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Fixed Point Results for $(\alpha, \eta, \psi, \xi)$-Contractive Multi-Valued Mappings in Menger PM-Spaces and their Applications

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Abstract. In this paper, we obtain some new fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings in α-η-$\mathcal{T}$-complete Menger PM-spaces, which turn out to generalize many results in existing literatures. Some examples are also given to illustrate the definition and support our new results. Moreover, some interesting fixed point results endowed with binary relations or graphs are derived as applications of our main results.

1. Introduction

Karl Menger is a pioneer in probabilistic analysis who first introduced the concept of a probabilistic metric space (for short, a PM-space) which laid a foundation for this branch [1]-[2]. Since then, many scholars are devoted to the theory of PM-space and apply such theory to other branches of mathematics [3]-[6]. During the past few years, fixed point problems under various conditions in Menger PM-spaces have been studied extensively (see *e.g.* [7]-[10]).

Samet *et al.* [11] first introduced the concepts of α-ψ-contractive and α-admissible mappings in metric spaces, while Asl *et al.* [12] initiated the notion of α_*-admissible mappings for multi-valued mappings in metric spaces, which was later extended by Mohammadi *et al.* [13] to α-admissible multi-valued mappings. In 2013, Salimi *et al.* [14] modified the definitions of α-ψ-contractive and α-admissible mappings, and Hussain *et al.* [15] further introduced the concept of α-η-ψ-contractive mappings for both single-valued and set-valued mappings. Later, Gopal *et al.* [16] introduced the concepts of α-admissible mappings and β-admissible mappings in Menger PM-spaces and obtained some fixed point theorems for α-ψ-type contractive mappings, while S. H. Hong [17] introduced the concept of α_*-η_*-admissible mappings in set-valued case in fuzzy metric spaces. Inspired by these results, we introduced the new concepts of α-admissible mappings with respect to η in single-valued case and α_*-admissible mappings with respect to η_* in set-valued case in Menger PM-spaces and studied the existence of fixed points for both singled-valued and set-valued mappings under some contractive conditions [18].

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On the other hand, Ali *et al.* [19] introduced the concept of (α, ψ, ξ)-contractive multi-valued mappings and proved fixed point results for such mappings on complete metric spaces. In [20], the authors asked if it is possible to prove corresponding results under weaker conditions. To this end, they made use of the concept of α-completeness of a metric space raised by Hussain *et al.* in [21] and introduced α-continuity for multi-valued mappings in a metric space to study the existence of fixed points for (α, ψ, ξ)-contractive multi-valued mappings and obtained some general yet interesting results [20].

In this paper, we first introduce some new concepts such as α-η-$\mathcal{T}$-completeness of a Menger PM-space, α-η-$\mathcal{T}$-continuity of a multi-valued mapping, and $(\alpha, \eta, \psi, \xi)$-contraction in the setting of Menger PM-spaces. Then we prove some new fixed point results for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings in α-η-$\mathcal{T}$-complete Menger PM-spaces, which generalize the main results of [20] and many other results in the literatures. Some examples are also given to illustrate the definitions and to show the validity of our new results. Finally, some fixed point results endowed with binary relations or graphs are given as consequences of our main results.

2. Preliminaries

In this section, we will recall some known definitions, notations and results and introduce some new concepts and establish some new lemmas which will be needed in the sequel. We first recall some basic definitions and results about Menger PM-spaces.

Throughout this paper, we will denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ the set of real numbers, positive real numbers and natural numbers. For a nonempty set X, denote by $N(X)$ the class of all nonempty subsets of X.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We will denote by $\mathcal{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Let $F_1, F_2 \in \mathcal{D}$. The algebraic sum $F_1 \oplus F_2$ is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\},$$

for all $t \in \mathbb{R}$.

Definition 2.1 ([6]). A mapping $\Delta : [0,1]\times[0,1] \to [0,1]$ is called a *triangular norm* (for short, a t-norm) if the following conditions are satisfied: $\Delta(a,1) = a$; $\Delta(a,b) = \Delta(b,a)$; $\Delta(a,c) \ge \Delta(b,d)$ for $a \ge b, c \ge d$;$\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

A typical example of a t-norm is Δ_{min} defined by $\Delta_{min}(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

Definition 2.2 ([6]). A triplet $(X,\mathcal{F},\Delta)$ is called a *Menger probabilistic metric space* (for short, a *Menger PM-space*) if X is a nonempty set, Δ is a t-norm and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}$ satisfying the following conditions (we denote $\mathcal{F}(x,y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(0) = 0$;

(PM-2) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;

(PM-4) $F_{x,y}(t+s) \ge \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $t,s \ge 0$.

Remark 2.1. If $(X,\mathcal{F},\Delta)$ satisfies the condition $\sup_{0<t<1} \Delta(t,t) = 1$, then $(X,\mathcal{F},\Delta)$ is a Hausdorff topological space in the (ϵ,λ)-topology $\mathcal{T}$, *i.e.*, the family of sets $\{U_x(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1]\}(x \in X)$ is a basis of neighborhoods of a point x for $\mathcal{T}$, where $U_x(\epsilon,\lambda) = \{y \in X : F_{x,y}(\epsilon) > 1-\lambda)\}$[6].

By virtue of the topology $\mathcal{T}$, a sequence $\{x_n\}$ is said to be $\mathcal{T}$-convergent to $x \in X$(we write $x_n \xrightarrow{\mathcal{T}} x(n \to \infty)$) if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$

whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathcal{T}$-Cauchy sequence in $(X, \mathcal{F}, \Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$; $(X, \mathcal{F}, \Delta)$ is said to be $\mathcal{T}$-complete if each $\mathcal{T}$-Cauchy sequence in X is $\mathcal{T}$-convergent in X. Note that in a Menger PM-space, when we write $\lim_{n\to\infty} x_n = x$, it means that $x_n \xrightarrow{\mathcal{T}} x(n \to \infty)$.

Remark 2.2 ([6]). (1) $(CB(X), \delta)$ is a metric space. If (X, d) is complete, then $(CB(X), \delta)$ is complete;

(2) Let (X, d) be a metric space. Define a mapping $\mathcal{F} : X \times X \to \mathcal{D}$ by

$$\mathcal{F}(x, y)(t) = F_{x,y}(t) = H(t - d(x, y)), \ \ \forall x, y \in X, \ t \in \mathbb{R}. \tag{1}$$

Then $(X, \mathcal{F}, \Delta_{min})$ is a Menger PM-space induced by (X, d) with $\Delta_{min}(a, b) = \min\{a, b\}, \forall a, b \in [0, 1]$. If (X, d) is complete, then $(X, \mathcal{F}, \Delta_{min})$ is $\mathcal{T}$-complete.

