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Multidecomposition of Cartesian Product of Some Graphs into Even Cycles and Matchings

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Abstract. Let C_{2p} and pK_2 denote a cycle with $2p$ edges and p vertex-disjoint edges, respectively. For graphs G, H' and H'', a (H', H'') -multidecomposition of G is a partition of the edge set of G into copies of H' and copies of H'' with at least one copy of H' and at least one copy of H''. In this paper, we investigate (C_{2p}, pK_2)-multidecomposition of the Cartesian product of paths, cycles and complete graphs, for some values $p \geq 3$.

1. Introduction

All graphs considered here are finite undirected simple graphs only. For the discussions, some terminologies and notations are needed. Let P_n for the path on n vertices, C_n for the cycle on n vertices, K_n for the complete graph on n vertices, and pK_2 for p vertex-disjoint edges. Let $V(P_n) = V(C_n) = V(K_n) = \{0, 1, 2, \ldots, n-1\}$, $E(P_n) = \{\{i, i+1\} : i \in \{0, 1, 2, \ldots, n-2\}\}$ and $E(C_n) = E(P_n) \cup \{\{n-1, 0\}\}$.

A *decomposition* of a graph G is a collection $\mathscr{G} = \{G_1, G_2, \ldots, G_s\}$ of nonempty subgraphs of G such that the sets $E(G_1), E(G_2), \ldots, E(G_s)$ form a partition of $E(G)$, where $E(G_i)$ and $E(G)$ are, respectively, the edge sets of G_i and G; denote this by $G = G_1 \oplus G_2 \oplus \cdots \oplus G_s$.

Consider a decomposition $\mathscr{G} = \{G_1, G_2, \ldots, G_s\}$ of G. If, for every $i \in \{1, 2, \ldots, s\}$, $G_i \cong H$, then say that H *divides* G and denote it by $H|G$, and the collection $\mathscr{G}$ is called a *H-decomposition of* G or a *H-design of* G.

Consider a decomposition $\mathscr{G} = \{G_1, G_2, \ldots, G_s\}$ of G; $s \geq 2$. If there exists $\ell \in \{1, 2, \ldots, s-1\}$ such that, for every $i \in \{1, 2, \ldots, \ell\}$, $G_i \cong H'$ and for every $i \in \{\ell+1, \ell+2, \ldots, s\}$, $G_i \cong H''$, and if $H' \not\cong H''$, then say that the graph-pair (H', H'') *divides* G, and the collection $\mathscr{G}$ is called a (H', H'')*-multidecomposition of* G or a (H', H'')*-multidesign of* G.

The *Cartesian product* $H_1 \square H_2$ of two graphs H_1 and H_2 is the simple graph with $V(H_1) \times V(H_2)$ as its vertex set and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $H_1 \square H_2$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in H_2, or u_1 is adjacent to u_2 in H_1 and $v_1 = v_2$.

The study of the (G, H)-multidecomposition was introduced by Abueida and Daven in [2]. Abueida and Daven [4] investigated the problem of the (K_k, S_k)-multidecomposition of the complete graph K_n. In [5] Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of the (G_n, H_n)-multidecomposition of λK_n where $G_n, H_n \in \{C_n, P_{n-1}, S_{n-1}\}$, where S_n denote the star on $n+1$ vertices. The graph multidecomposition problems has been widely studied (see [6 − 10]). Abueida and Daven

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[3] have recently established necessary and sufficient conditions for $(C_4, 2K_2)$-multidecomposition of the Cartesian products $P_m\Box P_n$, $P_m\Box C_n$, $P_m\Box K_n$, $C_m\Box C_n$, $C_m\Box K_n$ and $K_m\Box K_n$. On this extension, we have consider (C_{2p}, pK_2)-multidecomposition of the above Cartesian products, for some values of $p \geq 3$.

2. Cartesian Product of Paths

In this section, we have proved that $P_m\Box P_n$ admits a (C_{2p}, pK_2)-multidecomposition, for some values of $p \geq 3$.

As $g.c.d.(|E(C_{2p})|, |E(pK_2)|) = g.c.d.(2p, p) = p$ and $|E(P_m\Box P_n)| = 2mn - m - n$. If $P_m\Box P_n$ admits a (C_{2p}, pK_2)-multidecomposition, then p divides $2mn-m-n$. Observe that, if either $m \equiv 0 \, mod\, p \equiv n$ or $m \equiv 1 \, mod\, p \equiv n$, then $p|(2mn - m - n)$. Note that, for $p = 3$, $3|(2mn - m - n)$ if and only if either $m \equiv 0 \, mod\, 3 \equiv n$ or $m \equiv 1 \, mod\, 3 \equiv n$. For $p = 4$, $4|(2mn - m - n)$ if and only if $(m\, mod\, 4, n\, mod\, 4) \in \{(0,0),(1,1),(2,2),(3,3)\}$. For $p = 5$, $5|(2mn - m - n)$ if and only if $(m\, mod\, 5, n\, mod\, 5) \in \{(0,0),(1,1),(2,4),(4,2)\}$.

Theorem 2.1. *For integers $m, n \geq p$ and $(m,n) \neq (3,3)$, either $m \equiv 0 \, mod\, p \equiv n$ or $m \equiv 1 \, mod\, p \equiv n$ then $P_m\Box P_n$ admits a (C_{2p}, pK_2)-multidecomposition for all $p \geq 3$.*

Theorem 2.2. *For integers $m, n \geq 3$ and $(m,n) \neq (3,3)$, $P_m\Box P_n$ admits a $(C_6, 3K_2)$-multidecomposition if and only if $(m,n) \neq (3,3)$ and either $m \equiv 0 \, mod\, 3 \equiv n$ or $m \equiv 1 \, mod\, 3 \equiv n$.*

Theorem 2.3. *For integers $m, n \geq 2$ and $(m,n) \neq (2,2)$, $P_m\Box P_n$ admits a $(C_8, 4K_2)$-multidecomposition if and only if $(m\, mod\, 4, n\, mod\, 4) \in \{(0,0),(1,1),(2,2),(3,3)\}$.*

Theorem 2.4. *For integers $m, n \geq 2$ and $(m,n) \neq (2,2)$, $P_m\Box P_n$ admits a $(C_{10}, 5K_2)$-multidecomposition if and only if $(m\, mod\, 5, n\, mod\, 5) \in \{(0,0),(1,1),(2,4),(4,2)\}$.*

Proof of Theorem 2.1 follows from Lemmas 2.5 to 2.9; proof of Theorem 2.2 follows from Theorem 2.1 and Lemma 2.10; proof of Theorem 2.3 follows from Theorem 2.1 and Lemmas 2.11 and 2.12; proof of Theorem 2.4 follows from Theorem 2.1 and Lemma 2.13.

Lemma 2.5. *If $n \equiv 0 \, mod\, p$, and if $(p,n) \neq (3,3)$, then $P_p\Box P_n$ admits a (C_{2p}, pK_2)-multidecomposition.*

Proof. Consider two cases.
Case 1. $n \equiv 0 \, mod\, 2$.

