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The Eigenvalue Problem with Interaction Conditions at One Interior Singular Point

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Abstract. Some physical processes, both classical physics and quantum physics reduced to eigenvalue problems for Sturm-Liouville equations. In the recent years there has been an increasing interest in discontinuous eigenvalue problems for various Sturm-Liouville type equations. Such problems are connected with heat transfer problems, vibrating string problems, diffraction problems and etc. In this study we shall investigate a class of two order eigenvalue problem with supplementary transmission conditions at one interior singular point. We give an operator-theoretic interpretation in suitable Hilbert space.

1. Introduction

Boundary value problems arise directly as mathematical models of motion according to Newton's law, but more often as a result of using the method of separation of variables to solve the classical partial differential equations of physics, such as Laplace's equation, the heat equation, and the wave equation. Many topics in mathematical physics require investigations of eigenvalues and eigenfunctions of boundary value problems. These investigations are of utmost importance for theoretical and applied problems in mechanics, the theory of vibrations and stability, hydrodynamics, elasticity, acoustics, electrodynamics, quantum mechanics, and many other branches of natural science (cf. [11, 12, 21]). Such problems are formulated in many different ways. For example, the Schrodinger equation in quantum mechanics is a famous example of an eigenvalue problem where the energy levels are determined by a self-adjoint operator. The Hamiltonian H is the operator of the energy in a quantum system. In the case of molecules it is conveniently divided into three parts

$$H = T_N + T_e + U(q, Q).$$

T_N and T_e are the kinetic energies of the nuclei and the electrons , respectively . The total potential energy U(q,Q) comprises the mutual repulsion of the electrons , the mutual repulsion of the nuclei and the attracting potential between electrons and nuclei . The coordinates of the electrons and the nuclei are represented by q and Q, respectively . The eigenvalues E_m and the corresponding eigenfunctions ψ_m are the solutions of the Schrodinger equation $(H - E_m)\psi_m = 0$.

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In this study we shall investigate a new class of Sturm-Liouville type problem which consist of a Sturm-Liouville equation

$$\mathcal{L}(y) := -p(x)y''(x) + q(x)y(x) = \mu^2 y(x) \tag{1}$$

to hold in disjoint intervals $[a,c)$ and (c,b] where discontinuity in y and y' at the interface point x=c are prescribed by two the transmission conditions

$$V_j(y) := \beta^-_{j1}y'(c-) + \beta^-_{j0}y(c-) + \beta^+_{j1}y'(c+) + \beta^+_{j0}y(c+) = 0, \quad j = 1,2 \tag{2}$$

together with eigenparameter-dependent boundary conditions

$$U_1(y) := \alpha_{10}y(a) - \alpha_{11}y'(a) - \mu^2(\alpha'_{10}y(a) - \alpha'_{11}y'(a)) = 0, \tag{3}$$

$$U_2(y) := \alpha_{20}y(b) - \alpha_{21}y'(b) + \mu^2(\alpha'_{20}y(b) - \alpha'_{21}y'(b)) = 0, \tag{4}$$

where $p(x) = p^- > 0$ for $x \in [a,c)$, $p(x) = p^+ > 0$ for $x \in (c,b]$, the potential $q(x)$ is real-valued continuous function in each of the intervals $[a,c)$ and $(c,b]$, and has a finite limits $q(c\mp 0)$, λ is a complex eigenparameter, $p^\mp$, α_{ij}, $\beta^\mp_{ij}$, α'_{ij} ($i = 1,2$ and $j = 0,1$) are real numbers. These boundary-transmission conditions are of great importance for theoretical and applied studies and have a definite mechanical or physical meaning (see, for example, [1–10, 13–19, 23, 24]). Also the problems with transmission conditions arise in mechanics, such as thermal conduction problems for a thin laminated plate, which studied in [21]. This class of problems essentially differs from the classical case, and its investigation requires a specific approaches.

2. Definitions and integral equations of the one-hand eigensolutions

Let $B_1 = \begin{bmatrix} \alpha_{11} & \alpha_{10} \\ \alpha'_{11} & \alpha'_{10} \end{bmatrix}$, $B_2 = \begin{bmatrix} \alpha_{21} & \alpha_{20} \\ \alpha'_{21} & \alpha'_{20} \end{bmatrix}$ and T=$\begin{bmatrix} \beta^+_{10} & \beta^+_{11} & \beta^-_{10} & \beta^-_{11} \\ \beta^+_{20} & \beta^+_{21} & \beta^-_{20} & \beta^-_{21} \end{bmatrix}$. Denote the determinant of the matrix B_i by $\theta_i (i = 1,2)$ and the determinant of the k-th and j-th columns of the matrix T by Δ_{kj}. Note that throughout this study we shall assume that $\theta_1 > 0$, $\theta_2 > 0$, $\Delta_{12} > 0$ and $\Delta_{34} > 0$. With a view to constructing the characteristic function we define four one-hand eigensolutions $\varphi^\mp(x,\mu)$ and $\psi^\mp(x,\mu)$ by own procedure as follows. At first we shall define one left solution $\varphi^-(x,\mu)$ and one right solution $\psi^+(x,\mu)$ of the equation (1) on left interval (a,c) and on right interval (c,b) satisfying the initial conditions