(3) Define $\tilde{\mathcal{F}} : CB(X) \times CB(X) \to \mathcal{D}$ by

$$\tilde{\mathcal{F}}(A, B)(t) = \tilde{F}_{A,B}(t) = H(t - \delta(A, B)), \ \ \forall A, B \in CB(X), \ t \in \mathbb{R}. \tag{2}$$

Then $\tilde{\mathcal{F}}$ is the Menger-Hausdorff metric induced by $\mathcal{F}$. Moreover, if $(X, \mathcal{F}, \Delta)$ is a $\mathcal{T}$-complete Menger PM-space with the t-norm $\Delta \geq \Delta_m$, where $\Delta_m(a, b) = \max\{a + b - 1, 0\}, \forall a, b \in [0, 1]$, then $(\Omega, \tilde{\mathcal{F}}, \Delta)$ is also a $\mathcal{T}$-complete Menger PM-space.

Let (X, d) be a metric space, $CB(X)$ be the family of all nonempty bounded closed subsets of X and δ be the *Hausdorff metric* induced by d, that is, $\delta(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$, for any $A, B \in CB(X)$, where $d(x, A) = \inf_{y\in A} d(x, y)$.

Let $(X, \mathcal{F})$ be a PM-space and A be a nonempty subset of X. Then the function

$$D_A(t) = \sup_{s<t} \inf_{x,y\in A} F_{x,y}(s), \quad t \in \mathbb{R}$$

is called the *probabilistic diameter* of A. If $\sup_{t>0} D_A(t) = 1$, then A is said to be *probabilistically bounded*.

Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space and Ω be the family of all nonempty probabilistically bounded $\mathcal{T}$-closed subsets of X. For any $A, B \in \Omega$, define the distribution functions as follows:

$$\tilde{\mathcal{F}}(A, B)(t) = \tilde{F}_{A,B}(t) = \sup_{s<t} \Delta(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)), \quad s, t \in \mathbb{R},$$

$$\mathcal{F}(x, A)(t) = F_{x,A}(t) = \sup_{s<t} \sup_{y\in A} F_{x,y}(s), \quad s, t \in \mathbb{R},$$

where $\tilde{\mathcal{F}}$ is called the *Menger-Hausdorff metric* induced by $\mathcal{F}$.

The following lemmas will play an important role in proving our main results in Section 3.

Lemma 2.1 ([6]). Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with a continuous t-norm Δ on $[0, 1] \times [0, 1]$, $x, y \in X$, $\{x_n\}, \{y_n\} \subset X$ and $x_n \xrightarrow{\mathcal{T}} x$, $y_n \xrightarrow{\mathcal{T}} y$. Then $\liminf_{n\to\infty} F_{x_n,y_n}(t) \geq F_{x,y}(t)$ for all $t > 0$. Particularly, if $F_{x,y}(\cdot)$ is continuous at the point t_0, then $\lim_{n\to\infty} F_{x_n,y_n}(t_0) = F_{x,y}(t_0)$.

Lemma 2.2 ([6]). Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. Then for any $A, B, C \in \Omega$ and any $x, y \in X$, we have the following:

(i) $F_{x,A}(t) = 1$ for all $t > 0$ if and only if $x \in A$;

(ii) For any $x \in A$, $F_{x,B}(t) \geq \tilde{F}_{A,B}(t)$, for all $t > 0$;

(iii) $F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,y}(t_1), F_{y,A}(t_2))$, for all $t_1, t_2 > 0$;

(iv) $F_{x,A}(t_1 + t_2) \geq \Delta(F_{x,B}(t_1), \tilde{F}_{A,B}(t_2))$, for all $t_1, t_2 > 0$.

In [18], we introduced the following concept which is the generalization of the one in a metric space to a Menger PM-space.

Definition 2.3 ([18]). Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space, $T : X \to N(X)$ be a set-valued mapping and $\alpha, \eta : X \times X \times (0,+\infty) \to [0,+\infty)$ be two functions, where η is bounded. T is called an *α_*-admissible mapping with respect to η_**, if for all $t > 0$, we have

$$\alpha(x, y, t) \leq \eta(x, y, t) \Longrightarrow \alpha_*(Tx, Ty, t) \leq \eta_*(Tx, Ty, t), x, y \in X,$$

where $\alpha_*(A, B, t) = \sup\limits_{x\in A, y\in B} \alpha(x, y, t)$, and $\eta_*(A, B, t) = \inf\limits_{x\in A, y\in B} \eta(x, y, t)$.

Now we introduce the following concept which is more general than the above one.

Definition 2.4. Let $X \neq \emptyset$, $T : X \to N(X)$, and $\alpha, \eta : X \times X \times (0,+\infty) \to [0,+\infty)$. T is called *α-admissible with respect to η*, if for each $x \in X, y \in Tx$ and $t > 0$ with $\alpha(x, y, t) \leq \eta(x, y, t)$, we have $\alpha(y, z, t) \leq \eta(y, z, t)$, for all $z \in Ty$ and $t > 0$.

Remark 2.3. An α_*-admissible mapping with respect to η_* is an α-admissible mapping with respect to η, but the converse is not true as is shown in the following example.

Example 2.1. Let $X = [-1, 1]$, and $\alpha, \eta : X \times X \times (0,+\infty) \to [0,+\infty)$ and $T : X \to N(X)$ be defined by

$$\alpha(x, y, t) = \begin{cases} 2, & x = y, \\ 4, & x \neq y, \end{cases} t > 0,$$

$$\eta(x, y, t) = \begin{cases} 1, & x = y, \\ 6, & x \neq y, \end{cases} t > 0,$$

$$Tx = \begin{cases} \{x\}, & x \notin \{-1, 0\}, \\ \{0, 1\}, & x = -1, \\ \{1\}, & x = 0. \end{cases}$$

For $x = -1, y = 0 \in Tx = \{0, 1\}$, $\alpha(x, y, t) \leq \eta(x, y, t), \forall t > 0$. Note that $\alpha_*(Tx, Ty, t) = \alpha_*(\{0, 1\}, \{1\}, t) = 4 > 1 = \eta_*(\{0, 1\}, \{1\}, t) = \eta_*(Tx, Ty, t)$, so T is not an α_*-admissible mapping with respect to η_*. But T is α-admissible with respect to η. In fact, consider the following cases.