For $j \in \{0, 1, \ldots, \frac{n-2}{2}\}$, the cycle $C_{2p}(j) = (0,2j)(0,2j+1)(1,2j+1)(2,2j+1)\cdots(p-1,2j+1)(p-1,2j)(p-2,2j)(p-3,2j)\cdots(1,2j)(0,2j)$ is isomorphic to C_{2p}. For $j \in \{0,1,\ldots,\frac{n-4}{2}\}$, the graph $M_p^1(j) = \bigoplus_{i=0}^{p-1}(i,2j+1)(i,2j+2)$ is isomorphic to pK_2. For $j \in \{0,1,\ldots,\frac{n-2}{2}\}$, the graph $M_p^2(j) = \bigoplus_{i=1}^{p-2}(i,2j)(i,2j+1)$ is a matching of cardinality $p-2$. Furthermore, $\bigcup_{j=0}^{\frac{n-2}{2}} M_p^2(j)$ is a matching of cardinality $\frac{n}{2}(p-2)$. If p is odd, then $n \equiv 0 \, mod\, p$ and $n \equiv 0 \, mod\, 2$ implies that $\frac{n}{2} \equiv 0 \, mod\, p$ and therefore $\frac{n}{2}(p-2) \equiv 0 \, mod\, p$. If p is even, then $n \equiv 0 \, mod\, p$ implies that $\frac{n}{2} \equiv 0 \, mod\, \frac{p}{2}$; this together with $p-2 \equiv 0 \, mod\, 2$ implies that $\frac{n}{2}(p-2) \equiv 0 \, mod\, p$. In any case, $\frac{n}{2}(p-2) \equiv 0 \, mod\, p$. Consequently, $(pK_2)|(\bigcup_{j=0}^{\frac{n-2}{2}} M_p^2(j))$. Hence, $\{C_{2p}(j) : j \in \{0,1,\ldots,\frac{n-2}{2}\}\} \cup \{M_p^1(j) : j \in \{0,1,\ldots,\frac{n-4}{2}\}\} \cup \{\bigcup_{j=0}^{\frac{n-2}{2}} M_p^2(j)\}$ form a (C_{2p}, pK_2)-multidecomposition of $P_p\Box P_n$.
Case 2. $n \equiv 1 \, mod\, 2$.
Subcase 2.1. $p \neq 3$.

For $j \in \{0,1,\ldots,\frac{n-3}{2}\}$, the cycle $C_{2p}(j) = (0,2j)(0,2j+1)(1,2j+1)(2,2j+1)\cdots(p-1,2j+1)(p-1,2j)(p-2,2j)(p-3,2j)\cdots(0,2j)$ is isomorphic to C_{2p}. For $j \in \{0,1,\ldots,\frac{n-3}{2}\}$, the graph $M_p^1(j) = \bigoplus_{i=0}^{p-1}(i,2j+1)(i,2j+2)$ is isomorphic to pK_2. For $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$, the graph $M_p^2(j) = \bigoplus_{i=1}^{p-2} (i, 2j)(i, 2j+1)$ is a matching of cardinality $p-2$. For each $j \in \{0, 1, \ldots, \frac{p-3}{2}\}$, $M_p^2(j) \bigcup \{(j, n-1)(j+1, n-1), (\frac{p-1}{2}+j, n-1)(\frac{p+1}{2}+j, n-1)\} = M_p^3(j)$ is isomorphic to pK_2. Furthermore, $\bigcup_{j=\frac{p-1}{2}}^{\frac{n-3}{2}} M_p^2(j)$ is a matching of cardinality $\frac{n-p}{2}(p-2)$. $n \equiv 0 \, mod \, p$ and $n \equiv 1 \, mod \, 2$ implies that $p \equiv 1 \, mod \, 2$. Thus $\frac{n-p}{2} \equiv 0 \, mod \, p$. Consequently, $(pK_2) | (\bigcup_{j=\frac{p-1}{2}}^{\frac{n-3}{2}} M_p^2(j))$. Hence, $\{C_{2p}(j) : j \in \{0, 1, \ldots, \frac{n-3}{2}\}\} \cup \{M_p^1(j) : j \in \{0, 1, \ldots, \frac{n-3}{2}\}\} \cup \{\bigcup_{j=\frac{p-1}{2}}^{\frac{n-3}{2}} M_p^2(j)\} \cup \{M_p^3(j) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\}$ form a (C_{2p}, pK_2)-multidecomposition of $P_p \square P_n$.

Subcase 2.2. $p = 3$.

$n \equiv 0 \, mod \, 3$ and $n \equiv 1 \, mod \, 2$ implies that $n \equiv 3 \, mod \, 6$.

For $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$, the cycle $C_6(j) = (0, 2j)(0, 2j+1)(1, 2j+1)\,(2, 2j+1)(2, 2j)(1, 2j)(0, 2j)$ is isomorphic to C_6. For $j \in \{0, 1, \ldots, \frac{n-3}{2}\}$, the graph $M_3^1(j) = (0, 2j+1)(0, 2j+2) \oplus (1, 2j+1)(1, 2j+2) \cup (2, 2j+1)\,(2, 2j+2)$, the graph $M_3^2 = (0, n-1)(1, n-1) \oplus (1, n-2)(1, n-3) \oplus (1, n-4)\,(1, n-5)$, the graph $M_3^3 = (1, n-1)(2, n-1) \oplus (1, n-6)(1, n-7) \oplus (1, n-8)(1, n-9)$, and for $n \geq 15$ and $j \in \{0, 1, \ldots, \frac{n-15}{6}\}$, the graph $M_3^4(j) = (1, 6j)(1, 6j+1) \oplus (1, 6j+2)(1, 6j+3) \oplus (1, 6j+4)(1, 6j+5)$ are all isomorphic to $3K_2$.

$\{C_6(j) : j \in \{0, 1, \ldots, \frac{n-3}{2}\}\} \cup \{M_3^1(j) : j \in \{0, 1, \ldots, \frac{n-3}{2}\}\} \cup \{M_3^2\} \cup \{M_3^3\} \cup \{M_3^4(j) : n \geq 15$ and $j \in \{0, 1, \ldots, \frac{n-15}{6}\}\}$ form a $(C_6, 3K_2)$-multidecomposition of $P_3 \square P_n$.

Lemma 2.6. *If $m \equiv 0 \, mod \, p \equiv n$ and if $(m, n) \neq (3, 3)$, then $P_m \square P_n$ admits a (C_{2p}, pK_2)-multidecomposition.*

Proof. As $(m, n) \neq (3, 3)$, either $(p, n) \neq (3, 3)$ or $(m, p) \neq (3, 3)$. Without loss of generality assume that $(p, n) \neq (3, 3)$. Observe that $P_m \square P_n = \frac{m}{p}(P_p \square P_n) \oplus \frac{m-p}{p}(nK_2)$. By Lemma 2.5, $P_p \square P_n$ admits a (C_{2p}, pK_2)-multidecomposition. As $n \equiv 0 \, mod \, p$, $(pK_2) | (nK_2)$ and hence, $(pK_2) | [\frac{m-p}{p}(nK_2)]$. Thus $P_m \square P_n$ admits a (C_{2p}, pK_2)-multidecomposition.

Lemma 2.7. *$P_4 \square P_4$ admits a $(C_6, 3K_2)$-multidecomposition.*

Proof. $P_4 \square P_4$ = the 6-cycle $(0, 0)(0, 1)(0, 2)(1, 2)(1, 1)(1, 0)(0, 0)$ $\oplus$ the 6-cycle $(2, 1)(2, 2)(2, 3)(3, 3)(3, 2)(3, 1)(2, 1)$ $\oplus$ the $3K_2$ $\{(0, 1)(1, 1), (0, 2)(0, 3), (1, 2)(1, 3)\}$ $\oplus$ the $3K_2$ $\{(2, 0)(2, 1), (3, 0)(3, 1), (2, 2)(3, 2)\}$ $\oplus$ the $3K_2$ $\{(0, 3)(1, 3), (1, 2)(2, 2), (2, 0)(3, 0)\}$ $\oplus$ the $3K_2$ $\{(1, 0)(2, 0), (1, 1)(2, 1), (1, 3)(2, 3)\}$.