$$y(a) = \alpha_{11} - \mu^2\alpha'_{11}, \; y'(a) = \alpha_{10} - \mu^2\alpha'_{10} \tag{5}$$

and

$$y(b) = \alpha_{21} + \mu^2\alpha'_{21}, \; y'(b) = \alpha_{20} + \mu^2\alpha'_{20} \tag{6}$$

respectively. It is known that these solutions are entire functions of parameter $\mu \in \mathbb{C}$ for each fixed x (see, for example, [12]). After defining this solutions we shall define the other solutions $\varphi^+(x,\mu)$ and $\psi^-(x,\mu)$ in terms of $\varphi^-(x,\mu)$ and $\psi^+(x,\mu)$. Namely employing the same method as in [19] we can prove that the equation (1) under initial conditions

$$y(c+) = \frac{1}{\Delta_{12}}(\Delta_{23}\varphi^-(c-,\mu) + \Delta_{24}\frac{\partial\varphi^-(c-\mu)}{\partial x}) \tag{7}$$

$$y'(c+) = \frac{-1}{\Delta_{12}}(\Delta_{13}\varphi^-(c-,\mu) + \Delta_{14}\frac{\partial\varphi^-(c-,\mu)}{\partial x}). \tag{8}$$

and

$$y(c-) = \frac{-1}{\Delta_{34}}(\Delta_{14}\psi^+(c+,\mu) + \Delta_{24}\frac{\partial\psi^+(c+,\mu)}{\partial x}), \tag{9}$$

$$y'(c-) = \frac{1}{\Delta_{34}}(\Delta_{13}\psi^+(c+,\mu) + \Delta_{23}\frac{\partial\psi^+(c+,\mu)}{\partial x}). \tag{10}$$

has unique solutions $\varphi^{+}(x,\mu)$ and $\psi^{-}(x,\mu)$ which are entire functions of parameters $\mu \in \mathbb{C}$ for each fixed $x \in (c,b)$ and $x \in (a,c)$ respectively. Below, for shorting we shall use also notations; $\varphi^{\pm}(x,\mu) := \varphi^{\pm}_{\mu}(x)$, $\psi^{\pm}(x,\mu) := \psi^{\pm}_{\mu}(x)$:

Lemma 2.1. *The next integral and integro-differential equations are hold for $k = 0$ and $k = 1$.*

$$\frac{d^k}{dx^k}\varphi^{-}_{\mu}(x) = \sqrt{p^-}\frac{(\alpha_{10} - \mu^2\alpha'_{10})}{\mu}\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-a)}{\sqrt{p^-}}\right]$$
$$+(\alpha_{11} - \mu^2\alpha'_{11})\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-a)}{\sqrt{p^-}}\right] + \frac{1}{\sqrt{p^-}\mu}\int\limits_a^x \frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^-}}\right] q(z)\varphi^{-}_{\mu}(z)dz$$

$$\begin{aligned}
\frac{d^k}{dx^k}\psi^{-}_{\mu}(x) &= -\frac{1}{\Delta_{34}}(\Delta_{14}\psi^{+}(c,\mu) + \Delta_{24}\frac{\partial\psi^{+}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right] \\
&+\frac{\sqrt{p^-}}{\mu\Delta_{34}}(\Delta_{13}\psi^{+}(c,\mu) + \Delta_{23}\frac{\partial\psi^{+}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right] \\
&+\frac{1}{\sqrt{p^-}\mu}\int\limits_x^{c-}\frac{d^k}{dx^k}\sin\left[\mu\,(x-z)\right] q(z)\psi^{-}_{\mu}(z)dz
\end{aligned} \tag{11}$$

for $x \in [a,c)$ and

$$\begin{aligned}
\frac{d^k}{dx^k}\varphi^{+}_{\mu}(x) &= \frac{1}{\Delta_{12}}(\Delta_{23}\varphi^{-}(c,\mu) + \Delta_{24}\frac{\partial\varphi^{-}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right] \\
&-\frac{\sqrt{p^+}}{\mu\Delta_{12}}(\Delta_{13}\varphi^{-}(c,\mu) + \Delta_{14}\frac{\partial\varphi^{-}(c,\mu)}{\partial x}))\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right] \\
&+\frac{1}{\sqrt{p^+}\mu}\int\limits_{c+}^{x}\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^+}}\right] q(z)\varphi^{+}_{\mu}(z)dz
\end{aligned} \tag{12}$$

$$\frac{d^k}{dx^k}\psi^{+}_{\mu}(x) = \frac{\sqrt{p^+}}{\mu}(\alpha_{20} + \mu^2\alpha'_{20})\frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-b)}{\sqrt{p^+}}\right]$$
$$+(\alpha_{21} + \mu^2\alpha'_{21})\frac{d^k}{dx^k}\cos\left[\frac{\mu\,(x-b)}{\sqrt{p^+}}\right] + \frac{1}{\sqrt{p^+}\mu}\int\limits_x^b \frac{d^k}{dx^k}\sin\left[\frac{\mu\,(x-z)}{\sqrt{p^+}}\right] q(z)\psi^{+}_{\mu}(z)dz$$

for $x \in (c,b]$.