Case 1. $x = 0, y = 1 \in Tx$. Then $\alpha(x, y, t) \leq \eta(x, y, t), \forall t > 0$. Thus, $\alpha(y, z, t) \leq \eta(y, z, t), \forall t > 0$, since $z = -1 \in Ty = \{-1\}$.

Case 2. $x = -1, y \in \{0, 1\}$. Then $\alpha(x, y, t) \leq \eta(x, y, t), \forall t > 0$. Thus, $\alpha(y, z, t) \leq \eta(y, z, t), \forall t > 0$, since $z = 1$ when $y = 0$ and $z = -1$ when $y = 1$.

Case 3. $x \notin \{-1, 0\}, y = -x$. Then $\alpha(x, y, t) \leq \eta(x, y, t), \forall t > 0$. Thus, $\alpha(y, z, t) \leq \eta(y, z, t), \forall t > 0$, since $z \in \{0, 1\}$ when $y = -1$ and $z = x \in Ty = \{x\}$ when $y \neq -1$.

The following definition generalizes the concept of α-completeness in a metric space by Hussain *et al.* [21] to α-η-$\mathcal{T}$-completeness in a Menger PM-space.

Definition 2.5. Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space and $\alpha, \eta : X \times X \times (0,+\infty) \to [0,+\infty)$. $(X,\mathcal{F},\Delta)$ is called *α-η-$\mathcal{T}$-complete* if each $\mathcal{T}$-Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$, $\mathcal{T}$-converges in X.

Remark 2.4. If $(X,\mathcal{F},\Delta)$ is a $\mathcal{T}$-complete Menger PM-space, then it is α-η-$\mathcal{T}$-complete Menger PM-space. But the following example shows that the converse is not true.

Example 2.2. Let $X = (0,+\infty)$, $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ and $\mathcal{F}$ be defined by (1). Then $(X,\mathcal{F},\Delta_{min})$ is a Menger PM-space. Define $\alpha, \eta : X \times X \times (0,+\infty) \to [0,+\infty)$ by

$$\alpha(x, y, t) = \begin{cases} \frac{1}{2}, & x, y \in [1, 3], \\ 3, & otherwise, \end{cases} t > 0,$$

$$\eta(x, y, t) = \begin{cases} \frac{3}{2}, & x, y \in [1, 3], \\ 2, & otherwise, \end{cases} t > 0.$$

It is easy to see that $(X, \mathcal{F}, \Delta_{min})$ is not $\mathcal{T}$-complete, but it is α-η-$\mathcal{T}$-complete. In fact, if $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in X such that $\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$. Then $x_n \in [1,3]$ for all $n \in \mathbb{N}$. Since $[1,3]$ is a $\mathcal{T}$-closed subset of $\mathbb{R}$, it follows that $([1,3], \mathcal{F}, \Delta_{min})$ is $\mathcal{T}$-complete, and thus there exists $x_* \in [1,3]$, such that $x_n \xrightarrow{\mathcal{T}} x_*(n \to \infty)$.

We also introduce the following concept which extends the one of α-continuity in a metric space [20] to the setting of a Menger PM-space.

Definition 2.6. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space, $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$, $T : X \to \Omega$, T is called an *α-η-$\mathcal{T}$-continuous* multi-valued mapping, if for sequence $\{x_n\} \subset X$,

$$x_n \xrightarrow{\mathcal{T}} x_*(n \to \infty), \alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t) \text{ for all } n \in \mathbb{N} \text{ and } t > 0 \Longrightarrow \lim_{n\to\infty} \tilde{F}_{Tx_n, Tx}(t) = 1, \forall t > 0.$$

Remark 2.5. It is easy to see that the $\mathcal{T}$-continuity of T implies the α-η-$\mathcal{T}$-continuity of T for all mappings α and η, but the converse is not true in general, which can be shown by the following example.

Example 2.3. Let $X = [0, +\infty)$, $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$, and $\mathcal{F}, \tilde{\mathcal{F}}$ be defined by (1) and (2) respectively. Then $(X, \mathcal{F}, \Delta_{min})$ and $(\Omega, \tilde{\mathcal{F}}, \Delta_{min})$ are both Menger PM-spaces. Define $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ and $T : X \to \Omega$ by

$$\alpha(x, y, t) = \begin{cases} \frac{2}{3}, & x, y \in [0, 1], \\ 4, & otherwise, \end{cases} \quad t > 0,$$

$$\eta(x, y, t) = \begin{cases} \frac{5}{3}, & x, y \in [0, 1], \\ 1, & otherwise, \end{cases} \quad t > 0.$$

$$Tx = \begin{cases} [0, 10x^2], & x \in [0, 1], \\ [1, x], & x > 1, \end{cases} \quad t > 0.$$

It is obvious that T is not a $\mathcal{T}$-continuous mapping. Indeed, for $\{x_n\} = \{1 + \frac{1}{n}\}$ in X, $\lim_{n\to\infty} F_{1+\frac{1}{n},1}(t) = \lim_{n\to\infty} H(t - d(1 + \frac{1}{n}, 1)) = \lim_{n\to\infty} H(t - \frac{1}{n})) = 1, \forall t > 0$, i.e., $1 + \frac{1}{n} \xrightarrow{\mathcal{T}} 1(n \to \infty)$. However, since $\lim_{n\to\infty} \tilde{F}_{Tx_n,T1}(t) = \lim_{n\to\infty} H(t - \delta(Tx_n, T1)) = \lim_{n\to\infty} H(t - (9 - \frac{1}{n})), \forall t > 0$, there exists $t_0 = 8 > 0$, such that $\lim_{n\to\infty} \tilde{F}_{Tx_n,T1}(t_0) = \lim_{n\to\infty} H(\frac{1}{n} - 1) = 0 \neq 1$.

Now we show that T is α-η-$\mathcal{T}$-continuous. Let $\{x_n\} \subset X$ such that $x_n \xrightarrow{\mathcal{T}} x_*(n \to \infty), \alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N}$ and $t > 0$. Then we have $x_n, x \in [0, 1]$ for all $n \in \mathbb{N}$. Therefore, $\lim_{n\to\infty} \tilde{F}_{Tx_n,Tx}(t) = \lim_{n\to\infty} H(t - \delta([0, 10x_n^2], [0, 10x^2])) = 1, \forall t > 0$.

We denote by Ψ the class of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(ψ_1) ψ is a nondecreasing function;

(ψ_1) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Such class of functions are referred to as Bianchini-Grandolfi gauge functions in some literatures (see *e.g.*[22]-[23]). It is easy to see that for each $\psi \in \Psi$, the following assertions hold [6]:

(1) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;

(2) $\psi(t) < t$ for each $t > 0$.