Lemma 2.8. *If $k \equiv 1 \, mod \, p$, and if $k \neq p+1$, then $(pK_2) | P_k$.*

Proof. For each $j \in \{0, 1, \ldots, \frac{k-1-p}{p}\}$, consider $\bigcup_{i=0}^{p-1} \{i(\frac{k-1}{p}) + j, i(\frac{k-1}{p}) + 1 + j\}$. It is a matching of cardinality p. Hence $(pK_2) | P_k$.

Lemma 2.9. *If $m \equiv 1 \, mod \, p \equiv n$ with $m, n \geq 4$, then $P_m \square P_n$ admits a (C_{2p}, pK_2)-multidecomposition for all $p \geq 3$.*

Proof. If $(m, n) = (4, 4)$, then $p = 3$ and hence the lemma follows by Lemma 2.7. Hence, assume that $(m, n) \neq (4, 4)$. Observe that $P_m \square P_n$ = a path $(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0)(0, 1)(0, 2) \ldots (0, n-2)(0, n-1)$ $\oplus$ a matching $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-1\}\}$ $\oplus$ a matching $\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-1\}\}$ $\oplus$ $(P_{m-1} \square P_{n-1})$.

Since $m-1 \equiv 0 \, mod \, p \equiv n-1$, by Lemma 2.6, $P_{m-1} \square P_{n-1}$ admits a (C_{2p}, pK_2)-multidecomposition if $(m, n) \neq (4, 4)$. Again, since $m-1 \equiv 0 \, mod \, p \equiv n-1$, the matchings $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-1\}\}$ and $\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-1\}\}$ are each divisible by pK_2. Finally, by Lemma 2.8, the path $(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0)(0, 1)(0, 2) \ldots (0, n-2)(0, n-1)$ is divisible by pK_2 since its order $\equiv 1 \, mod \, p$ and $\neq p+1$.

Lemma 2.10. *There is no $(C_6, 3K_2)$-multidecomposition for $P_3 \Box P_3$.*

Proof. Suppose $P_3 \Box P_3$ admits a $(C_6, 3K_2)$-multidecomposition. Then the removal of the edges of any C_6 from $P_3 \Box P_3$ is a forest and it contains three mutually adjacent edges. These three mutually adjacent edges are edges of two $3K_2$'s in the multidecompostion, a contradiction.

Lemma 2.11. *If $m \equiv 2 \, mod \, 4 \equiv n, m, n \geq 2$ and $(m, n) \neq (2, 2)$, then $P_m \Box P_n$ admits a $(C_8, 4K_2)$-multidecomposition.*

Proof. If $m = 2$ and $n \geq 6$, then $P_2 \Box P_n$ admits a $(C_8, 4K_2)$- multidecomposition as follows. $P_2 \Box P_n$ = a cycle $\bigoplus_{j=0}^{\frac{n-6}{4}} \{(0, 4j+1)(0, 4j+2)(0, 4j+3)(0, 4j+4)(1, 4j+4)(1, 4j+3)(1, 4j+2)(1, 4j+1)(0, 4j+1)\} \oplus$ a matching $\{(0, 0)(1, 0), (0, 2)(1, 2), (0, 3)(1, 3), (0, n-1)(1, n-1)\} \oplus$ the remaining edges are form a matching with cardinality $\frac{n+2}{4}$ which is divisible by 4. Now for the remaining values of (m, n), observe that $P_m \Box P_n$ = a path $(m-1, 1)(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0)(0, 1)(0, 2) \ldots (0, n-2)(0, n-1)(1, n-1) \oplus$ a matching $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-2\}\} \oplus$ a matching $\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-2\}\} \oplus (P_{m-1} \Box P_{n-1})$.

Since $m - 1 \equiv 1 \, mod \, 4 \equiv n - 1$, by Lemma 2.9, $P_{m-1} \Box P_{n-1}$ admits a $(C_8, 4K_2)$-multidecomposition. Again, since $m - 2 \equiv 0 \, mod \, 4 \equiv n - 2$, the matchings $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-2\}\}$ and $\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-2\}\}$ are each divisible by $4K_2$. Finally, by Lemma 2.8, the path $(m-1, 1)(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0)(0, 1)(0, 2) \ldots (0, n-2)(0, n-1)(1, n-1)$ is divisible by $4K_2$ since its order $\equiv 1 \, mod \, 4$.

Lemma 2.12. *If $m \equiv 3 \, mod \, 4 \equiv n, m, n \geq 3$ and $(m, n) \neq (3, 3)$, then $P_m \Box P_n$ admits a $(C_8, 4K_2)$-multidecomposition.*

Proof. Observe that $P_m \Box P_n$ = a path $(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0) (0, 1)(0, 2) \ldots (0, n-2)(0, n-1) \oplus$ a matching $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-3\}\} \oplus$ a matching $[\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-1\}\} \cup \{(m-1, 0)(m-1, 1)\} \cup \{(m-2, 0)(m-2, 1)\}] \oplus (P_{m-1} \Box P_{n-1})$.

Since $m - 1 \equiv 2 \, mod \, 4 \equiv n - 1$, by Lemma 2.11, $P_{m-1} \Box P_{n-1}$ admits a $(C_8, 4K_2)$-multidecomposition. Again, since $m - 3 \equiv 0 \, mod \, 4 \equiv n - 3$, the matchings $\{(i, 0)(i, 1) : i \in \{1, 2, \ldots, m-3\}\}$ and $[\{(0, j)(1, j) : j \in \{1, 2, \ldots, n-1\}\} \cup \{(m-1, 0)(m-1, 1)\} \cup \{(m-2, 0)(m-2, 1)\}]$ are each divisible by $4K_2$. Finally, by Lemma 2.8, the path $(m-1, 0)(m-2, 0) \ldots (2, 0)(1, 0)(0, 0)(0, 1)(0, 2) \ldots (0, n-2)(0, n-1)$ is divisible by $4K_2$ since its order $\equiv 1 \, mod \, 4$.

Lemma 2.13. *If $m \equiv 2 \, mod \, 5$ and $n \equiv 4 \, mod \, 5$, then $P_m \Box P_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.*

Proof. Observe that $P_m \Box P_n$ = a path $(1, 0)(0, 0)(0, 1) \ldots (0, n-2)(0, n-1)(1, n-1) \oplus (2, 0)(1, 0)(1, 1) \ldots (1, n-2)(1, n-1)(2, n-1) \oplus$ the $2(\frac{m-2}{5})$-cycle $\{(5i+2, 2j)(5i+2, 2j+1)(5i+3, 2j+1)(5i+4, 2j+1)(5i+5, 2j+1)(5i+6, 2j+1)(5i+6, 2j)(5i+5, 2j)(5i+4, 2j)(5i+3, 2j)(5i+2, 2j) : i \in \{0, 1, \ldots, (\frac{m-2}{5}) - 1\}, j \in \{0, 1\}\} \oplus$ a matching $\{(i, j)(i+1, j) : i \in \{0, 1\}, j \in \{1, 2, \ldots, n-2\}\} \cup \{(5i+k, 2j)(5i+k, 2j+1) : i \in \{0, 1, \ldots, (\frac{m-2}{5}) - 1\}, j \in \{0, 1\}, k \in \{3, 4, 5\} \cup \{(5i+1, 2j)(5i+2, 2j+1) : i \in \{1, 2, \ldots, (\frac{m-2}{5}) - 1\}, j \in \{0, 1\}\} \oplus$ a matching $\{(i, 2j-1)(i, 2j) : i \in \{2, 3, \ldots, m-1\}, j \in \{1, 2\}\} \oplus (P_{m-2} \Box P_{n-4})$.