Proof. For proving of these formulas it is enough substitute $\mu^2\varphi^{\mp}_{\mu}(z) + p^{\mp}\frac{d^k}{dx^k}\varphi^{\mp}_{\mu}(z)$ and$\mu^2\psi^{\mp}_{\mu}(z) + p^{\mp}\frac{d^k}{dx^k}\psi^{\mp}_{\mu}(z)$ instead of $q(z)\varphi^{\mp}_{\mu}(z)$ and $q(z)\psi^{\mp}_{\mu}(z)$ respectively in the corresponding integral terms and then integrate by parts twice. □

3. Asymptotic expressions of eigensolutions

Now we are ready to prove the following theorems.

Theorem 3.1. *Let , $Im\mu = t$. Then if $\alpha'_{11} \neq 0$ the asymptotic estimates*

$$\frac{d^k}{dx^k}\varphi^-_\mu(x) = -\alpha'_{11}\mu^{k+2}\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-a)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k+1} e^{\frac{|t|(x-a)}{\sqrt{p^-}}}\right) \tag{13}$$

$$\frac{d^k}{dx^k}\varphi^+_\mu(x) = \frac{\Delta_{24}}{\Delta_{12}}\frac{\alpha'_{11}}{\sqrt{p^-}}\mu^{k+3}\sin\left[\frac{\mu(c-a)}{\sqrt{p^-}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right] + O\left(|\mu|^{k+2}e^{|t|(\frac{(x-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{14}$$

are valid as $|\mu| \to \infty$, while if $\alpha'_{11} = 0$ the asymptotic estimates

$$\frac{d^k}{dx^k}\varphi^-_\mu(x) = -\alpha'_{10}\sqrt{p^-}\mu^{k+1}\frac{d^k}{dx^k}\sin\left[\frac{\mu(x-a)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k} e^{\frac{|t|(x-a)}{\sqrt{p^-}}}\right) \tag{15}$$

$$\frac{d^k}{dx^k}\varphi^+_\mu(x) = -\frac{\Delta_{24}}{\Delta_{12}}\alpha'_{10}\mu^{k+2}\cos\left[\frac{\mu(c-a)}{\sqrt{p^-}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^+}}\right] + O\left(|\mu|^{k+1} e^{|t|(\frac{(x-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{16}$$

are valid as $|\mu| \to \infty$ ($k = 0, 1$). All asymptotic estimates are holds uniformly with respect to x.

Proof. The asymptotic formulas (13) and (15) for $\varphi^-_\mu(x)$ follows immediately from the Titchmarsh's Lemma ([22], Lemma 1.7). But the corresponding formulas for $\varphi^+_\mu(x)$ need individual consideration. Let $\alpha_{11} \neq 0$. Put

$$\varphi^+_\mu(x) = e^{|t|(\frac{(x-c)}{\sqrt{p^+}}}Y(x,\mu) \tag{17}$$

it follows from (12) that

$$\begin{aligned} Y(x,\mu)\,| \leq\; & e^{|t|(\frac{(x-c)}{\sqrt{p^+}}}\{\frac{1}{\Delta_{12}}(\Delta_{23}M_1|\mu|^2 + \Delta_{24}M_2|\mu|^3)e^{-|t|(\frac{(x-c)}{\sqrt{p^+}}}e^{\frac{|t|(c-a)}{\sqrt{p^-}}} \\ & -\frac{\sqrt{p^+}}{\mu\Delta_{12}}(\Delta_{13}M_1|\mu|^2 + \Delta_{14}M_2|\mu|^3)e^{-|t|(\frac{(x-c)}{\sqrt{p^+}}}e^{\frac{|t|(c-a)}{\sqrt{p^-}}} \\ & +\frac{1}{\sqrt{p^+}\mu}\int_{c+}^{x} e^{|t|(\frac{(x-z)}{\sqrt{p^+}}}q(z)Y(z,\mu)e^{-|t|(\frac{(x-c)}{\sqrt{p^+}}}dz\} \end{aligned} \tag{18}$$

for some $M_1 > 0$ and $M_2 > 0$. (17) and (18) we get

$$\varphi^+_\mu(x) = O\left(|\mu|^3 e^{|t|(\frac{(x-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{19}$$

as $|\mu| \to \infty$. Now, the estimate (14) for the case $k = 0$ is obtained by substituting (19) in the integral term on the right-hand side of (12) . The case $k = 1$ of the (14) follows at once on differentiating (12) and making the same procedure as in the case $k = 0$. The proof of (16) is similar. □

Similarly we can prove the following Theorem.

Theorem 3.2. *Let $Im\mu = t$. Then if $\alpha'_{21} \neq 0$*

$$\frac{d^k}{dx^k}\psi^+_\mu(x) = \alpha'_{21}\mu^{k+2}\frac{d^k}{dx^k}\cos\left[\frac{\mu(b-x)}{\sqrt{p^+}}\right] + O\left(|\mu|^{k+1} e^{\frac{|t|(b-x)}{\sqrt{p^+}}}\right) \tag{20}$$

$$\frac{d^k}{dx^k}\psi^-_\mu(x) = -\frac{\Delta_{24}}{\Delta_{34}}\frac{\alpha'_{21}}{\sqrt{p^+}}\mu^{k+3}\sin\left[\frac{\mu(b-c)}{\sqrt{p^+}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k+2}e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-x)}{\sqrt{p^-}})}\right) \tag{21}$$