Let Ξ be the family of functions $\xi : [0, 1] \to [0, 1]$ satisfying the following conditions:

(ξ_1) ξ is continuous;

(ξ_2) ξ is strictly increasing on $[0, 1]$;

(ξ_3) $\xi(t) = 0$ if and only if $t = 0$ and $\xi(t) = 1$ if and only if $t = 1$;

(ξ_4) ξ is subadditive.

(ξ_5) $\xi(t) < t$ for all $t > 0$ and $\xi(\Delta(a, b)) \geq \Delta(\xi(a), \xi(b)), \forall a, b \in [0, 1]$.

We prove the following lemma which will be needed in proving our main results.

Lemma 2.3. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space, $\xi \in \Xi$, $B \in \Omega$ and $c < 1$. Suppose that there exists $x \in X$, such that $\xi(F_{x,B}(t)) > 0, \forall t > 0$. Then there exists $y \in B$, such that

$$\xi(F_{x,y}(t)) > c\xi(F_{x,B}(t)), \ \ \forall t > 0.$$

Proof. It follows from $\xi(F_{x,B}(t)) > 0, \forall t > 0$ and (ξ_3) that $F_{x,B}(t) > 0, \forall t > 0$. By the definition of $F_{x,B}(\cdot)$, there exist $\{y_n\} \subset B$, such that

$$\lim_{n\to\infty} F_{x,y_n}(t) = F_{x,B}(t), \forall t > 0. \tag{3}$$

By (ξ_4), we have

$$\xi(F_{x,B}(t)) \leq \xi(F_{x,B}(t) - F_{x,y_n}(t)) + \xi(F_{x,y_n}(t)), \forall t > 0.$$

From (ξ_1) and (3), we obtain

$$\limsup_{n\to\infty} \xi(F_{x,B}(t)) - \xi(F_{x,y_n}(t)) \leq \limsup_{n\to\infty} \xi(F_{x,B}(t) - F_{x,y_n}(t)) \leq \xi(0) = 0, \forall t > 0.$$

Thus, we get

$$\xi(F_{x,B}(t)) - \liminf_{n\to\infty} \xi(F_{x,y_n}(t)) \leq 0, \forall t > 0,$$

which implies that

$$\liminf_{n\to\infty} \xi(F_{x,y_n}(t)) \geq \xi(F_{x,B}(t)) > c\xi(F_{x,B}(t)), \forall t > 0.$$

Therefore, there exists $N \in \mathbb{N}$, such that $\xi(F_{x,y_N}(t)) > c\xi(F_{x,B}(t)), \forall t > 0$.

We next extend the concept of (α, ψ, ξ)-contractive mapping in metric spaces to $(\alpha, \eta, \psi, \xi)$-contractive mapping in Menger PM-spaces.

Definition 2.7. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. $T : X \to \Omega$ is called an $(\alpha, \eta, \psi, \xi)$*-contractive mapping* if there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$, such that for all $t > 0$,

$$x, y \in X, \alpha(x, y, t) \leq \eta(x, y, t) \Longrightarrow \xi(\tilde{F}_{Tx,Ty}(\psi(t))) \geq \xi(M(x, y, t)), \tag{4}$$

where $M(x, y, t) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

In order to prove our main results in the next section, we need some lemmas. To this end, we first recall the following definition.

Definition 2.8 ([6]). Let X be a nonempty set, $\{d_\alpha : \alpha \in (0, 1)\}$ be a family of mappings from $X \times X$ into $\mathbb{R}^+$. The ordered pair $(X, d_\alpha : \alpha \in (0, 1))$ is called a *generating space of quasi-metrics family*, and $\{d_\alpha : \alpha \in (0, 1)\}$ is called the *family of quasi-metrics on* X, if the following conditions are satisfied:

(QM-1) $d_\alpha(x, y) = 0$ for all $\alpha \in (0, 1)$ if and only if $x = y$;

(QM-2) $d_\alpha(x, y) = d_\alpha(y, x)$ for all $\alpha \in (0, 1)$ and $x, y \in X$;

(QM-3) for any given $\alpha \in (0, 1)$, there exists $\mu \in (0, \alpha)$, such that

$$d_\alpha(x, y) \leq d_\mu(x, z) + d_\mu(z, y), \forall x, y, z \in X;$$

(QM-4) for any given $x, y \in X$, the function $\alpha \longmapsto d_\alpha(x, y)$ is nondecreasing and left-continuous.

By (ξ_3) and (ξ_5), we can prove the following lemma by imitating the proof of Lemma 1.7 in [24].

Lemma 2.4. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with Δ satisfying the condition $\sup_{0<t<1} \Delta(t, t) = 1$. For any given $\lambda \in (0, 1)$, define a mapping $E_{\lambda,F}(x, y) : X \times X \to \mathbb{R}^+$ as follows:

$$E_{\lambda,\xi,F}(x, y) = \inf\{t > 0 : \xi(F_{x,y}(t)) > 1 - \lambda\}. \tag{5}$$

Then $\{E_{\lambda,\xi,F} : \lambda \in (0, 1)\}$ is a family of quasi-metrics on X and $(X, E_{\lambda,\xi,F} : \lambda \in (0, 1))$ is a generating space of the quasi-metrics family $\{E_{\lambda,\xi,F} : \lambda \in (0, 1)\}$.

Remark 2.6. By (ξ_5), it is also easy to see that $E_{\lambda,\xi,F}(x, y) < \epsilon \Longrightarrow F_{x,y}(\epsilon) > 1 - \lambda$. This implies that if $E_{\lambda,\xi,F}(x_n, x) \to 0 (n \to \infty), \forall \lambda \in (0, 1)$, then $\{x_n\} \mathcal{T}$-converges to x in $(X, \mathcal{F}, \Delta)$. Also, if $E_{\lambda,\xi,F}(x_n, x_m) \to 0 (n, m \to \infty), \forall \lambda \in (0, 1)$, then $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in $(X, \mathcal{F}, \Delta)$.

Also, we define the mapping $E^c_{\lambda,\xi,F}(x,y) : X \times X \to \mathbb{R}^+$ as follows:

$$E^c_{\lambda,\xi,F}(x,y) = \inf\{t > 0 : c\xi(F_{x,y}(t)) > 1 - \lambda\}.$$

Imitating the proof in [25], we can similarly prove the following lemma.