Since $m-2 \equiv 0 \, mod \, 5$ and $n-4 \equiv 0 \, mod \, 5$, by Lemma 2.9, $P_{m-2} \Box P_{n-4}$ admits a $(C_{10}, 5K_2)$-multidecomposition. Again, since $2(m-2) \equiv 0 \, mod \, 5$, the matchings $\{(i, 2j-1)(i, 2j) : i \in \{2, 3, \ldots, m-1\}, j \in \{1, 2\}\}$ and $(n-2) + 3(\frac{m-2}{5}) + 2((\frac{m-2}{5}) - 1) = (n-4) + (m-2) \equiv 0 \, mod \, 5$, $\{(i, j)(i+1, j) : i \in \{0, 1\}, j \in \{1, 2, \ldots, n-2\}\} \cup \{(5i+k, 2j)(5i+k, 2j+1) : i \in \{0, 1, \ldots, (\frac{m-2}{5}) - 1\}, j \in \{0, 1\}, k \in \{3, 4, 5\}\} \cup \{(5i+1, 2j)(5i+2, 2j+1) : i \in \{1, 2, \ldots, (\frac{m-2}{5}) - 1\}, j \in \{0, 1\}\}$ are each divisible by $5K_2$. Finally, by Lemma 2.8, the path $(1, 0)(0, 0)(0, 1) \ldots (0, n-2)(0, n-1)(1, n-1)$ $(2, 0)(1, 0)(1, 1) \ldots (1, n-2)(1, n-1)(2, n-1)$ is divisible by $5K_2$ since its order $\equiv 1 \, mod \, 5$.

3. Cartesian Product of a Path and a Cycle

In this section, we have proved that $P_m \Box C_n$ admits a (C_{2p}, pK_2)-multidecomposition, fpr some values of $p \geq 3$.

As $g.c.d.(|E(C_{2p})|, |E(pK_2)|) = g.c.d.(2p, p) = p$ and $|E(P_m \Box C_n)| = (2m-1)n$. If $P_m \Box C_n$ admits a (C_{2p}, pK_2)-multidecomposition, then p divides $(2m-1)n$. Note that, if either $2m \equiv 1 \, mod \, p$ or $n \equiv 0 \, mod \, p$ then $p|((2m-1)n)$. For $p = 3$, $3|((2m-1)n)$ if and only if either $m \equiv 2 \, mod \, 3$ or $n \equiv 0 \, mod \, 3$. For $p = 4$, $4|((2m-1)n)$ if and only if $n \equiv 0 \, mod \, 4$. For $p = 5$, $5|((2m-1)n)$ if and only if either $m \equiv 3 \, mod \, 5$ or $n \equiv 0 \, mod \, 5$.

Theorem 3.1. *For integers $m \geq 2$, $n \geq 3$, and $p \geq 3$, if $n \equiv 0 \bmod p$, then $P_m \Box C_n$ admits a (C_{2p}, pK_2)-multidecomposition.*

Lemma 3.2. *If $k \equiv 0 \bmod p$, and if $k \neq p$ then $(pK_2)|C_k$.*

Proof. Let $k = pr$, r is a positive integer. For each $j \in \{0, 1, \ldots, r-1\}$, consider $\bigcup_{i=0}^{p-1}\{ri + j, ri + j + 1\}$. It is a matching of cardinality p. Hence, $(pK_2)|C_k$.

Proof of Theorem 3.1. Let $m = \ell p + s$, where $s \in \{0, 1, \ldots, p-1\}$. Decompose $P_m \Box C_n$ as follows: (i) $P_{\ell p} \Box P_n$ with vertex set $\{0, 1, \ldots, \ell p - 1\} \times \{0, 1, \ldots, n-1\} \oplus$ (ii) a matching $\{(i, 0)(i, n-1) : i \in \{0, 1, \ldots, \ell p - 1\}\}$ of cardinality $\ell p \oplus$ (iii) $\bigoplus_{i=\ell p-1}^{m-1}$ (a matching $\{(i, j)(i+1, j) : j \in \{0, 1, \ldots, n-1\}\}$ of cardinality n) $\oplus$ (iv) $\bigoplus_{i=\ell p}^{m-1}$ (a cycle $(i, 0)(i, 1)(i, 2) \ldots (i, n-1)(i, 0)$ of cardinality n). If $(\ell p, n) \neq (3, 3)$, i.e., $(\ell, p, n) \neq (1, 3, 3)$, then by Theorem 2.1 graph (i) admits a (C_{2p}, pK_2)-multidecomposition. Clearly, graph (ii) and each graph in (iii) admits a pK_2-decomposition. By Lemma 3.2, if $n \neq p$, then each graph in (iv) admits a pK_2-decomposition. Thus it is enough to consider the following two cases.

Case 1. $n = p$.

Consider the following subcases

Subcase 1.1. For $n = p$, assume p and s are odd. Let $m = \ell p + s$, where $s \in \{0, 1, \ldots, p-1\}$. Decompose $P_m \Box C_n$ as follows: (i) $P_{\ell p} \Box P_p$ with vertex set $\{0, 1, \ldots, \ell p - 1\} \times \{0, 1, \ldots, p-1\}$, by Theorem 2.1 graph (i) admits a (C_{2p}, pK_2)-multidecomposition $\oplus$ (ii) a matching $\{(\ell p, 2j)(\ell p, 2j+1) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p - 1, \ell p - 1)(\ell p, \ell p - 1)\} \cup \{(i, 0)(i, p-1) : i \in \{0, 1, \ldots, \frac{p-3}{2}\}\}$ of cardinality $p \oplus$ (iii) a matching $\{(\ell p, 2j+1)(\ell p, 2j+2) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p - 1, 0)(\ell p, 0)\} \cup \{(i, 0)(i, p-1) : i \in \{\frac{p-1}{2}, \frac{p+1}{2}, \ldots, p-2\}\}$ of cardinality $p \oplus$ (iv) a matching $\{(\ell p - 1, j)(\ell p, j) : j \in \{1, 2, \ldots, p-2\}\} \cup \{(\ell p, 0)(\ell p, p-1), (p-1, 0)(p-1, p-1)\}$ of cardinality $p \oplus$ (v) the subgraphs $\frac{s-1}{2}$ times $P_2 \Box C_p$, for each $t \in \{1, 2, \ldots, \frac{s-1}{2}\}$, decompose $P_2 \Box C_p$ into pK_2 as follows: (a) a matching $\{(\ell p + 2t, 2j)(\ell p + 2t, 2j+1) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t + 1, 2j)(\ell p + 2t + 1, 2j + 1) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t, \ell p - 1)(\ell p + 2t + 1, \ell p - 1)\} \oplus$ (b) a matching $\{(\ell p + 2t, 2j+1)(\ell p + 2t, 2j+2) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t + 1, 2j+1)(\ell p + 2t + 1, 2j+2) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t, 0)(\ell p + 2t + 1, 0)\}$ of cardinality $p \oplus$ (c) a matching $\{(\ell p + 2t, j)(\ell p + 2t + 1, j) : j \in \{1, 2, \ldots, p-2\}\} \cup \{(\ell p + 2t, 0)(\ell p + 2t, p-1), (\ell p + 2t + 1, 0)(\ell p + 2t + 1, p-1)\}$ of cardinality $p \oplus \bigoplus_{i=0}^{\frac{s-3}{2}}$ (a matching $\{(\ell + 2i + 1, j)(\ell + 2i + 2, j) : j \in \{0, 1, \ldots, p-1\}\}$ of cardinality n).