as $|\mu| \to \infty$, *while if* $\alpha'_{21} = 0$

$$\frac{d^k}{dx^k}\psi^+_\mu(x) = -a'_{20}\sqrt{p^+}\mu^{k+1}\frac{d^k}{dx^k}\sin\left[\frac{\mu(b-x)}{\sqrt{p^+}}\right] + O\left(|\mu|^k e^{|t|\frac{(b-x)}{\sqrt{p^+}}}\right) \tag{22}$$

$$\frac{d^k}{dx^k}\psi^-_\mu(x) = -\frac{\Delta_{24}}{\Delta_{34}}\alpha'_{20}\mu^{k+2}\cos\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right]\frac{d^k}{dx^k}\cos\left[\frac{\mu(x-c)}{\sqrt{p^-}}\right] + O\left(|\mu|^{k+1} e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-x)}{\sqrt{p^-}})}\right) \tag{23}$$

as $|\mu| \to \infty$ $(k = 0, 1)$. *Each of this asymptotic equalities hold uniformly for* x.

4. The characteristic function and eigenvalues

It is well-known from ordinary differential equation theory that the Wronskians $W[\varphi^-_\mu(x), \psi^-_\mu(x)]$ and $W[\varphi^+_\mu(x), \psi^+_\mu(x)]$ are independent of variable x. Let $w^\pm(\mu) := W[\varphi^\pm_\mu(x), \psi^\pm_\mu(x)]$.

Lemma 4.1. *The equality* $\Delta_{34}w^-(\mu) = \Delta_{12}w^+(\mu)$ *holds for each* $\mu \in \mathbb{C}$.

Proof. By using (7)-(8) and (9)-(10) we have

$$\begin{aligned} w^+(\mu) &= \varphi^+(c,\mu)\frac{d\psi^+(c,\mu)}{dx} - \frac{d\varphi^+(c,\mu)}{dx}\psi^+(c,\mu) \\ &= \frac{\Delta_{34}}{\Delta_{12}}(\varphi^-(c,\mu)\frac{d\psi^-(c,\mu)}{dx} - \frac{d\varphi^-(c,\mu)}{dx}\psi^-(c,\mu)) \\ &= \frac{\Delta_{34}}{\Delta_{12}}w^-(\mu). \end{aligned}$$

□

We shall define the characteristic function four our problem as

$$w(\mu) := \Delta_{34}w^-(\mu) = \Delta_{12}\, w^+(\mu).$$

Theorem 4.2. *The eigenvalues of the problem* (1)-(4) *are consist of the zeros of the characteristic function* $w(\mu)$.

Proof. Let $w(\mu_0) = 0$. Then $W[\varphi^-_{\mu_0}, \psi^-_{\mu_0}]_x = 0$. Thus, the functions $\varphi^-_{\mu_0}(x)$ and $\psi^-_{\mu_0}(x)$ are linearly depended, i.e., there is $k \neq 0$ such that

$$\psi^-_{\mu_0}(x) = k\varphi^-_{\mu_0}(x), x \in [a, c). \tag{24}$$

It is easy to see that the function $\psi_{\mu_0}(x)$ defined on whole $[a, c) \cup (c, b]$ by $\psi_{\mu_0}(x) = \psi^-_{\mu_0}(x)$ for $[a, c)$ and $\psi_{\mu_0}(x) = \psi^+_{\mu_0}(x)$ for $(c, b]$ satisfies the second boundary condition (4) and both transmission conditions (2). Moreover, in view of (24) $\psi_{\mu_0}(x)$ would satisfy also the first boundary condition (3). Consequently the function $\psi_{\mu_0}(x)$ is an eigenfunction of the problem(1)- (4) corresponding to the eigenvalue μ_0. Hence each zero of $w(\mu)$ is an eigenvalue. Now let $y_0(x)$ be any eigenfunction corresponding to eigenvalue μ_0. Suppose, it possible that $w(\mu_0) \neq 0$. Then the couples of the functions φ^-, ψ^- and φ^+, ψ^+ would be linearly independent on $[a, c)$ and $(c, b]$ respectively. Therefore, the solution $y_0(x)$ may be represented in the form

$$y_0(x) = \begin{cases} k_{11}\varphi^-_{\mu_0}(x) + k_{12}\psi^-_{\mu_0}(x) \text{ for } x \in [a, c) \\ k_{21}\varphi^+_{\mu_0}(x) + k_{22}\psi^+_{\mu_0}(x) \text{ for } x \in (c, b] \end{cases} \tag{25}$$

where at least one of the coefficients k_{11}, k_{12}, k_{21} and k_{22} is not zero. Considering the equations

$$U_1(y_0) = U_2(y_0) = V_1(y_0) = V_2(y_0) = 0 \tag{26}$$

as the homogenous system of linear equations of the variables $k_{11}, k_{12}, k_{21}, k_{22}$ and taking into account the conditions (2) – (4) we obtain homogenous linear simultaneous equation of the variables k_{ij} $(i, j = 1, 2)$ the determinant of which is equal to $\frac{1}{\Delta_{12}\Delta_{34}}\omega^3(\mu)$ and therefore does not vanish by assumption. Consequently this linear simultaneous equation has only trivial solution $k_{ij} = 0$ $(i, j = 1, 2)$. We thus arrive at a contradiction, which completes the proof. □

Now by modifying the classical method we shall prove the next Theorem.

