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Oscillation Results for Second Order Matrix Differential Equations with Damping

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Abstract. By using the positive linear functional, including the general means and Riccati technique, some new oscillation criteria are established for the second order matrix differential equations

$$(r(t)P(t)\psi(X(t))K(X'(t)))' + p(t)R(t)\psi(X(t))K(X'(t)) + Q(t)F(X'(t))G(X(t)) = 0, t \geq t_0 > 0.$$

The results improve and generalize those given in some previous papers.

1. Introduction

Consider the second order matrix differential equations of the form

$$(r(t)P(t)\psi(X(t))K(X'(t)))' + p(t)R(t)\psi(X(t))K(X'(t)) + Q(t)F(X'(t))G(X(t)) = 0, t \geq t_0, \tag{1.1}$$

where $t_0 \geq 0$ and r, p, P, ψ, K, R, Q and G satisfy the following conditions:

1) $r \in C^1([t_0, \infty); (0, \infty))$, $p \in C([t_0, \infty); (-\infty, \infty))$;
2) $P(t) = P^T(t) > 0$, $Q(t) \geq 0$, $R(t) = R^T(t) > 0$ for $t \geq t_0$, P, Q and R are $n \times n$ matrices real valued continuous functions on the interval $[t_0, \infty)$, and $P(t)$ and $R(t)$ are commutative. By A^T we mean the transpose of the matrix A;
3) ψ, K, G, $F \in C^1(\mathbb{R}^{n^2}; \mathbb{R}^{n^2})$, and $\psi^{-1}(X(t))$, $K^{-1}(X'(t))$ and $G^{-1}(X(t))$ exist for all $X \neq 0$, and $F(X') \geq 0$ for all $X \neq 0$.

We now denote by M the linear space of $n \times n$ real matrices, $I_n \in M$ the identity matrix and S the subspace of all symmetric matrices in M. For any A, B, $C \in S$, we write $A \geq B$ to mean that $A - B \geq 0$, that is, $A - B$ is positive semi-definite and $A > B$ to mean that $A - B > 0$, that is, $A - B$ is positive definite. If A and B are positive definite matrices, then $B^{-1} - A^{-1}$ is positive definite matrix. Note that $A \pm B$ and A' are also symmetric matrices, where $'$ denotes the first derivative. We will use some properties of this ordering, that

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is, $A \geq B$ implies that $C^T AC \geq C^T BC$.

We call a matrix function solution $X(t) \in C^2((t_0, \infty); \mathbb{R}^{n^2})$ of (1.1) is prepared nontrivial if $detX(t) \neq 0$ for at least one $t \in [t_0, \infty)$ and $X(t)$ satisfies the equation

$$G^T(X(t))P(t)\psi(X(t))K(X'(t)) - (K(X'(t)))^T\psi^T(X(t))P(t)G(X(t)) \equiv 0, \tag{1.2}$$

$$G^T(X(t))R(t)\psi(X(t))K(X'(t)) - (K(X'(t)))^T\psi^T(X(t))R(t)G(X(t)) \equiv 0, \tag{1.3}$$

and

$$\psi^T(X(t))G'(X(t))X'(t)K^{-1}(X'(t)) - (K^T(X'(t)))^{-1}(X'(t))^T(G'(X(t)))^T\psi(X(t)) \equiv 0, \quad t \geq t_0. \tag{1.4}$$

A prepared solution $X(t)$ of (1.1) is called oscillatory if $detX(t)$ has arbitrarily large zeros; otherwise, it is called nonoscillatory.

For $n = 1$, oscillatory and nonoscillatory behavior of solutions for various classes of second-order differential equations have been widely discussed in the literature (see, for example, [3,5,10,13-16,19-21,23,24,27,37-39,46,52,53,56,57,63] and references quoted there in). In the absence of damping, there is great number of papers [24,29,30,33,34,38,46,52], dealing with particular cases of equation (1.1) for $n = 1$ such as the linear equations

$$x''(t) + q(t)x(t) = 0, \tag{1.5}$$

$$(r(t)x'(t))' + q(t)x(t) = 0, \tag{1.6}$$

and the nonlinear equations

$$(r(t)x'(t))' + q(t)g(x(t)) = 0, \tag{1.7}$$

$$(r(t)\psi(x(t))x'(t))' + q(t)g(x(t)) = 0. \tag{1.8}$$

In 2000, the second order nonlinear differential equation

$$x''(t) + q(t)f(x(t))g(x'(t)) = 0, \tag{1.9}$$

has been studied by Li and Agarwal [29]. Motivated by the ideas of Li [28] and Rogovchenko [43], they obtained new oscillation criteria for oscillation by using a generalized Riccati technique.

