\mu(\mu+\varphi+\theta) \\ &> \mu+\varphi+\theta+\beta I^*-(\mu+\alpha)\beta I^*-(\mu+\theta)\beta I^* \\ &> \varphi+(\mu+\theta)+\beta I^*-\beta I^*-(\mu+\theta) = \varphi, \end{aligned}$$

thus $-b_1-b_2+b_3 > 0$ and $a_3 > -1$. Moreover, relations

$$\begin{aligned} 2+b_1 &= 2-(2\mu+\varphi+\theta+\beta I^*) \\ &> 1-\mu-(\mu+\varphi+\theta)+\mu(\mu+\varphi+\theta) \\ &= (1-\mu)(1-(\mu+\varphi+\theta)) > 0, \end{aligned}$$

and

$$b_2-b_3 = ((\mu+\alpha)+(\mu+\theta)-(\mu+\alpha)(\mu+\theta))\beta I^* > 0,$$

imply that $2+b_1+b_2-b_3 > 0$ and $a_3 < 1$. Hence $|a_3| < 1$ and condition (iii) holds too.
Condition (iv) can be written as

$$a_2 - 1 = a_1a_3 - a_3^2 = a_3(a_1-a_3),$$

and it is equivalent to

$$b_3 + (b_1 + b_2 - b_3)(b_2 - b_3) = 0.$$

Solving nonlinear programming problem

$$Maximization \quad b_3 + (b_1 + b_2 - b_3)(b_2 - b_3),$$

subject to assumptions of the model as constraints, we find that $b_3 + (b_1 + b_2 - b_3)(b_2 - b_3) < 0$ and thus condition (iv) is not satisfied. This implies that the Neimark-Sacker bifurcation can not be occurred. □

4. The Model with Constant Population Size

The population size N_t in the model (1) is non-constant although it is bounded. Assuming $\Lambda = \mu N$ and $\alpha = 0$, the population size remains constant value $N = N(0)$. We restrict our attention to the reduced model described by the system of following two equations:

$$\begin{aligned} S_{t+1} &= [(1-q)\mu + \theta]N - \beta S_t I_t + [1 - (\mu + \varphi + \theta)]S_t + (\gamma - \theta)I_t, \\ I_{t+1} &= \beta S_t I_t + [1 - (\mu + \gamma)]I_t. \end{aligned} \tag{19}$$

Sufficient conditions to have non-negative solutions are $0 < \mu + \varphi + \theta + \beta < 1$ and $0 < \mu + \gamma < 1$.

4.1. Stability Analysis

There are two equilibria for the model (19): the disease-free equilibrium $Q_0 = (I_0, S_0)^T = \left(0, \frac{[(1-q)\mu+\theta]N}{\mu+\varphi+\theta}\right)^T$ and the endemic equilibrium $Q^* = (I^*, S^*)^T = \left(\frac{[(1-q)\mu+\theta]\beta N-(\mu+\varphi+\theta)(\mu+\gamma)}{\beta(\mu+\theta)}, \frac{\mu+\gamma}{\beta}\right)^T$. The endemic equilibrium exists if $\mathcal{R}_0 = \frac{\beta N[(1-q)\mu+\theta]}{(\mu+\varphi+\theta)(\mu+\gamma)} > 1$, where $\mathcal{R}_0$ represents the basic reproduction number of the model (19). Moreover, the Jacobian matrix of the model in $\bar{X} = (\bar{I}, \bar{S})^T$ is:

$$J(X) = \begin{bmatrix} 1 - (\mu + \gamma) + \beta\bar{S} & \beta\bar{I} \\ -\beta\bar{S} + (\gamma - \theta) & 1 - (\mu + \varphi + \theta) - \beta\bar{I} \end{bmatrix}. \tag{20}$$