Lemma 2.5. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. Suppose that the function $\psi : [0,+\infty) \to [0,+\infty)$ is onto and strictly increasing. Then

$$\inf\{\psi^n(t) > 0 : c\xi(F_{x,y}(t)) > 1 - \lambda\} \le \psi^n(\inf\{t > 0 : c\xi(F_{x,y}(t)) > 1 - \lambda\}),$$

for any $x, y \in X$, $\lambda \in (0,1)$ and $n \in \mathbb{N}$, where $c < 1$.

Based on the above two lemmas, we can further prove the following lemma.

Lemma 2.6. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with Δ satisfying $\sup\limits_{0<t<1} \Delta(t,t) = 1$ and $\{x_n\}$ be a sequence in X such that

$$\xi(F_{x_n,x_{n+1}}(\psi^n(t))) > c\xi(F_{x_0,x_1}(t), \forall t > 0 \text{ and } n \in \mathbb{N}, \tag{6}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is onto, strictly increasing and $\psi \in \Psi$, and $\xi \in \Xi$. Suppose further that

$$E^c_{\xi,F}(x_0,x_1) := \sup_{\lambda\in(0,1)} \{E^c_{\lambda,\xi,F}(x_0,x_1)\} < +\infty.$$

Then $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in X.

Proof. For any $\lambda \in (0,1)$, it follows from Lemma 2.5 and (6) that

$$\begin{aligned}
E_{\lambda,\xi,F}(x_n,x_{n+1}) &= \inf\{t > 0 : \xi(F_{x_n,x_{n+1}}(t)) > 1 - \lambda\} \\
&= \inf\{\psi^n(\psi^n)^{-1}(t) > 0 : \xi(F_{x_n,x_{n+1}}(\psi^n(\psi^n)^{-1}(t))) > 1 - \lambda\} \\
&\le \inf\{t > 0 : c\xi(F_{x_0,x_1}((\psi^n)^{-1}(t))) > 1 - \lambda\} \\
&\le \psi^n(\inf\{t > 0 : c\xi(F_{x_0,x_1}(t)) > 1 - \lambda\}) \\
&= \psi^n(E^c_{\lambda,\xi,F}(x_0,x_1)) \\
&\le \psi^n(E^c_{\xi,F}(x_0,x_1)).
\end{aligned}$$

For any given $n, m \in \mathbb{N}$ with $n > m$ and for any given $\lambda \in (0,1)$, it follows from the above inequality and Lemma 2.4 that there exists $\mu \in (0,\lambda]$ such that

$$\begin{aligned}
E_{\lambda,\xi,F}(x_n,x_m) &\le E_{\mu,\xi,F}(x_n,x_{n+1}) + E_{\mu,\xi,F}(x_{n+1},x_{n+2}) + \cdots + E_{\mu,\xi,F}(x_{n-1},x_m) \\
&\le \sum_{j=n}^{m-1} \psi^j(E^c_{\xi,F}(x_0,x_1)) \to 0 (n,m \to \infty).
\end{aligned}$$

By Remark 2.5, $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in $(X, \mathcal{F}, \Delta)$. This completes the proof.

3. Main Results

In this section, we will prove some new fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive mappings in α-η-$\mathcal{T}$-complete Menger PM-spaces.

Theorem 3.1. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with $\Delta = \Delta_{min}$ and $T : X \to \Omega$ be an $(\alpha, \eta, \psi, \xi)$-contractive mapping. Suppose that the following conditions hold:

(1) $(X, \mathcal{F}, \Delta)$ is α-η-$\mathcal{T}$-complete;

(2) T is an α-admissible multi-valued mapping with respect to η;

(3) there exist $x_0 \in X$ and $x_1 \in Tx_0$, such that $\alpha(x_0,x_1,t) \le \eta(x_0,x_1,t), \forall t > 0$;

(4) T is an α-η-$\mathcal{T}$-continuous mapping.

Then T has a fixed point in X.

Proof. Take $x_0 \in X$ and $x_1 \in Tx_0$ satisfying condition (3), i.e., $\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t), \forall t > 0$. If $x_0 = x_1$, then the conclusion follows immediately. Now assume that $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. By (4), (ξ_2) and Lemma 2.2 (iii), for all $t > 0$ and any $\delta \in (0, 2t)$, we have

$$\begin{aligned}\xi(\tilde{F}_{Tx_0,Tx_1}(\psi(t))) &\geq \xi(\min\{F_{x_0,x_1}(t), F_{x_0,Tx_0}(t), F_{x_1,Tx_1}(t), [F_{x_0,Tx_1} \oplus F_{x_1,Tx_0}](2t)\}) \\ &\geq \xi(\min\{F_{x_0,x_1}(t), F_{x_0,Tx_0}(t), F_{x_1,Tx_1}(t), \min\{F_{x_0,Tx_1}(2t-\delta), F_{x_1,Tx_0}](\delta)\}) \\ &= \xi(\min\{F_{x_0,x_1}(t), F_{x_1,Tx_1}(t), F_{x_0,Tx_1}(2t-\delta)\}) \\ &\geq \xi(\min\{F_{x_0,x_1}(t), F_{x_1,Tx_1}(t), \Delta_{min}(F_{x_0,x_1}(t-\frac{\delta}{2}), F_{x_1,Tx_1}(t-\frac{\delta}{2})\}).\end{aligned}$$

Letting $\delta \to 0$, by the left-continuity of the distribution function, we get

$$\xi(\tilde{F}_{Tx_0,Tx_1}(\psi(t))) \geq \xi(\min\{F_{x_0,x_1}(t), F_{x_1,Tx_1}(t)\}), \forall t > 0. \tag{7}$$

Suppose that $\min\{F_{x_0,x_1}(t), F_{x_1,Tx_1}(t)\} = F_{x_1,Tx_1}(t), \forall t > 0$. Noting that $x_1 \in Tx_0$, it follows from (7), (ξ_2) and Lemma 2.2 (ii) that

$$\xi(\tilde{F}_{x_1,Tx_1}(\psi(t))) \geq \xi(\tilde{F}_{Tx_0,Tx_1}(\psi(t))) \geq \xi(F_{x_1,Tx_1}(t)\}), \forall t > 0. \tag{8}$$

which by induction yields that

$$\xi(\tilde{F}_{x_1,Tx_1}(\psi^k(t))) \geq \xi(F_{x_1,Tx_1}(t)), \forall k \in \mathbb{N} \text{ and } t > 0. \tag{9}$$

Letting $k \to \infty$, by the property of ψ and (ξ_3), we obtain $F_{x_1,Tx_1}(t) = 0, \forall t > 0$, which is in contradiction with the definition of the distribution function. So $\min\{F_{x_0,x_1}(t), F_{x,Tx_1}(t)\} = F_{x_0,x_1}(t)$. Thus if follows from (7) that

$$\xi(F_{x_1,Tx_1}(\psi(t))) \geq \xi(\tilde{F}_{Tx_0,Tx_1}(\psi(t))) \geq \xi(F_{x_0,x_1}(t)\}), \forall t > 0. \tag{10}$$

Fix $c < 1$. By Lemma 2.3, there exists $x_2 \in Tx_1$, such that

$$\xi(F_{x_1,x_2}(\psi(t))) > c\xi(F_{x_1,Tx_1}(\psi(t))) > c\xi(F_{x_0,x_1}(t)\}), \forall t > 0. \tag{11}$$

Let $c_1 = \frac{c\xi(F_{x_0,x_1}(t)}{\xi(F_{x_1,x_2}(\psi(t)))}$. Then $c_1 < 1$.