For n=1, oscillation of nonlinear differential equations with a linear damping term of the form (1.1), that is,

$$(r(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1.10}$$

has been addressed in the monograph of Agarwal et al. [2] and papers by Elabbasy et al. [11], Grace and Lalli [19], Hao and Lu [22], Kirane and Rogovchenko [25], Li and Agarwal [30], Li et al. [32], Rogovchenko [39], Rogovchenko and Tuncay [41,42], Yang [61], to mention a few, whereas oscillation criteria for the general equation

$$(r(t)\psi(x(t))x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0, \tag{1.11}$$

were suggested, for instance,Grace [13,15], Grace and Lalli [17,18] and Manojlovic [35], Rogovchenko and Tuncay [40], Tiryaki and Zafer [49].

In 2014, the oscillation of a second-order nonlinear differential equation with damping

$$(r(t)(x'(t))^{\gamma})' + p(t)(x'(t))^{\gamma} + q(t)f(x(t)) = 0, \tag{1.12}$$

studied by Li et. al [31] for $\gamma \geq 1$ is a ratio of odd positive integers. They extended the results of Rogovchenko and Tuncay [41].

Recently, there has been an increasing interest in studying oscillation and nonoscillation of different classes of differential equations. For $n \geq 1$, p-Laplace equations have some applications in continuum mechanics as seen from [3,4].

In 2014, Zhang et al. [64] concerned with the problem of oscillation and asymptotic behavior of a higher-order delay damped differential equation with p-Laplacian like operators

$$(a(t)\left|x^{(n-1)}(t)\right|^{p-2}x^{(n-1)}(t))' + r(t)\left|x^{(n-1)}(t)\right|^{p-2}x^{(n-1)}(t) + q(t)\left|x^{(n-1)}(g(t))\right|^{p-2}x(g(t)) = 0, \tag{1.13}$$

where $p > 1$ is a real number. They obtained oscillation results by using the integral averaging technique for Eq.(1.13). Also, they obtained asymptotic results by using the comparison technique. In the special case when $p = 2$ and $n = 2$, Eq.(1.13) reduces to Eq.(1.10).

In recent years, there has been an increasing interest in studying oscillatory behavior of solutions to various classes of dynamic equations on time scales. In particular, oscillation of dynamic equations with damping has become an important area of research due to the fact such equations arise in many real life problems; see, e.g.[6-8,44,65] and the references cited therein. In 2015, the following dynamic equations with damping studied by several authors:

$$(a(x^{\Delta})^{\gamma})^{\Delta}(t) + p(t)(x^{\Delta})^{\gamma}(t) + q(t)x^{\gamma}(\delta(t)) = 0, \tag{1.14}$$

$$(r(x^{\Delta})^{\gamma})^{\Delta}(t) + p(t)(x^{\Delta^{\sigma}})^{\gamma}(t) + q(t)f(x(\tau(t))) = 0 \tag{1.15}$$

and

$$(a(t)\psi(x(t))x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta^{\sigma}}(t) + q(t)f(x^{\sigma}(t)) = 0, \tag{1.16}$$

where $t \in [t_0,\infty)_{\mathbb{T}} := [t_0,\infty) \cap \mathbb{T}$, $\mathbb{T}$ is a time scale which is unbounded above. In the special case when $\mathbb{T} = \mathbb{R}$, equations (1.14) and (1.16) reduce to equations (1.10) and (1.11).

Agarwal et al.[1] studied for Eq.(1.14). They obtained new oscillation criteria for Eq.(1.14) by using the generalized Riccati transformation technique.

Bohner and Li [9] established a new Kamenev-type theorem for Eq.(1.15) by using the generalized Riccati transformation technique.

Wang et al. [51] considered with Eq.(1.16). They obtained several sufficient conditions for the oscillation of solutions for Eq.(1.16) by using the Riccati transformation and integral averaging technique.

For $n > 1$, self-adjoint second order matrix differential systems arise in many dynamical problems studied by several authors (e.g., see [12,26,50,52,56,58-60,62] and references quoted therein). In the special cases of (1.1), Eq.(1.1) reduces to the following second-order matrix differential equations:

$$(P(t)X'(t))' + Q(t)X(t) = 0, t \geq t_0 > 0, \tag{1.17}$$

$$(P(t)X'(t))' + p(t)P(t)X'(t) + Q(t)X(t) = 0, t \geq t_0 > 0, \tag{1.18}$$

$$(P(t)X'(t))' + R(t)X'(t) + Q(t)X(t) = 0, t \geq t_0 > 0, \tag{1.19}$$

$$(r(t)X'(t))' + p(t)X'(t) + Q(t)G(X(t)) = 0, t \geq t_0 > 0, \tag{1.20}$$

$$(r(t)X'(t))' + p(t)X'(t) + Q(t)F(X'(t))G(X(t)) = 0, t \geq t_0 > 0 \tag{1.21}$$

and

$$(r(t)P(t)X'(t))' + p(t)P(t)X'(t) + Q(t)F(X'(t))G(X(t)) = 0, t \geq t_0 > 0. \tag{1.22}$$

The oscillatory solution properties of equations (1.1) and (1.17)-(1.22) are important in the mechanical systems associated with (1.1). Therefore, such properties have been studied quite extensively (see [12,29,30,50,54,58-60,62] and references quoted therein).