Now assume p is odd and s is even. Except the last decomposition of the above, the remaining are same, that is decomposition of $P_m \Box C_n$ is (i)$\oplus$ (ii) $\oplus$ (iii) $\oplus$ (iv) $\oplus$ (v) the subgraphs $\frac{s}{2}$ times $P_2 \Box C_p$, for each $t \in \{1, 2, \ldots, \frac{s}{2} - 1\}$, decompose $P_2 \Box C_p$ into pK_2 as follows: (a) a matching $\{(\ell p + 2t, 2j)(\ell p + 2t, 2j+1) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t + 1, 2j)(\ell p + 2t + 1, 2j+1) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t, \ell p - 1)(\ell p + 2t + 1, \ell p - 1)\} \oplus$ (b) a matching $\{(\ell p + 2t, 2j+1)(\ell p + 2t, 2j+2) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t + 1, 2j+1)(\ell p + 2t + 1, 2j+2) : j \in \{0, 1, \ldots, \frac{p-3}{2}\}\} \cup \{(\ell p + 2t, 0)(\ell p + 2t + 1, 0)\}$ of cardinality $p \oplus$ (c) a matching $\{(\ell p + 2t, j)(\ell p + 2t + 1, j) : j \in \{1, 2, \ldots, p-2\}\} \cup \{(\ell p + 2t, 0)(\ell p + 2t, p-1), (\ell p + 2t + 1, 0)(\ell p + 2t + 1, p-1)\}$ of cardinality $p \oplus \bigoplus_{i=1}^{\frac{s}{2}}$ (a matching $\{(\ell + 2i + 1, j)(\ell + 2i + 2, j) : j \in \{0, 1, \ldots, p-1\}\}$ of cardinality n).

Subcase 1.2. For $n = p$, assume p and s are even. Let $m = \ell p + s$, where $s \in \{0, 1, \ldots, p-1\}$. Decompose $P_m \Box C_n$ as follows: (i) $P_{\ell p} \Box P_p$ with vertex set $\{0, 1, \ldots, \ell p - 1\} \times \{0, 1, \ldots, p-1\}$, by Theorem 2.1 graph (i) admits a (C_{2p}, pK_2)-multidecomposition $\oplus$ (ii) the subgraphs $\frac{s}{2}$ times $P_2 \Box C_p$, for each $t \in \{0, 1, \ldots, \frac{s}{2} - 1\}$, decompose $P_2 \Box C_p$ into (C_{2p}, pK_2) as follows: (a) a matching $\{(\ell p + 2t, j)(\ell p + 2t + 1, j) : j \in \{1, 2, \ldots, p-2\}\} \cup \{(\ell p + 2t, 0)(\ell p + 2t, p-1), (\ell p + 2t + 1, 0)(\ell p + 2t + 1, p-1)\}$ of cardinality $p \oplus$ (b) a cycle $(\ell p + 2t, 0)(\ell p + 2t, 1)(\ell p + 2t, 2) \ldots (\ell p + 2t, n-1)(\ell p + 2t + 1, n-1) \ldots (\ell p + 2t + 1, 1)(\ell p + 2t + 1, 0)(\ell p + 2t, 0)$ of cardinality $2p \oplus$ (iii) for each $t \in \{0, 1, \ldots, \frac{s}{2} - 1\}$, a matching $\{(\ell p + 2t - 1, j)(\ell p + 2t, j) : j \in \{0, 1, \ldots, p-1\}\}$

of cardinality p ⊕ (iv) a matching $\bigoplus_{i=0}^{\ell p-1} \{((i,0)(i,p-1))\}$ of cardinality ℓp.

Now assume p is even and s is odd. Decompose $P_m \square C_n$ as follows: (i) $P_{\ell p} \square P_p$ with vertex set $\{0,1,\ldots,\ell p-1\} \times \{0,1,\ldots,p-1\}$, by Theorem 2.1 graph (i) admits a (C_{2p}, pK_2)-multidecomposition ⊕ (ii) the subgraphs $\frac{s-1}{2}$ times $P_2 \square C_p$, for each $t \in \{1,2,\ldots,\frac{s-1}{2}\}$, decompose $P_2 \square C_p$ into (C_{2p}, pK_2) as follows: (a) a matching $\{(\ell p+2t-1,j)(\ell p+2t,j) : j \in \{1,2,\ldots,p-2\}\} \cup \{(\ell p+2t-1,0)(\ell p+2t-1,p-1),(\ell p+2t,0)(\ell p+2t,p-1)\}$ of cardinality p ⊕ (b) a cycle $(\ell p+2t-1,0)(\ell p+2t-1,1)(\ell p+2t-1,2)\ldots(\ell p+2t-1,p-1)(\ell p+2t,p-1)\ldots(\ell p+2t,1)(\ell p+2t,0)(\ell p+2t-1,0)$ of cardinality $2p$ ⊕ (iii) for each $t \in \{1,2,\ldots,\frac{s-1}{2}\}$, a matching $\{(\ell p+2t-2,j)(\ell p+2t-1,j) : j \in \{0,1,\ldots,p-1\}\}$ of cardinality p ⊕ (iv)for each $t \in \{0,1,\ldots,\frac{s-3}{2}\}$, a matching $\{(\ell p+2t,j)(\ell p+2t+1,j) : j \in \{0,1,\ldots,p-1\}\}$ of cardinality p ⊕ (v) a matching $\{(\ell p-1,j)(\ell p,j) : j \in \{0,1,\ldots,p-1\}\}$ of cardinality p ⊕ (vi) a matching $\{(\ell p,2j)(\ell p,2j+1) : j \in \{0,1,\ldots,\frac{p}{2}-1\}\} \cup \{(i,0)(i,p-1) : i \in \{0,1,\ldots,\frac{lp}{2}-1\}\}$ of cardinality p ⊕ (vii) a matching $\{(\ell p,2j+1)(\ell p,2j+2) : j \in \{0,1,\ldots,\frac{p}{2}-1\}\} \cup \{(i,0)(i,p-1) : i \in \{\frac{lp}{2},\frac{lp}{2}+1,\ldots,lp-1\}\}$ of cardinality p

Case 2. $(\ell,p,n) = (1,3,3)$. $P_{\ell p+s} \square C_n = P_{3+s} \square C_3$, if $s = 0$, $P_3 \square C_3$, is decomposable into $(C_6, 3K_2)$-multidecomposition as
(i) a cycle $(0,0)(0,1)(0,2)(1,2)(1,1)(1,0)$ ⊕ (ii) a matching $\{(0,0)(0,2),(1,2)(2,2),(2,0)(2,1)\}$ ⊕ (iii) a matching $\{(0,1)(1,1),(1,0)(2,0),(2,1)(2,2)\}$ ⊕ (iv) a matching $\{(1,0)(1,2),(1,1)(2,1),(2,0)(2,2)\}$. Now assume that $s \geq 1$, consider the following two subcases.

Subcase 2.1. For m is even,$P_{3+s} \square C_3$, is decomposable into $(C_6, 3K_2)$-multidecomposition as (i) $\bigoplus_{i=0}^{\frac{m-2}{2}}$ (a cycle $(2i,0)(2i,1)(2i,2)(2i+1,2)(2i+1,1)(2i+1,0)$ of cardinality 6) ⊕ (ii) $\bigoplus_{i=0}^{\frac{m-2}{2}}$ (a matching $\{(2i,0)(2i,2),(2i,1)(2i+1,1),(2i+1,0)(2i+1,2)\}$ of cardinality 3) ⊕ (iii) $\bigoplus_{i=0}^{\frac{m-4}{2}}$ (a matching $\{(2i+1,0)(2i+2,0),(2i+1,1)(2i+2,1),(2i+1,2)(2i+2,2)\}$ of cardinality 3).