The Jacobian matrix at Q_0 is

$$J_0 = J(Q_0) = \begin{bmatrix} 1 - (\mu + \gamma) + (\mu + \gamma)\mathcal{R}_0 & 0 \\ (\mu + \gamma)\mathcal{R}_0 + (\gamma - \theta) & 1 - (\mu + \varphi + \theta) \end{bmatrix}, \tag{21}$$

therefore eigenvalues of the Jacobian matrix are obtained as $\lambda_1 = 1 - (\mu + \gamma)(1 - \mathcal{R}_0)$ and $\lambda_2 = 1 - (\mu + \varphi + \theta)$. $|\lambda_2| < 1$ holds from assumptions of the model and $|\lambda_1| < 1$ if and only if $\mathcal{R}_0 < 1$.
Also, the Jacobian matrix at Q^* is given by

$$J^* = J(Q^*) = \begin{bmatrix} 1 & \frac{(\mu+\varphi+\theta)(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0 - 1) \\ -(\mu + \theta) & 1 - (\mu + \varphi + \theta) - \frac{(\mu+\varphi+\theta)(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0 - 1) \end{bmatrix}. \tag{22}$$

The characteristic equation of J^* is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{23}$$

where

$$\begin{aligned} a_1 &= -tr(J^*) = 2 + b_1, \\ a_2 &= det(J^*) = 1 + b_1 + b_2, \end{aligned} \tag{24}$$

in which b_1 and b_2 are given by

$$\begin{aligned} b_1 &= -[(\mu+\varphi+\theta)(1+\tfrac{(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0-1))] < 0, \\ b_2 &= (\mu+\varphi+\theta)(\mu+\gamma)(\mathcal{R}_0-1) > 0. \end{aligned} \tag{25}$$

The Jury conditions for local asymptotic stability for characteristic equation of the Jacobian matrix J^* are compacted as the following inequalities[18]:

$$\mid tr(J^*) \mid < 1 + det(J^*) < 2. \tag{26}$$

First, $b_2 > 0$, thus $-2 - b_1 - b_2 < -2 - b_1$ and $-(1 + det(J^*)) < tr(J^*)$ holds. Second, by substituting from preceding values we see that $det(J^*) < 1$ is identical to $b_1 + b_2 < 0$. This inequality holds because

$$(\mu+\gamma)(\mathcal{R}_0-1) < 1 + \frac{(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0-1). \tag{27}$$

Thirdly, $tr(J^*) < 1 + det(J^*)$ holds if and only if $4 + 2b_1 + b_2 > 0$ and thus if and only if

$$\mathcal{R}_0 < 1 + \left(\frac{(\mu+\theta)}{(\mu+\varphi+\theta)(\mu+\gamma)}\right)\left(\frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}\right). \tag{28}$$

We summarize these results in the following theorem:

Theorem 4.1. *Under assumptions of the model (19),*

(I) If $\mathcal{R}_0 \leq 1$ then there exists only disease-free equilibrium Q_0 for the model (19). Also, the model has only two equilibria Q_0 and Q^*, if $\mathcal{R}_0 > 1$.

(II) Disease-free equilibrium Q_0 is asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 \geq 1$;

(III) Endemic equilibrium Q^* is asymptotically stable if $1 < \mathcal{R}_0 < 1 + \left(\frac{(\mu+\theta)}{(\mu+\varphi+\theta)(\mu+\gamma)}\right)\left(\frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}\right)$ and is unstable otherwise.

4.2. Bifurcations

As mentioned earlier, eigenvalues of the Jacobian matrix at Q_0 are $\lambda_1 = 1 - (\mu+\gamma)(1-\mathcal{R}_0)$ and $\lambda_2 = 1 - (\mu+\varphi+\theta)$. Because $\mid \lambda_2 \mid < 1$, bifurcations may occur when $\mid \lambda_1 \mid = 1$. Transcritical Bifurcation occurs when $\lambda_1 = 1$ that implies $\mathcal{R}_0 = 1$. On the other hand when $\lambda_1 = -1$ and thus $\mathcal{R}_0 = 1 - \frac{2}{\mu+\gamma}$, period-doubling bifurcation may take place. But, $\mathcal{R}_0 < 0$ because $0 < \mu + \gamma < 1$ and this concludes that period-doubling bifurcation doesn't occur for Q_0. Furthermore, since both eigenvalues are real, the Niemark-Sacker bifurcation can not be appeared for Q_0.
The coefficients of the characteristic equation (23) for Jacobian matrix at Q^* are

$$\begin{aligned} a_1 &= -2 + (\mu+\varphi+\theta) + \beta I^*, \\ a_2 &= 1 - (\mu+\varphi+\theta) - \beta I^* + \beta I^*(\mu+\theta), \end{aligned} \tag{29}$$