In 2002, Yang [60] extended some results of Li and Agarwal [29] to the matrix differential equation (1.21), here Li and Agarwal [29] presented criteria for oscillations for Eq.(1.7).

In 2003, Yang and Tang [59] obtained new oscillation criteria for Eq.(1.22). In this paper, the authors improved the theorems of Yang [60] and generalized the results of Li and Agarwal [30].

In 2004, Yang [62] extended and improved for Eq.(1.20) the results of Li and Agarwal [30] for scalar cases.

In 2005 and 2006, Sun and Meng [47,48] established some oscillation criteria by using the positive linear functional for (1.19). Also, in 2008, motivated by [26], Xu and Zhu [55] obtained several Wintner-type oscillation criteria for system (1.19). These results improved and generalized most known results.

In 2013, by using a matrix Riccati type transformation and matrix inequalities, Shi et al. [45] obtained some new oscillation criteria for the second order nonlinear matrix differential systems with damped term

$$(P(t)X'(t))' + R(t)X'(t) + F(t, X(t), X'(t)) = 0, t \geq t_0. \tag{1.23}$$

Motivated by the idea of Li and Agarwal [29], Yang and Tang [59] and Yang [60,62], in this paper we establish the Wintner type oscillation criterion for system of (1.1) by using matrix Riccati type transformation, the generalized averaging pairs and positive linear functionals, we establish the Wintner type oscillation criterion for system of (1.1).

In section 2 several definitions and Lemmas are given. Section 3 establish Wintner type oscillation criteria. Finally, in section 4 several examples that dwell upon the sharpness of our results are presented.

2. Definitions and Lemmas

Definition 2.1. *Denote by M the linear space of n n real matrices, by $I_n \in M$ the identity matrix and S the subspace of all symmetric matrices in M. A linear functional L on M is said to be " positive" if $L(A) > 0$ for any $A \in S$ and $A > 0$.*

Definition 2.2. [59] *A pair of real-valued functions (f, g) defined on $[t_0, \infty)$ is called an averaging pair if*

(i) f is nonnegative and locally integrable on $[t_0, \infty)$ satisfying $\int_{t_0}^{\infty} f(s)ds \neq 0$;
(ii) $g > 0$ is absolutely continuous on every compact subinterval of (t_0, ∞); and
(iii) for $0 \leq \kappa < 1$,

$$\lim_{t\to\infty} \int_{t_0}^{t} f(s)\Big[\Big(\int_{t_0}^{s} g(u)f^2(u)du\Big)^{-1}\Big(\int_{t_0}^{s} f(u)du\Big)^{\kappa}\Big]ds = \infty.$$

Definition 2.3. [59] *Let L be a positive linear functional and $B = B(t)$ a real valued matrix function which is invertible for each $t \in [t_0, \infty)$. A quartet of real-valued functions (f, g, L, B) defined on $[t_0, \infty)$ is a generalized averaging quartet if the conditions (i) and (ii) in Definition 2.2 and the following condition (iii) hold*

(iii) for $0 \leq \kappa < 1$,

$$\lim_{t\to\infty} \int_{t_0}^{t} f(s)\Big[\Big(\int_{t_0}^{s} g(u)f^2(u)L(B(u))du\Big)^{-1}\Big(\int_{t_0}^{s} f(u)du\Big)^{\kappa}\Big]ds = \infty.$$

Lemma 2.4. [59] *(I) Let conditions in Definition 2.3 hold; then $\int_{t_0}^{\infty} f(s)ds = \infty$.*
(II) Let $c \in C([t_0, \infty), R)$ and $\int_{t_0}^{\infty} f(s)ds = \infty$; then

$$\lim_{t\to\infty} \Big(\int_{t_0}^{\infty} f(s)ds\Big)^{-1} \int_{t_0}^{\infty} f(s)c(s)ds = \infty$$

implies

$$\lim_{t\to\infty}\Big(\int_\tau^\infty f(s)ds\Big)^{-1}\int_\tau^\infty f(s)c(s)ds=\infty, t\geq\tau\geq t_0.$$

Lemma 2.5. [36] *Let L be a positive linear functional on M. Then, for any $A, B\in S$, we have*

$$(L[A^TB])^2\leq L[A^TA]L[B^TB].$$

Lemma 2.6. *Let L be a positive linear functional on M. For any $R\in M$, $B\in S$ and $B>0$, then for all $v\in C((t_0,\infty),(0,\infty))$*

$$L\Big[\frac{1}{v}R^TBR\Big]\geq(vL[B^{-1}])^{-1}(L[R])^2.$$

Proof. In view of Lemma 2.5, we have

$$vL[B^{-1}]L\Big[\frac{1}{v}R^TBR\Big]=L[(B^{-1/2})^TB^{-1/2}]L[(B^{1/2}R)^T(B^{1/2}R)]$$

$$\geq(L[B^{-1/2}B^{1/2}R])^2=(L[R])^2.$$

Lemma 2.7. *Let $X(t)$ be a nontrivial prepared solution of (1.1) and $detX(t)\neq 0$ for $t_0\geq 0$. Then for all $a\in C^1((t_0,\infty),(0,\infty))$ the matrix function*

$$W(t)=a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t)) \tag{2.1}$$

satisfies the equation

$$W'(t)=\frac{a'(t)}{a(t)}W(t)-\frac{p(t)}{r(t)}R(t)P^{-1}(t)W(t)-a(t)Q(t)F(X'(t))$$
$$-\frac{W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t)}{a(t)r(t)}. \tag{2.2}$$