Theorem 4.3. *The eigenvalues of the boundary value transmission problem* (1) − (4) *are real.*

Proof. Let μ_0 be eigenvalue and y_0 be eigenfunction corresponding to this eigenvalue. Denoting $\begin{pmatrix} \ell_a(y) & \ell'_a(y) \\ \ell_b(y) & \ell'_b(y) \end{pmatrix} = \begin{pmatrix} \alpha_{10}y(a) - \alpha_{11}y'(a) & \alpha'_{10}y(a) - \alpha'_{21}y'(a) \\ \alpha_{20}y(b) - \alpha_{21}y'(b) & \alpha'_{20}y(b) - \alpha'_{21}y'(b) \end{pmatrix}$
we have

$$\begin{aligned}
& \frac{\Delta_{34}}{p^-}\int_a^{c-}(\mu_0 y_0(x))\overline{y_0(x)}dx + \frac{\Delta_{12}}{p^+}\int_{c+}^{b}((\mu_0 y_0(x))\overline{y_0(x)}dx + \frac{\Delta_{34}}{p^-\theta_1}\ell_a(y_0)\overline{\ell'_a(y_0)} \\
- & \frac{\Delta_{12}}{p^+\theta_2}\ell_b(y_0)\overline{\ell'_b(y_0)} - \{\frac{\Delta_{34}}{p^-}\int_a^{c-}(y_0)(x)\overline{\mu_0 y_0(x)}dx + \frac{\Delta_{12}}{p^+}\int_{c+}^{b}(y_0)(x)\overline{\mu_0 y_0(x)}dx \\
+ & \frac{\Delta_{34}}{p^-\theta_1}\ell'_a(y_0)\overline{\ell_a(y_0)} - \frac{\Delta_{12}}{p^+\theta_2}\ell'_b(y_0)\overline{\ell_b(y_0)}\} \\
= & \Delta_{34}\ W[y_0,\bar{z};c-] - \Delta_{34}\ W[y_0,\overline{y_0};a] + \Delta_{12}W[y_0,\overline{y_0};b] - \Delta_{12}\ W[y_0,\overline{y_0};c+] \\
+ & \frac{\Delta_{34}}{p^-\theta_1}\{\ell_a(y_0)\overline{\ell'_a(y_0)} - \ell'_a(y_0)\overline{\ell_a(y_0)}\} + \frac{\Delta_{12}}{p^+\theta_2}\{\ell'_b(y_0)\overline{\ell_b(y_0)} - \ell_b(y_0)\overline{\ell'_b(y_0)}\}
\end{aligned} \tag{27}$$

Since the eigenfunction $y_0(x)$ is satisfied the boundary and transmission conditions (2) − (4) it is easy to derive that

$$\ell_a(y_0)\overline{\ell'_a(y_0)} - \ell'_a(y_0)\overline{\ell_a(y_0)} = p^-\theta_1 W(y_0,\overline{y_0};a) \tag{28}$$

$$\ell'_b(y_0)\overline{\ell_b(y_0)} - \ell_b(y_0)\overline{\ell'_b(y_0)} = -p^+\theta_2 W(y_0,\overline{y_0};b) \tag{29}$$

$$W(y,\overline{y_0};c-) = \frac{\Delta_{12}}{\Delta_{34}}\ W(y_0,\overline{y_0};c+). \tag{30}$$

By substituting these equations in (27) we have

$$(\mu_0 - \overline{\mu_0})[\frac{\Delta_{34}}{p^-}\int_a^{c-}(y_0(x))^2dx + \frac{\Delta_{12}}{p^+}\int_{c+}^{b}(y_0(x))^2dx] = 0$$

Since $p^- > 0$ $p^+ > 0$, $\Delta_{12} > 0$ and $\Delta_{34} > 0$ we get $\mu_0 = \overline{\mu_0}$. Consequently all eigenvalues of the problem (1) − (4) are real. The proof is complete. □

Since the Wronskians of $\varphi^+_\mu(x)$ and $\psi^+_\mu(x)$ are independent of x, in particular, by putting $x = a$ we have

$$\begin{aligned}
\omega(\mu) &= \Delta_{34}\{\varphi^-(a,\mu)\frac{d\psi^-(a,\mu)}{dx} - \frac{d\varphi^-(a,\mu)}{dx}\psi^-(a,\mu)\} \\
&= \Delta_{34}\{(\alpha_{11} - \mu^2\alpha'_{11})\frac{d\psi^-(a,\mu)}{dx} + (\alpha_{10} - \mu^2\alpha'_{10})\psi^-(a,\mu)\}.
\end{aligned} \tag{31}$$

Let $Im\mu = t$. By substituting (20) and (23) in (31) we obtain easily the following asymptotic representations
(i) If $\alpha'_{21} \neq 0$ and $\alpha'_{11} \neq 0$, then

$$w(\mu) = -\frac{\Delta_{24}\alpha'_{11}\alpha'_{21}}{\sqrt{p^-}\sqrt{p^+}}\mu^6\sin\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\sin\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right] + O\left(|\mu|^5 e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{32}$$