Subcase 2.2. For m is odd,$P_{3+s} \square C_3$, is decomposable into $(C_6, 3K_2)$-multidecomposition as (i) $\bigoplus_{i=0}^{\frac{m-3}{2}}$ (a cycle $(2i,0)(2i,1)(2i,2)(2i+1,2)(2i+1,1)(2i+1,0)$ of cardinality 6) ⊕ (ii) $\bigoplus_{i=0}^{\frac{m-3}{2}}$ (a matching $\{(2i,0)(2i,2),(2i,1)(2i+1,1),(2i+1,0)(2i+1,2)\}$ of cardinality 3) ⊕ (iii) $\bigoplus_{i=0}^{\frac{m-7}{2}}$ (a matching $\{(2i+1,0)(2i+2,0),(2i+1,1)(2i+2,1),(2i+1,2)(2i+2,2)\}$ of cardinality 3) ⊕ (iv) a matching $\{(m-1,0)(m-1,1),(m-1,2)(m-2,2),(m-3,2)(m-4,2)\}$ of cardinality 3 ⊕ (v) a matching $\{(m-1,1)(m-1,2),(m-1,0)(m-2,0),(m-3,0)(m-4,0)\}$ of cardinality 3 ⊕ (vi) a matching $\{(m-1,0)(m-1,2),(m-1,1)(m-2,1),(m-3,1)(m-4,1)\}$ of cardinality 3.

Theorem 3.3. *For integers $m \geq 5$ and $n \geq 4$, $P_m \square C_n$ admits a $(C_6, 3K_2)$-multidecomposition if and only if $m \equiv 2 \bmod 3$ or $n \equiv 0 \bmod 3$.*

Theorem 3.4. *For integers $m, n \geq 4$, $P_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition if and only if $n \equiv 0 \bmod 4$.*

Theorem 3.5. *For integers $m, n \geq 3$, $P_m \square C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition if and only if $m \equiv 3 \bmod 5$ or $n \equiv 0 \bmod 5$.*

Proof of Theorem 3.3. follows from Lemmas 3.6., 3.7., Theorem 3.1. and $P_2 \square C_6$ = the 6-cycle $(0,0)(0,1)(0,2)(0,3)(0,4)(0,5)(0,0)$ ⊕ the 6-cycle $(1,0)(1,1)(1,2)(1,3)(1,4)(1,5)(1,0)$ ⊕ the $3K_2$ $\{(0,0)(1,0),(0,1)(1,1),(0,2)(1,2)\}$ ⊕ the $3K_2$ $\{(0,3)(1,3),(0,4)(1,4),(0,5)(1,5)\}$.; proof of Theorem 3.4. follows from Theorem 3.1.; proof of Theorem 3.5. follows from Lemmas 3.8. to 3.11. and Theorem 3.1..

Lemma 3.6. *If $m \equiv 2 \, mod \, 3 \equiv n$, with $m, n \geq 6$ then $P_m \Box C_n$ admits a $(C_6, 3K_2)$-multidecomposition.*

Proof. As $m-1 \equiv 1 \, mod \, 3 \equiv n-1$, by Theorem 2.1, $P_{m-1} \Box P_{n-1}$ admits a $(C_6, 3K_2)$-multidecomposition. The deletion of the edges of $P_{m-1} \Box P_{n-1}$ from $P_m \Box C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-1\}\} \cup \{(m-2,1)(m-1,1)\}$ of cardinality $m+1 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{0, 2, 3, \ldots, n-2\}$ of cardinality $n-2 \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{1,2,\ldots,m-2\}\}$ of cardinality $m-2 \oplus$ a path $(0,n-2)(0,n-1)(1,n-1)(2,n-1)\ldots(m-2,n-1)(m-1,n-1)(m-1,n-2)(m-1,n-3)\ldots(m-1,1)(m-1,0)$ of length $m+n-1$. All these matchings are divisible by $3K_2$ and by Lemma 2.8, the path is also divisible by $3K_2$. Thus $P_m \Box C_n$ admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 3.7. *If $m \equiv 2 \, mod \, 3$ and if $n \equiv 1 \, mod \, 3$, with $m \geq 5$, $n \geq 4$ then $P_m \Box C_n$ admits a $(C_6, 3K_2)$-multidecomposition.*

Proof. For $(m,n) \neq (5,4)$. As $m-1 \equiv 1 \, mod \, 3 \equiv n$, by Theorem 2.1, $P_{m-1} \Box P_n$ admits a $(C_6, 3K_2)$-multidecompos -ition. The deletion of the edges of $P_{m-1} \Box P_n$ from $P_m \Box C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-1\}\} \cup \{(m-2,1)(m-1,1)\}$ of cardinality $m+1 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{0, 2, 3, \ldots, n-1\}$ of cardinality $n-1 \oplus$ a path $(m-1,0)(m-1,1)(m-1,2)\ldots(m-1,n-2)(m-1,n-1)$ of length $n-1$. Both the matchings are divisible by $3K_2$ and by Lemma 2.8, the path is also divisible by $3K_2$. For $m = 5$ and $n = 4$. Since by Lemma 2.7, $P_4 \Box P_4$ admits a $(C_6, 3K_2)$-multidecomposition. The deletion of the edges of $P_4 \Box P_4$ from $P_5 \Box C_4$ results in the subgraph: a matching $\{(4,0)(4,3), (4,1)(4,2), (3,0)(3,3)\}$ of cardinality $3 \oplus$ a matching$\{(4,0)(4,1), (4,2)(4,3), (2,0)(2,3)\}$ of cardinality $3 \oplus$ a matching$\{(0,0)(0,3), (1,0)(1,3), (3,0)(4,0), (3,1)(4,1), (3,2)(4,2), (3,3)(4,3)\}$ of cardinality 6. Thus $P_m \Box C_n$ admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 3.8. *If $m \equiv 3 \, mod \, 5$ and if $n \equiv 1 \, mod \, 5$, then $P_m \Box C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.*

Proof. As $m-3 \equiv 0 \, mod \, 5 \equiv n-1$, by Theorem 2.1, $P_{m-3} \Box P_{n-1}$ admits a $(C_{10}, 5K_2)$-multidecomposition. The deletion of the edges of $P_{m-3} \Box P_{n-1}$ from $P_m \Box C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-4\}\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3 \oplus$ a path $(0,n-1)(1,n-1)(2,n-1)\ldots(m-4,n-1)(m-3,n-1)(m-3,0)(m-2,0)(m-2,n-1)(m-1,n-1)(m-1,0)$ of length $m+2 \equiv 0 \, mod \, 5 \oplus$ a path $(m-3,0)(m-3,1)\ldots(m-3,n-1)$ of length $n-1 \equiv 0 \, mod \, 5$ a path $(m-2,0)(m-2,1)\ldots(m-2,n-1)$ of length $n-1 \equiv 0 \, mod \, 5 \oplus$ a path $(m-1,0)(m-1,1)\ldots(m-1,n-1)$ of length $n-1 \equiv 0 \, mod \, 5 \oplus$ a matching $\{(m-4,j)(m-3,j) : j \in \{0,1,\ldots,n-2\}\}$ of cardinality $n-1 \oplus$ a matching $\{(m-3,j)(m-2,j) : j \in \{0,1,\ldots,n-2\}\}$ of cardinality $n-1 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{0,1,\ldots,n-2\}\}$ of cardinality $n-1$ All the matchings are divisible by $5K_2$ and by Lemma 2.8, all the paths are divisible by $5K_2$. Thus $P_m \Box C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.