Proof. From (1.1), we obtain

$$W'(t)=a'(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t))+a(t)(r(t)P(t)\psi(X(t))K(X'(t)))'G^{-1}(X(t))$$
$$-a(t)r(t)P(t)\psi(X(t))K(X'(t))(G^{-1}(X(t)))'$$
$$=\frac{a'(t)}{a(t)}W(t)-p(t)R(t)\psi(X(t))K(X'(t))G^{-1}(X(t))-a(t)Q(t)F(X'(t))$$
$$-a(t)r(t)P(t)\psi(X(t))K(X'(t))G^{-1}(X(t))G'(X(t))X'(t)G^{-1}(X(t))$$
$$=\frac{a'(t)}{a(t)}W(t)-\frac{p(t)}{r(t)}R(t)P^{-1}(t)W(t)-a(t)Q(t)F(X'(t))$$
$$-\frac{W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t)}{a(t)r(t)}.$$

3. Main Results

In this section, by using matrix Riccati type transformation, the generalized averaging pairs and positive linear functionals, we establish the Wintner type oscillation criterion for system of (1.1).

Theorem 3.1. *Assume that all conditions stated in Section 1 are satisfied; suppose for any solution $X(t)$ for (1.1),*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0$$

for $t \geq t_0$, and $P(t)$ and $R(t)$ are commutative with $G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$. Suppose further that there exists a function $a \in C^1([t_0,\infty),(0,\infty))$ and a generalized averaging quartet

$$(f, ar, L, P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}),$$

where L is a positive linear functional on M, satisfying

$$\lim_{x\to\infty} L\Big[\Xi_{t_0}^{t} J(t_0,t)\Big] = \infty \tag{3.1}$$

and the matrix J defined by

$$\begin{aligned} J(t_0,t) = {} & \frac{1}{2}\Big(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big)P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1} \\ & + \int_{t_1}^{t}\Bigg[a(s)Q(s)F(X'(s)) - \frac{\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{4a(s)r(s)} \\ & \times P(s)\psi(X(s)))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Bigg]ds \end{aligned} \tag{3.2}$$

and $\Xi_{t_0}^{t} : M \to M$ is the linear operator defined by

$$\Xi_{t_0}^{t} U(t) = \Big(\int_{t_0}^{t} f(s)ds\Big)^{-1}\int_{t_0}^{t} f(s)U(s)ds. \tag{3.3}$$

Then every prepared solution of (1.1) is oscillatory on $[t_0,\infty)$.

Proof. Suppose the Theorem 3.1 is not true and $X(t)$ is any nontrivial prepared solution of (1.1) in $[t_1,\infty)$ which is nonoscillatory. Without loss of generality , assume that $detX(t) \neq 0$, $t \geq t_1 \geq t_0$. Then by Lemma 2.7, $W(t)$ is symmetric and satisfies the Riccati equation (2.2). That is,

$$\begin{aligned} W'(t) = {} & \tfrac{a'(t)}{a(t)}W(t) - \tfrac{p(t)}{r(t)}R(t)P^{-1}(t)W(t) - a(t)Q(t)F(X'(t)) \\ & - \frac{W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t)}{a(t)r(t)}. \end{aligned} \tag{3.4}$$

Integrating both sides of (3.4) from t_1 to t, we obtain

$$\begin{aligned} W(t) = W(t_1) + {} & \int_{t_1}^{t}\Bigg[\tfrac{a'(s)}{a(s)}W(s) - \tfrac{p(s)}{r(s)}R(s)P^{-1}(s)W(s) - a(s)Q(s)F(X'(s)) \\ & - \frac{W(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)W(s)}{a(s)r(s)}\Bigg]ds. \end{aligned} \tag{3.5}$$

Denote

$$Z(t) = W(t) - \frac{1}{2}\Big(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big)P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}. \tag{3.6}$$

From (1.3), it can be seen that

$$R(t)P^{-1}(t)W(t)$$

is symmetric. Then from $R(t)P^{-1}(t)W(t)$ is symmetric, and $P(t)$ and $R(t)$ are commutative with