(ii) If $\alpha'_{21} \neq 0$ and $\alpha'_{11} = 0$, then

$$w(\mu) = -\frac{\Delta_{24}\alpha'_{10}\alpha'_{21}}{\sqrt{p^+}}\mu^5\cos\left[\frac{\mu\,(c-a)}{\sqrt{p^-}}\right]\sin\left[\frac{\mu\,(b-c)}{\sqrt{p^+}}\right] + O\left(|\mu|^4 e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{33}$$

(iii) If $\alpha'_{21} = 0$ and $\alpha'_{11} \neq 0$, then

$$w(\mu) = -\frac{\Delta_{24}\alpha'_{11}\alpha'_{20}}{\sqrt{p^-}}\mu^5 \sin\left[\frac{\mu(c-a)}{\sqrt{p^-}}\right]\cos\left[\frac{\mu(b-c)}{\sqrt{p^+}}\right] + O\left(\left|\mu\right|^4 e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{34}$$

(iv) If $\alpha'_{21} = 0$ and $\alpha'_{11} = 0$, then

$$w(\mu) = \Delta_{24}\alpha'_{10}\alpha'_{20}\mu^4 \cos\left[\frac{\mu(c-a)}{\sqrt{p^-}}\right]\cos\left[\frac{\mu(b-c)}{\sqrt{p^+}}\right] + O\left(\left|\mu^3\right| e^{|t|(\frac{(b-c)}{\sqrt{p^+}}+\frac{(c-a)}{\sqrt{p^-}})}\right) \tag{35}$$

5. Asymptotic formulas for eigenvalues and eigenfunctions

Now we are ready to derive the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 5.1. *The boundary-value-transmission problem* (1)-(4) *has an precisely numerable many real eigenvalues, whose behavior may be expressed by two sequence* $\{\mu_{n,1}\}$ *and* $\{\mu_{n,2}\}$ *with following asymptotic as* $n \to \infty$
(i) *If* $\alpha'_{21} \neq 0$ *and* $\alpha'_{11} \neq 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(n-3)\pi}{(c-a)} + O\left(\frac{1}{n}\right),\ \mu_{n,2} = \frac{\sqrt{p^+}n\pi}{(b-c)} + O\left(\frac{1}{n}\right), \tag{36}$$

(ii) *If* $\alpha'_{21} \neq 0$ *and* $\alpha'_{11} = 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(2n+1)\pi}{2(c-a)} + O\left(\frac{1}{n}\right),\ \mu_{n,2} = \frac{\sqrt{p^+}(n-2)\pi}{(b-c)} + O\left(\frac{1}{n}\right), \tag{37}$$

(iii) *If* $\alpha'_{21} = 0$ *and* $\alpha'_{11} \neq 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(n-2)\pi}{(c-a)} + O\left(\frac{1}{n}\right),\ \mu_{n,2} = \frac{\sqrt{p^+}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right), \tag{38}$$

(iv) *If* $\alpha'_{21} = 0$ *and* $\alpha'_{11} = 0$*, then*

$$\mu_{n,1} = \frac{\sqrt{p^-}(2n-3)\pi}{2(c-a)} + O\left(\frac{1}{n}\right),\ \mu_{n,2} = \frac{\sqrt{p^+}(2n+1)\pi}{2(b-c)} + O\left(\frac{1}{n}\right) \tag{39}$$

Proof. Let $\alpha'_{21} \neq 0$ and $\alpha'_{11} \neq 0$. By applying the well-known Rouche Theorem which asserts that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ then $f(z)$ and $f(z) + g(z)$ have the same number zeros inside Γ provided that the zeros are counted with multiplicity, it follows that $w(\mu)$ has the same number of zeros inside the sufficiently large appropriate contours as the leading term $w_0(\mu) = -\frac{\Delta_{24}\alpha'_{11}\alpha'_{21}}{\sqrt{p^-}\sqrt{p^+}}\mu^6 \sin\left[\frac{\mu(c-a)}{\sqrt{p^-}}\right]\sin\left[\frac{\mu(b-c)}{\sqrt{p^+}}\right]$ in (32). Now applying the similar technique which used in [20] we can found the needed asymptotic formulas (36) . Other cases can be proved similarly. □

Finally, using the fact that the function $\widetilde{\varphi}_{n,i}$ defined on whole $[a,c)\cup(c,b]$ and given by $\widetilde{\varphi}_{n,i} = \varphi^-_{\mu_{n,i}}(x)$ for $x \in [a,c)$ and $\widetilde{\varphi}_{n,i} = \varphi^+_{\mu_{n,i}}(x)$ for $x \in (c,b]$ is an eigenfunction according to the eigenvalue μ_n and putting (36)-(39) in the (13)-(16) and (20)-(23) we found the following asymptotic formulas for eigenfunctions:
(i) If $\alpha'_{21} \neq 0$ and $\alpha'_{11} \neq 0$, then