Lemma 3.9. *If $m \equiv 3 \, mod \, 5$ and if $n \equiv 2 \, mod \, 5$, then $P_m \Box C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.*

Proof. As $m-2 \equiv 1 \, mod \, 5 \equiv n-1$, by Theorem 2.1, $P_{m-2} \Box P_{n-1}$ admits a $(C_{10}, 5K_2)$-multidecomposition. The deletion of the edges of $P_{m-2} \Box P_{n-1}$ from $P_m \Box C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-4\}\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3$ $\oplus$ a path $(0,n-1)(1,n-1)(2,n-1)\ldots(m-3,n-1)(m-3,0)(m-2,0)(m-2,n-1)(m-1,n-1)(m-1,0)$ of length $m+2 \equiv 0 \, mod \, 5 \oplus$ a path $(m-2,0)(m-2,1)\ldots(m-2,n-2)$ of length $n-2 \equiv 0 \, mod \, 5 \oplus$ a path $(m-1,0)(m-1,1)\ldots(m-1,n-2)$ of length $n-2 \equiv 0 \, mod \, 5 \oplus$ a matching $\{(m-3,1)(m-2,1), (m-2,0)(m-1,0), (m-3,n-2)(m-3,n-1), (m-2,n-2)(m-2,n-1), (m-1,n-2)(m-1,n-1)\}$ of cardinality $5 \oplus$ a matching $\{(m-3,j)(m-2,j) : j \in \{2,3,\ldots,n-1\}\}$ of cardinality $n-2 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{1,2,\ldots,n-2\}\}$ of cardinality $n-2$. All the matchings are divisible by $5K_2$ and by Lemma 2.8, all the paths are divisible by $5K_2$. Thus $P_m \Box C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.

Lemma 3.10. *If $m \equiv 3 \, mod \, 5$ and if $n \equiv 3 \, mod \, 5$, then $P_m \Box C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.*

Proof. As $m-3 \equiv 0 \, mod \, 5 \equiv n-3$, by Theorem 2.1, $P_{m-3} \Box P_{n-3}$ admits a $(C_{10}, 5K_2)$-multidecomposition. The deletion of the edges of $P_{m-3} \Box P_{n-3}$ from $P_m \Box C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-4\}\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-4)(i,n-3) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3$

$\oplus$ a matching $\{(i,n-3)(i,n-2) : i \in \{2,3,\ldots,m-2\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3 \oplus$ a matching $\{(0,n-3)(0,n-2),(1,n-3)(1,n-2),(m-3,n-2)(m-3,n-1),(m-2,n-2)(m-2,n-1),(m-1,n-2)(m-1,n-1)\}$ of cardinality $5 \oplus$ a path $(0,n-3)(1,n-3)(2,n-3)\ldots(m-1,n-3)(m-1,n-2)(m-2,n-2)(m-3,n-2)\ldots(2,n-2)(1,n-2)(0,n-2)$ of length $2m-1 \equiv 0 \, mod \, 5$ $\oplus$ a path $(0,n-1)(1,n-1)(2,n-1)\ldots(m-3,n-1)(m-3,0)(m-2,0)(m-2,n-1)(m-1,n-1)(m-1,0)$ of length $m+2 \equiv 0 \, mod \, 5 \oplus$ a path $(m-3,0)(m-3,1)\ldots(m-3,n-4)(m-3,n-3)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a path $(m-2,0)(m-2,1)\ldots(m-2,n-4)(m-2,n-3)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a path $(m-1,0)(m-1,1)\ldots(m-1,n-4)(m-1,n-3)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a matching $\{(m-4,j)(m-3,j) : j \in \{0,1,\ldots,n-4\}\}$ of cardinality $n-3 \oplus$ a matching $\{(m-3,j)(m-2,j) : j \in \{1,2,\ldots,n-4\}\} \cup \{(m-3,n-1)(m-2,n-1)\}$ of cardinality $n-3 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{0,1,\ldots,n-4\}\}$ of cardinality $n-3$. All the matchings are divisible by $5K_2$ and by Lemma 2.8, all the paths are divisible by $5K_2$. Thus $P_m \square C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.

Lemma 3.11. *If $m \equiv 3 \, mod \, 5$ and if $n \equiv 4 \, mod \, 5$, then $P_m \square C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.*

Proof. As $m-3 \equiv 0 \, mod \, 5 \equiv n-4$, by Theorem 2.1, $P_{m-3} \square P_{n-4}$ admits a $(C_{10}, 5K_2)$-multidecomposition. The deletion of the edges of $P_{m-3} \square P_{n-4}$ from $P_m \square C_n$ results in the subgraph: a matching $\{(i,0)(i,n-1) : i \in \{0,1,\ldots,m-4\}\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-5)(i,n-4) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3$ $\oplus$ a matching $\{(i,n-4)(i,n-3) : i \in \{1,2,\ldots,m-3\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-3)(i,n-2) : i \in \{0,1,\ldots,m-4\}$ of cardinality $m-3 \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{1,2,\ldots,m-5\} \cup \{(m-3,n-2)(m-3,n-1),(m-2,n-3)(m-2,n-2)\}$ of cardinality $m-3 \oplus$ a matching $\{(m-2,n-4)(m-2,n-3),(m-3,n-3)(m-3,n-2),(m-4,n-2)(m-4,n-1),(m-2,n-2)(m-2,n-1),(m-1,n-2)(m-1,n-1)\}$ of cardinality $5 \oplus$ a path $(m-1,n-4)(m-2,n-4)\ldots(1,n-4)(0,n-4)(0,n-3)(1,n-3)\ldots(m-2,n-3)(m-1,n-3)(m-1,n-2)(m-2,n-2)\ldots(1,n-2)(0,n-2)(0,n-1)(1,n-1)\ldots(m-4,n-1)(m-3,n-1)(m-3,0)(m-2,0)(m-2,n-1)(m-1,n-1)(m-1,0)(m-1,1)$ of length $4m+3 \equiv 0 \, mod \, 5 \oplus$ a path $(m-3,0)(m-3,1)\ldots(m-3,n-5)(m-3,n-4)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a path $(m-2,0)(m-2,1)\ldots(m-2,n-5)(m-2,n-4)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a path $(m-1,1)(m-1,2)\ldots(m-1,n-4)(m-1,n-3)$ of length $n-3 \equiv 0 \, mod \, 5 \oplus$ a matching $\{(m-4,j)(m-3,j) : j \in \{0,1,\ldots,n-5\}\}$ of cardinality $n-4 \oplus$ a matching $\{(m-3,j)(m-2,j) : j \in \{0,1,\ldots,n-5\}\}$ of cardinality $n-4 \oplus$ a matching $\{(m-2,j)(m-1,j) : j \in \{0,1,\ldots,n-5\}\}$ of cardinality $n-4$. All the matchings are divisible by $5K_2$ and by Lemma 2.8, all the paths are divisible by $5K_2$. Thus $P_m \square C_n$ admits a $(C_{10}, 5K_2)$-multidecomposition.

4. Cartesian Product of Cycles

In this section, we have proved that $C_m \square C_n$ admits a (C_{2p}, pK_2)-multidecomposition, for some values of $p \geq 3$.