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)),$$

we obtain

$$\begin{aligned}
&Z^T(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)Z(t)\\
&= \Big(W(t) - \frac{1}{2}\Big[a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big]\\
&\qquad \times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}\Big)\\
&\qquad G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)\\
&\quad \Big(W(t) - \frac{1}{2}\Big[a'(t)r(t)I_n - a(s)p(t)R(t)P^{-1}(t)\Big]\\
&\qquad \times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}\Big)\\
&= W(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)W(t) - a'(t)r(t)W(t)\\
&\qquad + a(t)p(t)R(t)P^{-1}(t)W(t)\\
&\quad + \frac{1}{4}\Big(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big)^2 P(t)\psi(X(t)))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}.
\end{aligned} \tag{3.7}$$

So, from (3.6) and (3.7), (3.5) now becomes

$$\begin{aligned}
&Z(t) - W(t_1) + \frac{1}{2}\Big(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big)\\
&\qquad \times P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}\\
&\quad + \int_{t_1}^{t}\Big[a(s)Q(s)F(X'(s)) - \frac{\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{4a(s)r(s)}\\
&\qquad \times P(s)\psi(X(s)))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\Big]ds\\
&\quad + \int_{t_1}^{t}\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds = 0
\end{aligned}$$

for $t \geq t_1 \geq t_0$. Consequently, by the definition of $J(t_0, t)$, we obtain

$$Z(t) - W(t_1) + J(t_1, t) + \int_{t_1}^{t}\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds = 0. \tag{3.8}$$

By applying the operator $\Xi_{t_1}^t$, we have

$$\Xi_{t_1}^t Z(t) + \Xi_{t_1}^t J(t_1,t) + \Xi_{t_1}^t \int_{t_1}^t \frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds = W(t_1),$$

and hence

$$L[\Xi_{t_1}^t Z(t)] + L[\Xi_{t_1}^t J(t_1,t)] + L\Big[\Xi_{t_1}^t \int_{t_1}^t \frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds\Big] = L[W(t_1)].$$

Since Lemma 2.4 and $\lim_{t\to\infty} L[\Xi_t^{t_0} J(t_0,t)] = \infty$ imply $\lim_{t\to\infty} L[\Xi_{t_1}^t J(t_1,t)] = \infty$, it follows that

$$\begin{aligned} L[\Xi_{t_1}^t Z(t)] + L\Big[\Xi_{t_1}^t \int_{t_1}^t \frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds\Big] \\ = L[W(t_1)] - L[\Xi_{t_1}^t J(t_1,t)] < 0 \end{aligned} \tag{3.9}$$

on $[t_2,t)$ for some $t_2 \ge t_1$. Since $P(t) = P^T(t) > 0$,

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t)) > 0,$$

(1.4) and $P(t)$ is commutative with $G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))$, then

$$Z(t)G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t)Z(t)$$

is positive definite. Therefore, from (3.9) it follows that

$$L[\Xi_{t_1}^t Z(t)] < -L\Big[\Xi_{t_1}^t \int_{t_1}^t \frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds\Big] \le 0,$$

which implies

$$\begin{aligned} \int_{t_1}^t f(s)L[Z(s)]s &= L\Big[\int_{t_1}^t f(s)Z(s)ds\Big] \\ &< -L\Big[\int_{t_1}^t f(s)\Big(\int_{t_1}^s \frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(s)Z(u)}{a(u)r(u)}du\Big)ds\Big] \le 0 \end{aligned} \tag{3.10}$$

for any $t \in [t_2,\infty)$. By squaring both sides of (3.10) and using the linearity of L, we have

$$\begin{aligned} \Big(\int_{t_1}^t f(s)L[Z(s)]ds\Big)^2 &= \Big(L\Big[\int_{t_0}^t f(s)Z(s)ds\Big]\Big)^2 \\ &> \Big(L\Big[\int_{t_1}^t f(s)\Big(\int_{t_1}^s \frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big)ds\Big]\Big)^2 \\ &= \Big(\int_{t_1}^t f(s)\Big(L\Big[\int_{t_1}^s \frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big)ds\Big)^2. \end{aligned}$$

Using the Hölder inequality, we obtain

$$\Big(\int_{t_1}^{t} f(s)L[Z(s)]ds\Big)^2 \le \Big(\int_{t_1}^{t}(f^2ar)(s)L[P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]ds\Big)$$
$$\times\Big(\int_{t_1}^{t}\frac{\{L[Z(s)]\}^2}{a(s)r(s)L[P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]}ds\Big). \tag{3.11}$$

Denote

$$Y(t)=\int_{t_1}^{t} f(s)\Big(L\Big[\int_{t_1}^{s}\frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big)ds.$$

Then for $t \ge t_2 > t_1$, we have

$$Y(t) \ge \int_{t_2}^{t} f(s)\Big(L\Big[\int_{t_1}^{s}\frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big)ds$$
$$\ge \Big(\int_{t_2}^{t} f(s)ds\Big)\Big(L\Big[\int_{t_1}^{t_2}\frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big).$$