If $x_1 = x_2$ or $x_2 \in Tx_2$, then x_2 is a fixed point of T. So we now assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Since $x_1 \in Tx_0, x_2 \in Tx_1$, $\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t)$ for all $t > 0$ and T is an α-admissible mapping with respect to η, we have $\alpha(x_1, x_2, t) \leq \eta(x_1, x_2, t)$ for all $t > 0$. Using (4), (ξ_2) and Lemma 2.2(iii), we have

$$\begin{aligned}\xi(\tilde{F}_{Tx_1,Tx_2}(\psi(t))) &\geq \xi(\min\{F_{x_1,x_2}(t), F_{x_1,Tx_1}(t), F_{x_2,Tx_2}(t), [F_{x_1,Tx_2} \oplus F_{x_2,Tx_1}](2t)\}) \\ &\geq \xi(\min\{F_{x_1,x_2}(t), F_{x_1,Tx_1}(t), F_{x_2,Tx_2}(t), \min\{F_{x_1,Tx_2}(2t-\delta), F_{x_2,Tx_1}](\delta)\}) \\ &= \xi(\min\{F_{x_1,x_2}(t), F_{x_2,Tx_2}(t), F_{x_1,Tx_2}(2t-\delta)\}) \\ &\geq \xi(\min\{F_{x_1,x_2}(t), F_{x_2,Tx_2}(t), \Delta_{min}(F_{x_1,x_2}(t-\frac{\delta}{2}), F_{x_2,Tx_2}(t-\frac{\delta}{2})\}).\end{aligned}$$

Letting $\delta \to 0$, we get

$$\xi(\tilde{F}_{Tx_1,Tx_2}(\psi(t))) \geq \xi(\min\{F_{x_1,x_2}(t), F_{x_2,Tx_2}(t)\}), \forall t > 0. \tag{12}$$

Suppose that $\min\{F_{x_1,x_2}(t), F_{x_2,Tx_2}(t)\}) = F_{x_2,Tx_2}(t), \forall t > 0$. Then we can similarly deduce a contradiction. Thus, $\min\{F_{x_1,x_2}(t), F_{x_2,Tx_2}(t)\}) = F_{x_1,x_2}(t), \forall t > 0$.

$$\xi(F_{x_2,Tx_2}(\psi(t))) \geq \xi(\tilde{F}_{Tx_1,Tx_2}(\psi(t))) \geq \xi(F_{x_1,x_2}(t)\}), \forall t > 0. \tag{13}$$

Since $c_1 < 1$, by Lemma 2.3, there exists $x_3 \in Tx_2$, such that

$$\xi(F_{x_2,x_3}(\psi(t))) > c_1\xi(F_{x_2,Tx_2}(\psi(t))) > c_1\xi(F_{x_1,x_2}(t)\}), \forall t > 0, \tag{14}$$

which implies that

$$\xi(F_{x_2,x_3}(\psi^2(t))) > c_1\xi(F_{x_1,x_2}(\psi(t))) = c\xi(F_{x_0,x_1}(t), \forall t > 0. \tag{15}$$

Continuing this process, we can obtain a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1} \in Tx_n$,

$$\alpha(x_n, x_{n+1}, t) \leq \eta(x_n, x_{n+1}, t), \forall n \in \mathbb{N} \text{ and } t > 0 \tag{16}$$

and

$$\xi(F_{x_n,x_{n+1}}(\psi^n(t))) > c\xi(F_{x_0,x_1}(t), \forall n \in \mathbb{N} \text{ and } t > 0. \tag{17}$$

By Lemma 2.6, we obtain that $\{x_n\}$ is a $\mathcal{T}$-Cauchy sequence in $(X, \mathcal{F}, \Delta)$. By (16) and the α-η-$\mathcal{T}$-completeness of $(X, \mathcal{F}, \Delta)$, we obtain that there exists $x^* \in X$, such that $x_n \xrightarrow{\mathcal{T}} x^*(n \to \infty)$.

By the α-η-$\mathcal{T}$-continuity of T, we get

$$\lim_{n\to\infty} \tilde{F}_{Tx_n,Tx^*}(t) = 1, \forall t > 0.$$

By Lemma 2.2 (iv), we obtain

$$F_{x^*,Tx^*}(t) \geq \Delta(F_{x^*,Tx_n}(\frac{t}{2}), \tilde{F}_{Tx_n,Tx^*}(\frac{t}{2})) \geq \Delta(F_{x^*,x_{n+1}}(\frac{t}{2}), \tilde{F}_{Tx_n,Tx^*}(\frac{t}{2})), \forall t > 0.$$

Letting $n \to \infty$, we get $F_{x^*,Tx^*}(t) = 1, \forall t > 0$, which by Lemma 2.2 (i) implies that $x^* \in Tx^*$. Hence, T has a fixed point in X. This completes the proof.

From Theorem 3.1, we can obtain the following two corollaries.

Corollary 3.1. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with $\Delta = \Delta_{min}$ and $T : X \to \Omega$ be an $(\alpha, \eta, \psi, \xi)$-contractive mapping. Suppose that the following conditions hold:

(1) $(X, \mathcal{F}, \Delta)$ is α-η-$\mathcal{T}$-complete;

(2) T is an α^*-admissible multi-valued mapping with respect to η^*;

(3) there exist $x_0 \in X$ and $x_1 \in Tx_0$, such that $\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t), \forall t > 0$;

(4) T is an α-η-$\mathcal{T}$-continuous mapping.

Then T has a fixed point in X.