$$\widetilde{\varphi}_{n,1}(x) = \begin{cases} -\alpha'_{11}p^-\left[\frac{(n-3)\pi}{(c-a)}\right]^2\cos\left[\frac{(n-3)\pi(x-a)}{(c-a)}\right] + O(n), \quad \text{for } x \in [a,c) \\ \frac{\Delta_{24}\alpha'_{11}p^-}{\Delta_{12}}\left[\frac{(n-3)\pi}{(c-a)}\right]^3 \sin[(n-3)\pi]\cos\left[\frac{\sqrt{p^-}(n-3)\pi(x-c)}{\sqrt{p^+}(c-a)}\right] \\ +O\left(n^2\right), \ \text{for } x \in (c,b] \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases}-\alpha_{11}'p^{+}\left[\frac{n\pi}{(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}n\pi(x-a)}{\sqrt{p^{-}}(b-c)}\right]+O(n), & \text{for } x\in[a,c)\\ \frac{\Delta_{24}\alpha_{11}'}{\Delta_{12}\sqrt{p^{-}}}\left[\frac{\sqrt{p^{+}}n\pi}{(b-c)}\right]^{3}\sin\left[\frac{\sqrt{p^{+}}n\pi(c-a)}{\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{n\pi(x-c)}{(b-c)}\right] & \\ +O\left(n^{2}\right), \text{ for } x\in(c,b] & \end{cases}$$

(ii) If $\alpha_{21}'\neq 0$ and $\alpha_{11}'=0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases}-\alpha_{10}'p^{-}\left[\frac{(2n+1)\pi}{2(c-a)}\right]\sin\left[\frac{(2n+1)\pi(x-a)}{2(c-a)}\right]+O(1), & \text{for } x\in[a,c)\\ -\frac{\Delta_{24}\alpha_{10}'p^{-}}{\Delta_{12}}\left[\frac{(2n+1)\pi}{2(c-a)}\right]^{2}\cos\left[\frac{(2n+1)\pi}{2}\right]\cos\left[\frac{\sqrt{p^{-}}(2n+1)\pi(x-c)}{2\sqrt{p^{+}}(c-a)}\right] & \\ +O(n), \text{ for } x\in(c,b] & \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases}-\alpha_{10}'\sqrt{p^{-}}\left[\frac{\sqrt{p^{+}}(n-2)\pi}{(b-c)}\right]\sin\left[\frac{\sqrt{p^{+}}(n-2)\pi(x-a)}{\sqrt{p^{-}}(b-c)}\right]+O(1), & \text{for } x\in[a,c)\\ -\frac{\Delta_{24}\alpha_{10}'p^{+}}{\Delta_{12}}\left[\frac{(n-2)\pi}{(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}(n-2)\pi(c-a)}{\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{(n-2)\pi(x-c)}{(b-c)}\right] & \\ +O(n), \text{ for } x\in(c,b] & \end{cases}$$

(iii) If $\alpha_{21}'=0$ and $\alpha_{11}'\neq 0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases}-\alpha_{11}'p^{-}\left[\frac{(n-2)\pi}{(c-a)}\right]^{2}\cos\left[\frac{(n-2)\pi(x-a)}{(c-a)}\right]+O(n), & \text{for } x\in[a,c)\\ \frac{-\Delta_{24}\alpha_{11}'p^{-}}{\Delta_{12}}\left[\frac{(n-2)\pi}{(c-a)}\right]^{3}\sin\left[(n-2)\pi\right]\cos\left[\frac{\sqrt{p^{-}}(n-2)\pi(x-c)}{\sqrt{p^{+}}(c-a)}\right] & \\ +O\left(n^{2}\right), \text{ for } x\in(c,b] & \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases}-\alpha_{11}'p^{+}\left[\frac{(2n+1)\pi}{2(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}(2n+1)\pi(x-a)}{2\sqrt{p^{-}}(b-c)}\right]+O(n), & \text{for } x\in[a,c)\\ -\frac{\Delta_{24}\alpha_{11}'}{\Delta_{12}\sqrt{p^{-}}}\left[\frac{\sqrt{p^{+}}(2n+1)\pi}{2(b-c)}\right]^{3}\sin\left[\frac{\sqrt{p^{+}}(2n+1)\pi(c-a)}{2\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{(2n+1)\pi(x-c)}{2(b-c)}\right] & \\ +O\left(n^{2}\right), \text{ for } x\in(c,b] & \end{cases}$$

(iv) If $\alpha_{21}'=0$ and $\alpha_{11}'=0$, then

$$\widetilde{\varphi}_{n,1}(x)=\begin{cases}-\alpha_{10}'p^{-}\left[\frac{(2n-3)\pi}{2(c-a)}\right]\sin\left[\frac{(2n-3)\pi(x-a)}{2(c-a)}\right]+O(1) & \text{for } x\in[a,c)\\ -\frac{\Delta_{24}p^{-}\alpha_{10}'}{\Delta_{12}}\left[\frac{(2n-3)\pi}{2(c-a)}\right]^{2}\cos\left[\frac{(2n-3)\pi}{2}\right]\cos\left[\frac{\sqrt{p^{-}}(2n-3)\pi(x-c)}{2\sqrt{p^{+}}(c-a)}\right] & \\ +O(1), \text{ for } x\in(c,b] & \end{cases}$$

and

$$\widetilde{\varphi}_{n,2}(x)=\begin{cases}-\alpha_{10}'\sqrt{p^{-}}\left[\frac{\sqrt{p^{+}}(2n+1)\pi}{2(b-c)}\right]\sin\left[\frac{\sqrt{p^{+}}(2n+1)\pi(x-a)}{2\sqrt{p^{-}}(b-c)}\right]+O(1), & \text{for } x\in[a,c)\\ -\frac{\Delta_{24}p^{+}\alpha_{10}'}{\Delta_{12}}\left[\frac{(2n+1)\pi}{2(b-c)}\right]^{2}\cos\left[\frac{\sqrt{p^{+}}(2n+1)\pi(c-a)}{2\sqrt{p^{-}}(b-c)}\right]\cos\left[\frac{(2n+1)\pi(x-c)}{2(b-c)}\right] & \\ +O(n), \text{ for } x\in(c,b] & \end{cases}$$

All this asymptotic approximations are hold uniformly for x.