Therefore, using (3.11) and Lemma 2.6, we can write

$$\Big(\int_{t_2}^{t} f(s)ds\Big)^{\kappa}\Big(L\Big[\int_{t_1}^{t_2}\frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big)^{\kappa}$$
$$\le Y^{\kappa}(t)=Y^{\kappa-2}(t)Y^2(t)$$
$$\le Y^{\kappa-2}(t)\Big(\int_{t_1}^{t}(f^2ar)(s)L[P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]ds\Big)$$
$$\times\Big(\int_{t_1}^{t}\frac{\{L[Z(s)]\}^2}{a(s)r(s)L[P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]}ds\Big)$$
$$\le Y^{\kappa-2}(t)\Big(\int_{t_1}^{t}(f^2ar)(s)L[P(s)\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]ds\Big)$$
$$\times\Big(L\Big[\int_{t_1}^{t}\frac{Z(s)G'(X(s))X'(s)K^{-1}(X'(s))\psi^{-1}(X(s))P^{-1}(s)Z(s)}{a(s)r(s)}ds\Big]\Big).$$

That is,

$$f(t)\Big(\int_{t_2}^{t} f(s)ds\Big)^{\kappa}\Big(L\Big[\int_{t_1}^{t_2}\frac{Z(u)G'(X(u))X'(u)K^{-1}(X'(u))\psi^{-1}(X(u))Z(u)}{a(u)r(u)}du\Big]\Big)^{\kappa}$$
$$\le Y^{\kappa-2}(t)Y'(t)\Big(\int_{t_1}^{t}(f^2ar)(s)L[\psi(X(s))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}]ds\Big).$$

Integrating from t_2 to t, we have

$$\Big(L\Big[\int_{t_1}^{t_2}\frac{Z(u)G'(X(u))X'(s)K^{-1}(X'(s))\psi^{-1}(X(u))P^{-1}(u)Z(u)}{a(u)r(u)}du\Big]\Big)^{\kappa}$$

$$\times \int_{t_2}^{t} f(s)\Big(\int_{t_1}^{s} (f^2ar)(u)L[P(u)\psi(X(u))K(X'(u))(X'(u))^{-1}(G'(X(u)))^{-1}]du\Big)^{-1}\Big(\int_{t_2}^{s} f(u)du\Big)^{\kappa} ds$$
$$\leq \int_{t_2}^{t} Y^{\kappa-2}(s)Y'(s)ds \leq \frac{1}{1-\kappa}\frac{1}{Y^{1-\kappa}(t_2)} < \infty,$$

which contradicts the fact $(f, ar, L, P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1})$ is a generalized averaging quartet. This completes the proof of Theorem 3.1.

Corollary 3.2. *If the above conditions hold and*

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t) \geq A > 0$$

and

$$F(X'(t)) \geq B > 0, \qquad t \in [t_0, \infty),$$

where $A, B \in S$ are constant positive definite matrices, and A is commutative with $P(t)$ and $R(t)$. Suppose further that there exist an averaging pair (f, ar) , where $a \in C^1([t_0,\infty),(0,\infty))$ and L is a positive linear functional on M satisfying (3.1), where

$$J(t_0, t) = \frac{1}{2}\Big(a'(t)r(t)I_n - a(t)p(t)R(t)P^{-1}(t)\Big)A^{-1} + \int_{t_1}^{t}\Big[a(s)Q(s)B - \frac{\Big(a'(s)r(s)I_n - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{4a(s)r(s)}A^{-1}\Big]ds$$

and $\Xi_{t_0}^{t} : S \to S$ is the linear operator defined by (3.3). Then any prepared solution of (1.1) is oscillatory on $[t_0, \infty)$.

Remark 3.3. *Theorem 3.1 and Corollary 3.2 are improvement and generalize of Theorem 3.1 and Corollary 3.1 by Yang [60]. In fact, Theorem 3.1 in [60] is not applicable if we choose such that*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{a(s)r(s)L[P(s))\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(s)))^{-1}]} < \infty$$

or $P(t) \neq R(t)$.

Remark 3.4. *Theorem 3.1 is improvement and generalize of Theorem 3.1 by Xu and Zhu [55]. In fact, Theorem 3.1 in [55] is not applicable if we choose such that*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{a(s)r(s)L[P(s))\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1}]} < \infty.$$

Remark 3.5. *In [45], let $F(t, X(t), X'(t)) = Q(t)X(t)$. Also let $r(t) = 1$, $p(t) = 1$, $\psi(X(t)) = I_n$, $K(X'(t)) = X'(t)$, $F(X'(t)) = I_n$ and $G(X(t)) = X(t)$ in (1.1). Then (1.1) reduces to the second order nonlinear matrix differential system with damped term in [45]. Theorem 3.2 is improvement and generalize of Theorem 2.3 in Shi et al.[45 with $f(t) = 0$ nd $a(t) = \rho(t)$]. In fact, Theorem 2.3 in [45] is not applicable if we choose such that*

$$\lim_{t\to\infty} \int_{t_0}^{t} \frac{ds}{a(s)r(s)L[P(s))\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(s)))^{-1}]} < \infty.$$