Corollary 3.2. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with $\Delta = \Delta_{min}$ and $T : X \to \Omega$ be an $(\alpha, \eta, \psi, \xi)$-contractive mapping. Suppose that the following conditions hold:

(1) $(X, \mathcal{F}, \Delta)$ is $\mathcal{T}$-complete;

(2) T is an α^*-admissible multi-valued mapping with respect to η^*;

(3) there exist $x_0 \in X$ and $x_1 \in Tx_0$, such that $\alpha(x_0, x_1, t) \leq \eta(x_0, x_1, t), \forall t > 0$;

(4) T is an $\mathcal{T}$-continuous mapping.

Then T has a fixed point in X.

Remark 3.1. Setting $\eta(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ in the above theorems and corollaries in this section, we can obtain some corresponding fixed point results, which turned out to be the generalizations of the main results in [20] from metric spaces to the framework of Menger PM-spaces. So our main results also extend the results mentioned in Remark 2.9 of [20].

Now, we present an example to show the validity of our main result.

Example 3.1. Let $X = (-8, 8)$, $d : X \times X \to \mathbb{R}$ be defined by $d(x, y) = |x - y|$ for all $x, y \in X$, $\mathcal{F}_{x,y}(t) = F_{x,y}(t) = e^{-\frac{|x-y|}{t}}$ for $x, y \in X$ and $t > 0$, and $\tilde{\mathcal{F}}_{A,B}(t) = \tilde{F}_{A,B}(t) = e^{-\frac{\delta(A,B)}{t}}$ for $A, B \in CB(X)$ and $t > 0$. Then by Remark 2.2, $(X, \mathcal{F}, \Delta_{min})$ and $(\Omega, \tilde{\mathcal{F}}, \Delta_{min})$ are both Menger PM-spaces.

Define $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ and $T : X \to \Omega$ by

$$\alpha(x, y, t) = \begin{cases} \sqrt{xy}, & x, y \in [0, 3], \\ \frac{5}{2}, & otherwise, \end{cases} t > 0,$$

$$\eta(x,y,t)=\begin{cases}\frac{x+y}{2}, & x,y\in[0,3],\\ \frac{3}{2}, & otherwise,\end{cases}\quad t>0,$$

and

$$Tx=\begin{cases}[-7,|x|], & x\in(-8,0),\\ [0,\frac{x}{6}], & x\in[0,3],\\ [\frac{x+9}{3},7], & x\in(3,8).\end{cases}$$

It is clear that $(X,\mathcal{F},\Delta_{min})$ is not $\mathcal{T}$-complete. Now, we show that Theorem 3.1 or Theorem 3.2 can be applied. First of all, $(X,\mathcal{F},\Delta_{min})$ is α-η-$\mathcal{T}$-complete. Define $\psi:[0,+\infty)\to[0,+\infty)$ and $\xi:[0,1]\to[0,1]$ by $\psi(t)=\frac{t}{3}$ for all $t\in[0,+\infty)$ and $\xi(t)=\sqrt{t}$ for all $t\in[0,1]$. It is obvious that $\psi\in\Psi$ and $\xi\in\Xi$.

For $x,y\in X$ and $\alpha(x,y,t)\le\eta(x,y,t)$ for all $t>0$, we have $x,y\in[0,2]$. Then

$$\xi(\tilde{F}_{Tx,Ty}(\psi(t)))=\sqrt{e^{-\frac{|x-y|}{2t}}}\ge\sqrt{e^{-\frac{|x-y|}{t}}}=\xi(F_{x,y}(t))\ge\xi(M(x,y,t)).$$

Therefore, T is an (α,η,ψ,ξ)-contractive mapping.

Moreover, it is easy to check that T is an α-η-admissible multi-valued mapping as well as an α-η-$\mathcal{T}$-continuous mapping. Also, there exists $x_0=2\in X$ and $x_1=\frac{1}{3}\in Tx_0$, such that $\alpha(x_0,x_1,t)=\alpha(2,\frac{1}{3},t)\le\eta(2,\frac{1}{3},t)=\eta(x_0,x_1,t)$ for all $t>0$. Thus, all the conditions of Theorem 3.1 are satisfied and so T has a fixed point in X.

On the other hand, for each sequence $\{x_n\}$ in X with $x_n\xrightarrow{\mathcal{T}}x^*(n\to\infty)$ and $\alpha(x_n,x_{n+1},t)\le\eta(x_n,x_{n+1},t)$ for all $n\in\mathbb{N}$ and $t>0$, we have $\alpha(x_n,x,t)\le\eta(x_n,x,t)$ for all $n\in\mathbb{N}$ and $t>0$. Thus, we can also deduce from Theorem 3.2 that T has a fixed point in X. In fact, in this example, T has many fixed points such as $0,5,6$, etc.

4. Applications

In this section, we will apply our main results in Section 3 to obtain some interesting fixed point results endowed with binary relations or graphs. We first deal with the results concerning about binary relations for which the following notations and definitions are needed.

Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. Denote $\mathcal{S}:=\mathcal{R}\bigcup\mathcal{R}^{-1}$, i.e.,

$$x,y\in X,\ \ x\mathcal{S}y\Longleftrightarrow x\mathcal{R}y\ \text{ or }\ y\mathcal{R}x.$$

Definition 4.1. Let X be a nonempty set, and $\mathcal{R}_1$ and $\mathcal{R}_2$ be two binary relations over X. A multi-valued mapping $T:X\to N(X)$ is said to be *weakly comparative* if for each $x\in X$ and $y\in Tx$ with $x\mathcal{S}_1y$ and $x\mathcal{S}_2y$, we have $y\mathcal{S}_1z$ and $y\mathcal{S}_2z$, for all $z\in Ty$.

Definition 4.2. Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space and $\alpha,\eta:X\times X\times(0,+\infty)\to[0,+\infty)$. $(X,\mathcal{F},\Delta)$ is called $\mathcal{S}_1$-$\mathcal{S}_2$-$\mathcal{T}$*-complete* if each $\mathcal{T}$-Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{S}_1x_{n+1}$ and $x_n\mathcal{S}_2x_{n+1}$ for all $n\in\mathbb{N}$, $\mathcal{T}$-converges in X.