where $\beta I^* = \frac{(\mu+\varphi+\theta)(\mu+\gamma)}{(\mu+\theta)}(\mathcal{R}_0-1)$.
We have

$$\begin{aligned} a_1^2 - 4a_2 &= [-2 + (\mu+\varphi+\theta) + \beta I^*]^2 - 4[1 - (\mu+\varphi+\theta) - \beta I^* + \beta I^*(\mu+\theta)] \\ &= (\mu+\theta)^2 + (\beta I^*)^2 - 2(\mu+\theta)\beta I^* + \varphi^2 + 2(\mu+\theta)\varphi + 2\varphi\beta I^* \\ &= (\mu+\theta-\beta I^*)^2 + \varphi^2 + 2\varphi(\mu+\theta+\beta I^*) > 0. \end{aligned}$$

Hence, the characteristic equation (23) has two real roots and thus the Neimark-Sacker bifurcation can not be taken place for Q^* as for Q_0. On the other hand as discussed in previous section, tarnscritical

bifurcation occurs when $P(1) = 0$. This implies that $P(1) = b_2 = (\mu + \varphi + \theta)(\mu + \gamma)(\mathcal{R}_0 - 1) = 0$ and thus $\mathcal{R}_0 = 1$. Furthermore, $P(-1) = 0$ yields to period-doubling bifurcation. Thus $4 + 2b_1 + b_2 = 0$ and $4-2[(\mu+\varphi+\theta)+\beta I^*]+\beta I^*(\mu+\theta) = 0$. Hence, $\beta I^* = \frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}$ and as a result $\mathcal{R}_0 = 1+\Big(\frac{(\mu+\theta)}{(\mu+\varphi+\theta)(\mu+\gamma)}\Big)\Big(\frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}\Big)$. Consequently, we have the following:

Theorem 4.2. *For the SIS epidemic model with vaccination and constant population size,*

(I) Transcritical bifurcation appears when $\mathcal{R}_0 = 1$;

(II) Perid-doubling bifurcation occurs when $\mathcal{R}_0 = 1 + \Big(\frac{(\mu+\theta)}{(\mu+\varphi+\theta)(\mu+\gamma)}\Big)\Big(\frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}\Big)$;

(III) The Neimark-Sacker bifurcation can not be appeared.

5. Numerical Simulations

In this section, we present some numerical simulations of systems (1) and (19) to illustrate our results. A parameter is taken as bifurcation parameter and asymptotic stability of the equilibria is considered for it. Also, eigenvalues and the Lyapunov exponent of the Jacobian matrix are shown with respect to the bifurcation parameter. Moreover, dynamical behavior of the model compartments are given in some graphs.

Example 5.1. *Take parameters in model (1) as $q = 0.8, \beta = 0.6, \mu = 0.1, \varphi = 0.2, \alpha = 0.2, \gamma = 0.3$ and $\theta = 0.2$.*