Remark 3.6. *Theorem 3.1 and Corollary 3.2 are improved and generalize of Theorem 3.1 and Corollary 3.1 by Yang and Tang [59]. In fact, Theorem 3.1 in [59] is not applicable if we choose such that $P(t) \neq R(t)$. But when $P(t) = R(t)$, $\psi(X(t)) = I_n$ and $K(X'(t)) = X'(t)$ in Theorem 3.1 and Corollary 3.2 give Theorem 3.1 and Corollary 3.2 in [59], respectively. Also, when $G'(X(t)) > 0$ and $P(t) > 0$ in Theorem 3.1 [59], the product of these positive definite matrices is not necessarily positive definite.*

4. Examples

In this section, we will show the application of our oscillation criteria with two examples. We will see that the equations in the examples are oscillatory based on the results in Section 3, though the oscillations cannot be demonstrated by results of Xu and Zhu [55], Yang and Tang [59], Yang [60].

Example 4.1. *Let $t \geq t_0 \geq 1$. Consider the 2×2 matrix differential system (1.1) where*

$$\left(\frac{1}{t}\begin{bmatrix} t & 0 \\ 0 & \frac{t}{2} \end{bmatrix} X^2(t)X'(t)\right)' - \frac{3}{t^4}\begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} X^2(t)X'(t) + \begin{bmatrix} 2t & 0 \\ 0 & \frac{t}{2} \end{bmatrix} X^3(t) = 0. \tag{4.1}$$

Then $r(t) = \frac{1}{t}$, $p(t) = -\frac{3}{t^4}$, $P(t) = \begin{bmatrix} t & 0 \\ 0 & \frac{t}{2} \end{bmatrix}$, $R(t) = \begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix}$, $Q(t) = \begin{bmatrix} 2t & 0 \\ 0 & \frac{t}{2} \end{bmatrix}$, $\psi(X) = X^2$, $K(X') = X'$, $F(X'(t)) = I_2$, $G(X) = X^3$, $G'(X) = 3X^2$ and

$$G'(X)X'K^{-1}(X')\psi^{-1}(X) = 3X^2X'(X')^{-1}X^{-2} = 3I_2 > 0.$$

Choose $L[T] = tr(T)$ and $a(t) = 1$. So we obtain

$$\begin{aligned}
J(1,t) &= \frac{1}{2}\Big(a'(t)r(t)I_2 - a(t)p(t)R(t)P^{-1}(t)\Big)P(t)\psi(X(t))K(X'(t))(X'(t))^{-1}(G'(X(t)))^{-1} \\
&+ \int_1^t \left[a(s)Q(s)F(X'(s)) - \frac{\Big(a'(s)r(s)I_2 - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{4a(s)r(s)} P(s)\psi(X(s)))K(X'(s))(X'(s))^{-1}(G'(X(s)))^{-1}\right]ds \\
&= \frac{1}{2t^4}\begin{bmatrix} t & 0 \\ 0 & 2t \end{bmatrix} + \int_1^t \left(\begin{bmatrix} 2s & 0 \\ 0 & \frac{s}{2} \end{bmatrix} - \frac{3}{4s^8} s \begin{bmatrix} s^2 & 0 \\ 0 & 4s^2 \end{bmatrix}\begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{2}{s} \end{bmatrix}\right)ds \\
&= \begin{bmatrix} \frac{1}{2t^3} & 0 \\ 0 & \frac{1}{t^3} \end{bmatrix} + \begin{bmatrix} t^2 - 1 & 0 \\ 0 & \frac{t^2}{4} - \frac{1}{4} \end{bmatrix} - \frac{3}{4}\int_1^t \begin{bmatrix} \frac{1}{s^6} & 0 \\ 0 & \frac{8}{s^6} \end{bmatrix} ds \\
&= \begin{bmatrix} \frac{1}{2t^3} + t^2 - 1 + \frac{3}{20t^5} - \frac{3}{20} & 0 \\ 0 & \frac{1}{t^3} + \frac{t^2}{4} - \frac{1}{4} + \frac{6}{5t^5} - \frac{6}{5} \end{bmatrix}.
\end{aligned}$$

Also,

$$L[J(1,t)] = tr(J(1,t)) = \frac{3}{2t^3} + \frac{5t^2}{4} + \frac{27}{20t^5} - \frac{13}{5}.$$