Definition 4.3. Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space, $\alpha,\eta:X\times X\times(0,+\infty)\to[0,+\infty)$, $T:X\to\Omega$, T is called an $\mathcal{S}_1$-$\mathcal{S}_2$-$\mathcal{T}$*-continuous multi-valued mapping*, if for $x\in X$ and sequence $\{x_n\}\subset X$,

$$x_n\xrightarrow{\mathcal{T}}x_*(n\to\infty),x_n\mathcal{S}_1x_{n+1}\text{ and }x_n\mathcal{S}_2x_{n+1}\text{ for all }n\in\mathbb{N}\Longrightarrow\lim_{n\to\infty}\tilde{F}_{Tx_n,Tx}(t)=1,\forall t>0.$$

Definition 4.4. Let $(X,\mathcal{F},\Delta)$ be a Menger PM-space. $T:X\to\Omega$ is called an $(\mathcal{S}_1,\mathcal{S}_2,\psi,\xi)$*-contractive mapping* if there exist $\psi\in\Psi$, $\xi\in\Xi$, and $\alpha,\eta:X\times X\times(0,+\infty)\to[0,+\infty)$, such that for all $t>0$,

$$x,y\in X,x\mathcal{S}_1y\text{ and }x\mathcal{S}_2y\Longrightarrow\xi(\tilde{F}_{Tx,Ty}(\psi(t)))\ge\xi(M(x,y,t)),\tag{18}$$

where $M(x,y,t)=\min\{F_{x,y}(t),F_{x,Tx}(t),F_{y,Ty}(t),[F_{x,Ty}\oplus F_{y,Tx}](2t)\}$.

Theorem 4.1. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with $\Delta = \Delta_{min}$, $\mathcal{R}_1$ and $\mathcal{R}_2$ be two binary relations over X, and $T : X \to \Omega$ be an $(\alpha, \eta, \psi, \xi)$-*contractive mapping*. Suppose that the following conditions hold:

(1) $(X, \mathcal{F}, \Delta)$ is $\mathcal{S}_1$-$\mathcal{S}_2$-$\mathcal{T}$-complete;

(2) T is a weakly comparative mapping;

(3) there exist $x_0 \in X$ and $x_1 \in Tx_0$, such that $x_0\mathcal{S}_1x_1$ and $x_0\mathcal{S}_2x_1$;

(4) T is an $\mathcal{S}_1$-$\mathcal{S}_2$-$\mathcal{T}$-continuous mapping.

Then T has a fixed point in X.

Proof. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & x\mathcal{S}_1 y, \\ 5, & otherwise, \end{cases} \quad t > 0,$$

$$\eta(x, y, t) = \begin{cases} 2, & x\mathcal{S}_2 y, \\ 3, & otherwise, \end{cases} \quad t > 0.$$

Then it is easy to verify that the conditions of Theorem 3.1 are satisfied based on the conditions of this theorem. Thus, the conclusion follows. This completes the proof.

Next, we consider fixed point results in which graphs are taken into account. Let X be a nonempty set. First, recall that a set $\{(x, x) : x \in X\}$ is called a diagonal of $X \times X$ and is denoted by D. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $D \subset E(G)$. We may assume that G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may assign to each edge of G the distance between its vertices which is called a weighted graph.

Definition 4.5. Let X be a nonempty set endowed with two graphs G_1 and G_2. A multi-valued mapping $T : X \to N(X)$ is called *weakly preserves edges* if for each $x \in X$ and $y \in Tx$ with $x, y \in E(G_1)$ and $x, y \in E(G_2)$, we have $(y, z) \in E(G_1)$ and $(y, z) \in E(G_2)$ for all $z \in Ty$.

Definition 4.6. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space and $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$. $(X, \mathcal{F}, \Delta)$ is called $E(G_1)$-$E(G_2)$-$\mathcal{T}$-*complete* if each $\mathcal{T}$-Cauchy sequence $\{x_n\}$ in X with $x_n\mathcal{S}_1x_{n+1}$ and $x_n\mathcal{S}_2x_{n+1}$ for all $n \in \mathbb{N}$, $\mathcal{T}$-converges in X.

Definition 4.7. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space, $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$, $T : X \to \Omega$, T is called an $E(G_1)$-$E(G_2)$-$\mathcal{T}$-*continuous multi-valued mapping*, if for $x \in X$ and sequence $\{x_n\} \subset X$,

$$x_n \xrightarrow{\mathcal{T}} x_*(n \to \infty), (x_n, x_{n+1}) \in E(G_1) \text{ and } (x_n, x_{n+1}) \in E(G_2) \text{ for all } n \in \mathbb{N} \Longrightarrow \lim_{n\to\infty} \tilde{F}_{Tx_n,Tx}(t) = 1, \forall t > 0.$$

Definition 4.8. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space. $T : X \to \Omega$ is called an $(E(G_1), E(G_2), \psi, \xi)$-*contractive mapping* if there exist $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$, such that for all $t > 0$,

$$x, y \in X, (x, y) \in E(G_1) \text{ and } (x, y) \in E(G_2) \Longrightarrow \xi(\tilde{F}_{Tx,Ty}(\psi(t))) \geq \xi(M(x, y, t)), \tag{19}$$

where $M(x, y, t) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), [F_{x,Ty} \oplus F_{y,Tx}](2t)\}$.

Theorem 4.2. Let $(X, \mathcal{F}, \Delta)$ be a Menger PM-space with $\Delta = \Delta_{min}$ endowed with two graphs G_1 and G_2, and $T : X \to \Omega$ be an $(E(G_1), E(G_2), \psi, \xi)$-contractive mapping. Suppose that the following conditions hold:

(1) $(X, \mathcal{F}, \Delta)$ is $E(G_1)$-$E(G_2)$-$\mathcal{T}$-complete;

(2) T weakly preserves edges;

(3) there exist $x_0 \in X$ and $x_1 \in Tx_0$, such that $(x_0, x_1) \in E(G_1)$ and $(x_0, x_1) \in E(G_2)$;

(4) T is an $E(G_1)$-$E(G_2)$-$\mathcal{T}$-continuous mapping.

Then T has a fixed point in X.

Proof. Define the mappings $\alpha, \eta : X \times X \times (0, +\infty) \to [0, +\infty)$ by

$$\alpha(x, y, t) = \begin{cases} 1, & (x, y) \in E(G_1), \\ 5, & otherwise, \end{cases} \quad t > 0,$$

$$\eta(x,y,t)=\begin{cases}2, & (x,y)\in E(G_2),\\ 3, & otherwise,\end{cases}\quad t>0.$$

Then it follows from Theorem 3.1 that the conclusion holds. This completes the proof.

Remark 4.1. Setting $\mathcal{S}_1$=$\mathcal{S}_2$ or G_1=G_2 in the theorems and corollaries in this section, we obtain some corresponding fixed point results endowed with a binary relation or a graph, which extend the results in Section 3 of [20] from metric spaces to Menger PM-spaces. Therefore, our results also generalize a large number of results mentioned in Remark 3.13 of [20].

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