Remark 5.2. *Although eigenfunction looks as simple as that of standard Sturm-Liouville problems, it is a rather complicated because of the transmission conditions. To illustration this fact let us give the graphs of characteristic function $\omega(\mu)$ and eigenfunction $\varphi_{\mu}(x)$ for one simple special case of the considered problem.*

6. Example

Consider the following simple case of the BVTP's (1) − (4)

$$-y''(x) = \mu^2 y(x) \quad x \in [-\pi, 0) \cup (0, \pi] \tag{40}$$

$$y(-\pi) + \mu^2 y'(-\pi) = 0, \tag{41}$$

$$\mu^2 y(\pi) + y'(\pi) = 0, \tag{42}$$

$$y(0-) = 2y(+0), \quad y'(-0) = y'(+0) \tag{43}$$

We find easily that

$$\varphi^-(x,\mu) = \mu^2 \cos[\mu(\pi + x)] - \tfrac{1}{\mu}\sin[\mu(\pi + x)]$$

$$\varphi^+(x,\mu) = \tfrac{1}{2}(\mu^2 \cos \mu\pi - \tfrac{1}{\mu}\sin \mu\pi)\cos(\mu x) - (\mu^2 \sin \mu\pi + \tfrac{1}{\mu}\cos \mu\pi)\sin(\mu x)$$

$$\psi^-(x,\mu) = 2(\cos \mu\pi - \mu \sin \mu\pi)\cos(\mu x) + (\sin \mu\pi + \mu \cos \mu\pi)\sin(\mu x)$$

$$\psi^+(x,\mu) = \cos[\mu(\pi - x)] - \mu \sin[\mu(\pi - x)].$$

Using these formulas we have

$$w(\mu) = (\mu^4 + 2)\cos^2 \mu - (2\mu^4 + 1)\sin^2 \mu + 3(\mu^3 - \mu)\sin \mu \cos \mu.$$

The graphs of the functions $w(\mu)$ and $\varphi_\mu(x)$ is displayed in following Figures for different values of spectral parameter.

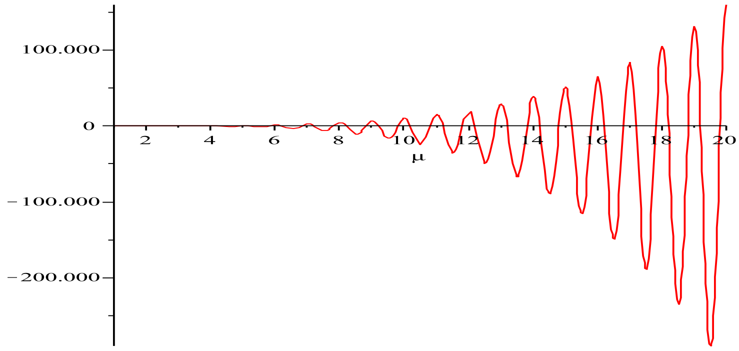

Figure1: Graph of the characteristic function $w(\mu)$ for real μ

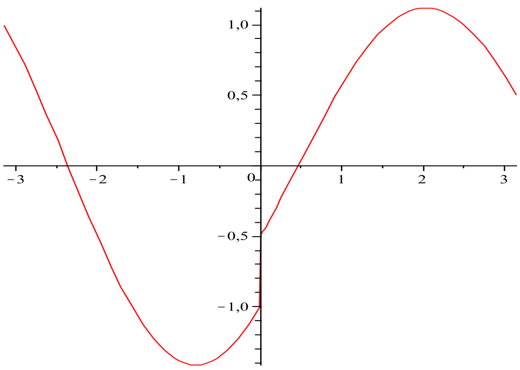

Figure2: Graph of the eigensolution $\varphi_\mu(x)$ for $\mu = 1$

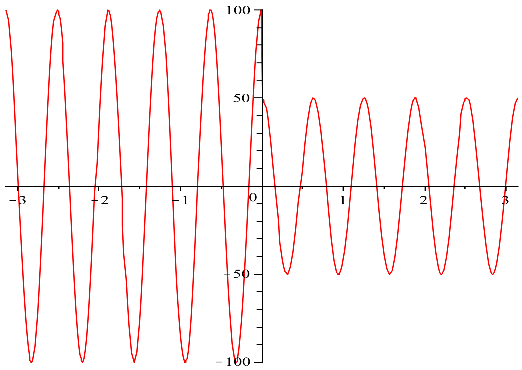

Figure3: Graph of the eigensolution $\varphi_\mu(x)$ for $\mu = 10$

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