If we take $f(t) = \frac{1}{t}$ and $\kappa = \frac{2}{3}$, then

$$\begin{aligned}
\lim_{t\to\infty} L\Big[\Xi_1^t J(1,t)\Big] &= \lim_{t\to\infty}\left(\int_1^t f(s)ds\right)^{-1}\int_1^t f(s)L[J(1,s)]ds \\
&= \lim_{t\to\infty}\left(\int_1^t \frac{1}{s}ds\right)^{-1}\int_1^t \frac{1}{s}\Big(\frac{3}{2s^3} + \frac{5s^2}{4} + \frac{27}{20s^5} - \frac{13}{5}\Big)ds = \infty
\end{aligned}$$

and

$$\lim_{t\to\infty}\int_1^t f(s)\Big[\Big(\int_1^s f^2(u)a(u)r(u)L[P(u)\psi(X(u))K(X(u))(X'(u))^{-1}(G'(X(u)))^{-1}]du\Big)^{-1}\Big(\int_1^s f(u)du\Big)^{\kappa}\Big]ds$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{s}\Big[\Big(\int_1^s \frac{1}{u^2}\frac{1}{u}L\Big[\frac{1}{3}P(u)\Big]du\Big)^{-1}\Big(\int_1^s \frac{1}{u}du\Big)^{2/3}\Big]ds$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{s}\Big(\int_1^s \frac{1}{2u^2}du\Big)^{-1}(\ln s)^{2/3}\,ds$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{s}\Big(-\frac{1}{2s}+\frac{1}{2}\Big)^{-1}(\ln s)^{2/3}\,ds$$

$$= \lim_{t\to\infty} \int_1^t \frac{2}{s-1}(\ln s)^{2/3}\,ds$$

$$\geq \lim_{t\to\infty} 2\int_1^t \frac{(\ln s)^{2/3}}{s}ds = \infty.$$

So, all the assumptions of Theorem 3.1 are satisfied and every prepared solutions of (4.1) is oscillatory.

Example 4.2. *Consider the 2×2 matrix differential system (1.1) where*

$$\Big(t^3X(t)X'(t)\Big)' - t^2\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}X(t)X'(t) + \begin{bmatrix}t & 0\\ 0 & t\end{bmatrix}[2X^2(t) - I_2] = 0, \quad t \geq t_0 \geq 1. \tag{4.2}$$

Then $r(t) = 1$, $p(t) = -t^2$, $P(t) = I_2$, $R(t) = 2I_2$, $Q(t) = tI_2$, $\psi(X) = X$, $K(X') = X'$, $F(X'(t)) = I_2$, $G(X) = 2X^2 - I_2$, $G'(X) = 4X$ and

$$G'(X(t))X'(t)K^{-1}(X'(t))\psi^{-1}(X(t))P^{-1}(t) = 4X(t)X'(t)(X'(t))^{-1}X^{-1}(t) = 4I_2 > 0.$$

So we can get $A = 4I_2$ and $B = I_2$. Choose $L[T] = tr(T)$ and $a(t) = \frac{1}{t^2}$. Therefore we obtain

$$J(1,t) = \frac{1}{2}\Big(a'(t)r(t)I_2 - a(t)p(t)R(t)P^{-1}(t)\Big)A^{-1}$$
$$+ \int_1^t \Big[a(s)Q(s)B - \frac{1}{4}\frac{\Big(a'(s)r(s)I_2 - a(s)p(s)R(s)P^{-1}(s)\Big)^2}{a(s)r(s)}A^{-1}\Big]ds$$
$$= \frac{1}{8}\Big(-2t^{-3}t^3I_2 + 2t^{-2}t^2I_2\Big) + \int_1^t \Big[\frac{1}{s^2}\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \frac{1}{16}\Big(-2t^{-3}t^3I_2 + 2t^{-2}t^2I_2\Big)^2\Big]ds$$
$$= \begin{bmatrix}\ln t & 0\\ 0 & \ln t\end{bmatrix}.$$

Also,

$$L[J(1,t)] = tr(J(1,t)) = 2\ln t.$$

If we take $f(t) = \frac{1}{t}$ and $\kappa = \frac{2}{3}$, then

$$\lim_{t\to\infty} L\Big[\sum_t J(1,t)\Big] = \lim_{t\to\infty}\Big(\int_1^t f(s)ds\Big)^{-1}\int_1^t f(s)L[J(1,s)]ds$$

$$= \lim_{t\to\infty}\Big(\int_1^t \frac{1}{s}ds\Big)^{-1}\int_1^t \frac{2}{s}\ln s ds = \infty$$

and

$$\lim_{t\to\infty}\int_1^t f(s)\Big(\Big(\int_1^s f^2(u)a(u)r(u)du\Big)^{-1}\Big(\int_1^s f(u)du\Big)^{\kappa}\Big)ds$$
$$= \lim_{t\to\infty}\int_1^t \frac{1}{s}\Big[\Big(\int_1^s \frac{1}{u^2}\frac{1}{u^2}u^3du\Big)^{-1}\Big(\int_1^s \frac{1}{u}du\Big)^{2/3}\Big]ds$$
$$= \lim_{t\to\infty}\int_1^t \frac{1}{s}(\ln s)^{-1/3}\,ds = \infty.$$

So, all the assumptions of Corollary 3.2 are satisfied and every prepared solutions of (1.1) is oscillatory. In fact, $X(t) = cos(lnt)I_2$ is an oscillatory solution of (4.2).

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