Above values satisfy the conditions assumed for parameters of the model. Assume that the unit population size suggests one million and initial number of individuals in each sub-population is $I(0) = 0.4$, $S(0) = 0.8$ and $V(0) = 0.5$. We take Λ as bifurcation parameter. Figure 1 shows a bifurcation diagram for infected population I_t as a function of $\Lambda \in [0, 2]$. When $\Lambda < 0.23$ the disease-free equilibrium E_0, where $\bar{I} = I_0 = 0$, is stable. At $\Lambda \approx 0.23$, the largest eigenvalue λ_1 is near one; $\lambda_1 \approx 1$, and a transcritical bifurcation occurs. For $0.23 < \Lambda < 1.59$ the endemic equilibrium E^*, where $\bar{I} = I^* > 0$, is stable. When $\Lambda \approx 1.59$ there is a period-doubling bifurcation, because one of eigenvalues of the Jacobian matrix is -1 (i.e., the spectral radius equals one). These observations is also confirmed by applying Theorem 3.2: the disease-free equilibrium is stable if $\mathcal{R}_0 < 1$ and otherwise is unstable. This implies that E_0 is stable for $\Lambda < 0.227$ and unstable otherwise. Part (2) of Theorem 3.2 gives conditions for stability of E^* as $0.227 < \Lambda < 1.585$. Figure 2 illustrates absolute value of eigenvalues of the Jacobian matrix evaluated at equilibria for $\Lambda \in [0, 2]$. When $\mathcal{R}_0 < 1$ the Jacobian matrix is evaluated at E_0 and absolute value of corresponding real eigenvalues are shown by open circles. However, E^* is used for evaluating the Jacobian matrix when $\mathcal{R}_0 > 1$ and closed circles show $\mid \lambda_i \mid, i = 1, 2, 3$. In this case the Jacobian matrix has one real and two complex pair eigenvalues for $\Lambda > 0.240$. Eigenvalues are shown in complex plane for $\Lambda \in [0.25, 1.85]$. Open and closed triangles show the eigenvalues at beginning and end values of Λ. The complex eigenvalues remain in the unit circle but the real eigenvalue crosses the unit circle at $\lambda = -1$ for $\Lambda = 1.59$. Figure 3 graphs the Lyapunov exponents of the model for values of Λ on the interval $[0, 2]$ when initial value mentioned in Example 5.1. It can be seen that the Lyapunov exponents are positive for many values of $\Lambda > 1.88$. Thus for these values the model is unstable and has chaotic behavior. In addition, the Lyapunov exponents are very close to zero for $\Lambda = 0.23$ and positive for $\Lambda = 1.58$ and 1.82. As it was seen in Figure 1, bifurcation occurs for these values. Figure 4 illustrates dynamical behavior of number of individuals in susceptible, infected and vaccinated sub-populations for various values of Λ. For $\Lambda = 0.15$, $\mathcal{R}_0 < 1$ and for $\Lambda = 0.6, 1.6$ and 1.9, $\mathcal{R}_0 > 1$. On the other hand, when $\Lambda = 0.15$ extinction occurs and the disease-free equilibrium is stable, for $\Lambda = 0.6$ disease will persist in population and the endemic equilibrium is stable, and for $\Lambda = 1.6$ and 1.9 the system is not stable and has oscillatory behavior.

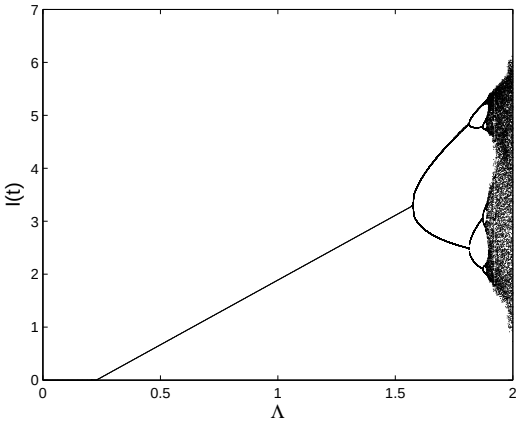

Figure 1: A bifurcation diagram for infected population I_t as a function of Λ in model (1) with parameter values in Example 5.1.

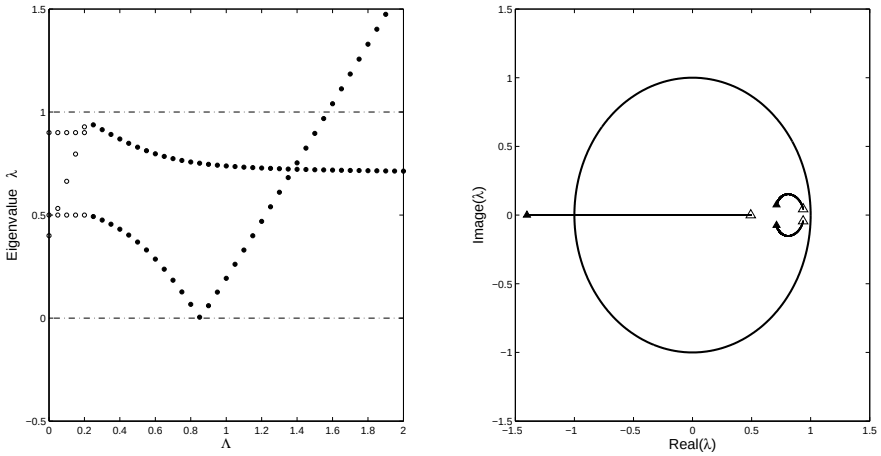

Figure 2: Eigenvalues of the Jacobian matrix as a function of Λ in model (1) with parameter values in Example 5.1.

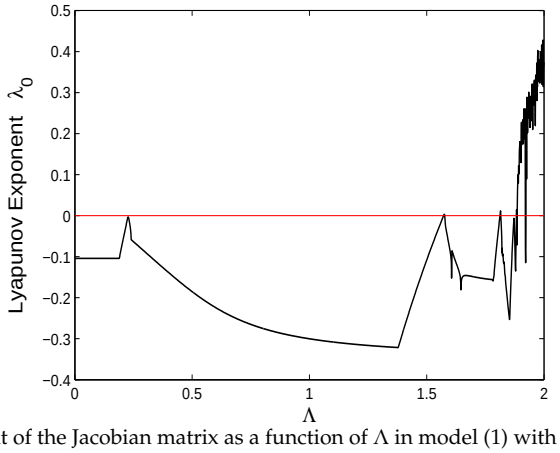

Figure 3: Lyapunov exponent of the Jacobian matrix as a function of Λ in model (1) with parameter values in Example 5.1.

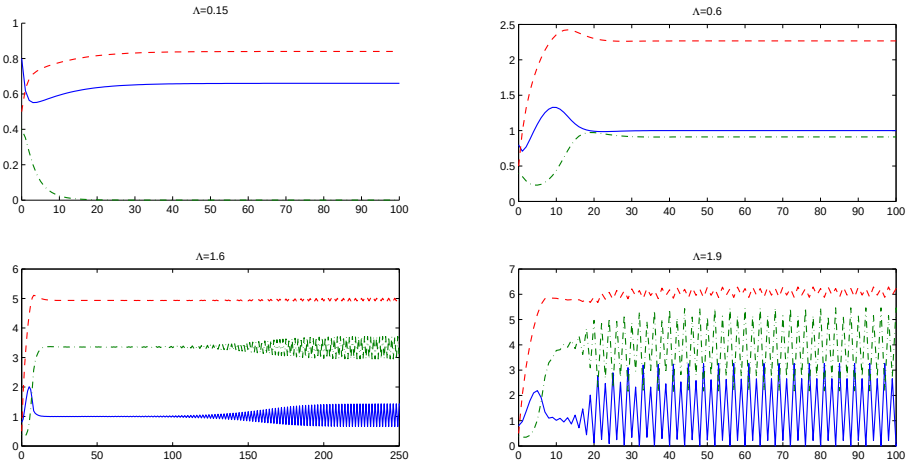

Figure 4: Diagrams of S_t ('-'line), I_t ('-.'line) and V_t ('- -'line) for values of Λ in model (1) with parameter values in Example 5.1.

Example 5.2. *Choose* $q = 0.6, \mu = 0.1, \varphi = 0.2, \gamma = 0.3$ *and* $\theta = 0.2$ *as parameters in model (19). Assume that the unit population size is one million and total population size is constant* $N = 4.2$ *with* $I(0) = 0.8$ *and* $S(0) = 2.4$.

In this example we choose β as bifurcation parameter. According to Theorem 4.2 for $\mathcal{R}_0 = 1$ and $\mathcal{R}_0 = 1 + \left(\frac{(\mu+\theta)}{(\mu+\varphi+\theta)(\mu+\gamma)}\right)\left(\frac{4-2(\mu+\varphi+\theta)}{2-(\mu+\theta)}\right) = 3.647$, transcritical bifurcation and period-doubling bifurcation appear respectively. Thus we have these bifurcations for $\beta = 0.198$ and $\beta = 0.724$ respectively. Furthermore, Theorem 4.1 says that Q_0 and Q^* are stable for $\beta \in (0, 0.198)$ and $\beta \in (0.198, 0.724)$ respectively. Figure 5 displays a bifurcation diagram for infected sub-population and eigenvalues of the Jacobian matrix (20) as a function of $\beta \in (0, 1)$. These diagrams confirm the theoretical results and the trajectories for system (19) with various values β can be seen in Figure 6.

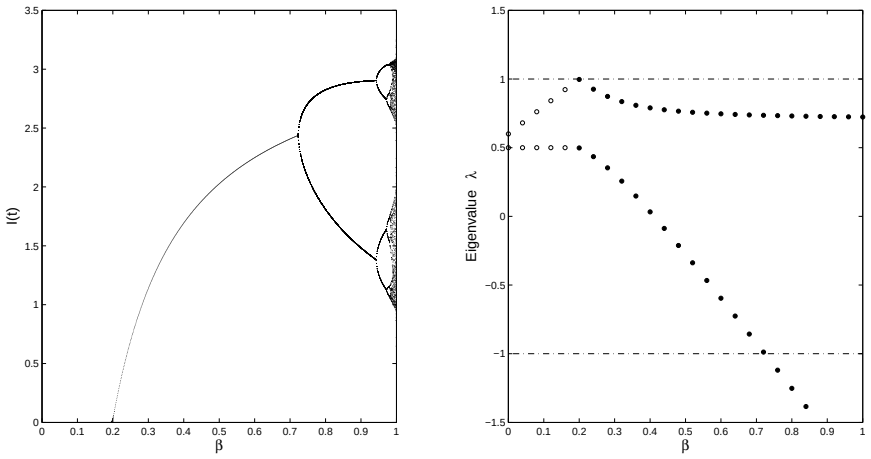

Figure 5: A bifurcation diagram for infected population I_t and eigenvalues of the Jacobian matrix as a function of β in model (19) with parameter values in Example 5.2.

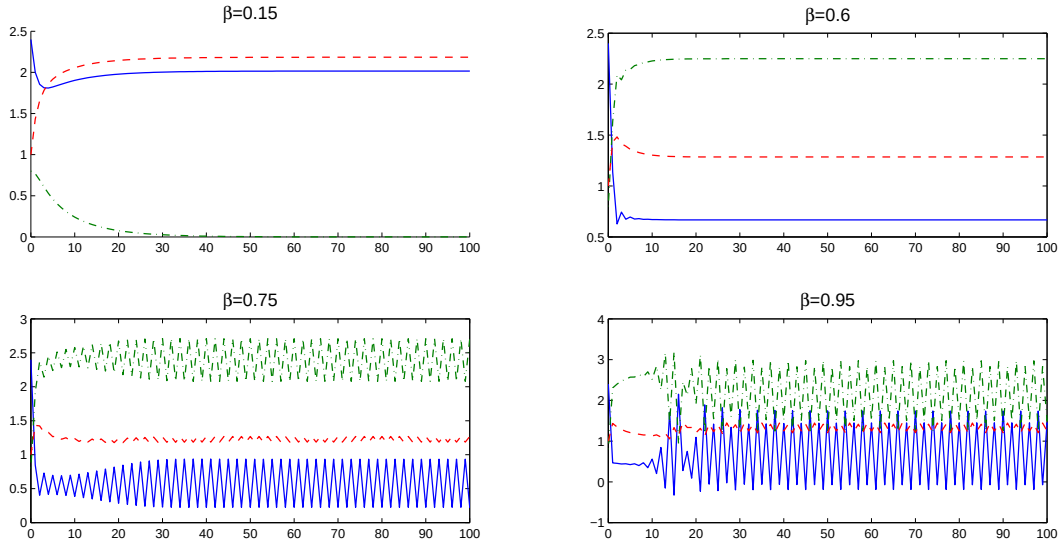

Figure 6: Diagrams of S_t ('-'line), I_t ('-.'line) and V_t ('- -'line) for values of Λ in model (19) with parameter values in Example 5.2.

6. Summary and Conclusions

A discrete-time SIS epidemic model with vaccination was described and considered, in which the model was formulated by a system of difference equations. Since a constant number of new members enters the population, the total population size is variable. In this study, asymptotic stability of the model was analyzed at the equilibria of the system, from which some conditions were obtained. Furthermore, a sufficient but not necessary condition for stability of the system was also expressed. Bifurcations of the system including transcritical bifurcations, period-doubling bifurcations, and the Neimark-Sacker bifurcation were studied, and their occurrence was considered. The same discussions were performed for the counterpart model with a constant population size. The theoretical results were confirmed in some numerical examples. The bifurcation diagrams, analysis of the eigenvalues, and Lyapunov exponents as functions of the bifurcation parameter were presented and exhibited the same behavior as those which were found in the previous sections. Several graphs of solutions of the models for various values of the parameters were also given. As shown by the study, bifurcations, oscillations and chaos phenomena generally exhibited more complicated and richer dynamical behaviors in comparison to the continuous-time counterpart model.

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