If $C_m \square C_n$ admits a (C_{2p}, pK_2)-multidecomposition, then p divides $|E(C_m \square C_n)| = 2mn$ and hence for prime p, either $m \equiv 0 \, mod \, p$ or $n \equiv 0 \, mod \, p$. By symmetry, assume that $n \equiv 0 \, mod \, p$. By Theorem 3.1, $P_m \square C_n$ admits a (C_{2p}, pK_2)-multidecomposition. The deletion of the edges of $P_m \square C_n$ from $C_m \square C_n$ results in nK_2. As $n \equiv 0 \, mod \, p$, $(pK_2)|(nK_2)$. Hence, $C_m \square C_n$ admits a (C_{2p}, pK_2)-multidecomposition. Thus,

Theorem 4.1. *For integers $m, n \geq p$ and for prime $p \geq 2$, $C_m \square C_n$ admits a (C_{2p}, pK_2)-multidecomposition if and only if either $m \equiv 0 \, mod \, p$ or $n \equiv 0 \, mod \, p$.*

For $p = 3, 5$, $C_m \square C_n$ admits a $(C_6, 3K_2)$, $(C_{10}, 5K_2)$-multidecomposition respectively by Theorem 4.1.

Theorem 4.2. *For integers $m, n \geq 4$, $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition if and only if either $m \equiv 0 \, mod \, 2$ or $n \equiv 0 \, mod \, 2$.*

Proof. By symmetry, assume that $n \equiv 0 \, mod \, 2$. Consider two cases.

Case 1. If $n \equiv 0 \, mod \, 4$, then $C_m \square C_n = P_m \square C_n \oplus nK_2$. By Theorem 3.4., $P_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition and by lemma 2.8., $(4K_2)|(nK_2)$. Thus $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition.

Case 2. If $n \equiv 2\, mod\, 4$, Consider four cases.

Sub case 2.1. If $n \equiv 2\, mod\, 4 \equiv m$ then $C_m \square C_n = P_m \square P_n \oplus nK_2 \oplus mK_2 = P_m \square P_n \oplus (n-2)K_2 \oplus (m-2)K_2 \oplus 4K_2$ by choosing the edges $\{(0,0)(m-1,0),(0,1)(m-1,1),(1,0)(1,n-1),(2,0)(2,n-1)\}$ of $4K_2$ from nK_2 and mK_2 and since $(n-2) \equiv 0\, mod\, 4 \equiv (m-2)$, by lemma 2.8., $(4K_2)|((n-2)K_2)$ and $(4K_2)|((m-2)K_2)$ and by Lemma 2.11., $P_m \square P_n$ admits a $(C_8, 4K_2)$-multidecomposition. Thus $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition.

Sub case 2.2. If $n \equiv 2\, mod\, 4$ and $m \equiv 0\, mod\, 4$ then $C_m \square C_n = C_m \square P_n \oplus mK_2$, since $m \equiv 0\, mod\, 4$, by Theorem 3.4, $C_m \square P_n$ admits a $(C_8, 4K_2)$-multidecomposition and by lemma 2.8., $(4K_2)|(mK_2)$. Thus $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition.

Sub case 2.3. If $n \equiv 2\, mod\, 4$ and $m \equiv 1\, mod\, 4$ then $C_m \square C_n = P_m \square P_{n-1} \oplus$ a matching $\{(0,j)(m-1,j) : j \in \{0,1,\dots,n-1\}\} \cup \{(1,n-2)(1,n-1),(2,n-2)(2,n-1)\}$ of cardinality $(n+2) \oplus$ a matching $\{(i,0)(i,n-1) : i \in \{0,2,3,\dots,m-1\}\}$ of cardinality $(m-1) \oplus$ a matching $\{(i,n-2)(i,n-1) : i \in \{0,3,4,\dots,m-1\}\} \cup \{(1,0)(1,n-1)\}$ of cardinality $(m-1)$. Since by lemma 2.9., $P_m \square P_{n-1}$ admits $(C_8, 4K_2)$-multidecomposition and by lemma 2.8., $4K_2|(n+2)K_2$ and $4K_2|(m-1)K_2$. Thus $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition.

Sub case 2.4. If $n \equiv 2\, mod\, 4$ and $m \equiv 3\, mod\, 4$ then $C_m \square C_n = P_{m-1} \square P_n \oplus$ a path $(m-2,n-1)(m-1,n-1)(m-1,n-2)\dots(m-1,0)(0,0)(0,n-1)$ of length $n+2 \oplus$ a matching $\{(m-1,j)(m-2,j) : j \in \{0,1,\dots,n-2\}\} \cup \{(1,0)(1,n-1),(2,0)(2,n-1),(0,n-1)(m-1,n-1)\}$ of cardinality $(n+2) \oplus$ a matching $\{(i,0)(i,n-1) : i \in \{3,4,\dots,m-1\}\}$ of cardinality $(m-3) \oplus$ a matching $\{(0,j)(m-1,j) : j \in \{1,2,\dots,n-2\}\}$ of cardinality $(n-2)$. Since by lemma 2.11., $P_{m-1} \square P_n$ admits $(C_8, 4K_2)$-multidecomposition, by lemma 2.8., $4K_2|P_{n+3}$ and $4K_2|(n+2)K_2$, $4K_2|(m-3)K_2$. Thus $C_m \square C_n$ admits a $(C_8, 4K_2)$-multidecomposition.

5. Cartesian Product of a Path and a Clique

In this section, we have proved that $P_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition, for some values of $p \geq 3$.

If $P_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition, then p divides $|E(P_m \square K_n)| = \frac{mn(n+1)}{2} - n$. Observe that, if $n \equiv 0\, mod\, p$, then $p|\left(\frac{mn(n+1)}{2} - n\right)$ and for all odd integers $p \geq 3$, if $m \equiv 1\, mod\, p \equiv n$ then $p|\left(\frac{mn(n+1)}{2} - n\right)$.

Theorem 5.1. *For integers $m \geq 2$, $n \geq 3$ and for an odd integer $p \geq 3$, then $P_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition if $m \equiv 1\, mod\, p \equiv n$.*

Proof. Consider two cases.
Case 1. If n is even.

As n is even, there is a decomposition of K_n into $\frac{n}{2}$ Hamilton paths. Note that each Hamilton path is of length $n-1 \equiv 0\, mod\, p$. First decompose each of the m disjoint K_n's in $P_m \square K_n$ into Hamilton paths and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into pK_2's. The deletion of the edges of these pK_2's results in $P_m \square P_n$ and, by Theorem 2.1, it clearly admits a (C_{2p}, pK_2)-multidecomposition.
Case 2. If n is odd.

As $n+1$ is even, there is a decomposition of K_{n+1} into $\frac{n-1}{2}$ Hamilton cycles and a 1-factor; consequently, there is a decomposition of K_n into $\frac{n-1}{2}$ Hamilton paths and a near 1-factor. Note that each Hamilton path is of length $n-1 \equiv 0\, mod\, p$ and the near 1-factor is a matching of cardinality $\frac{n-1}{2} \equiv 0\, mod\, p$. First decompose each of the m disjoint K_n's in $P_m \square K_n$ into Hamilton paths and a near 1-factor and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into pK_2's, also in each layer decompose the near 1-factor into pK_2's. The deletion of the edges of these pK_2's results in $P_m \square P_n$ and, by Theorem 2.1, it clearly admits a (C_{2p}, pK_2)-multidecomposition.

If $P_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition, then 3 divides $|E(P_m \square K_n)| = \frac{mn(n+1)}{2} - n$ and hence either $n \equiv 0\, mod\, 3$ or $m \equiv 1\, mod\, 3 \equiv n$.

Lemma 5.2. *For integers $m, n \geq 2$, $P_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 1 \, mod \, 3 \equiv n$.*

Proof. Consider two cases.
Case 1. For $n \equiv 4 \, mod \, 6$.
Subcase 1.1. $n \neq 4$.
Proof follows from Theorem 5.1.
Subcase 1.2. $n = 4$.
By lemma 2.9., $P_m \square P_4$ admits a $(C_6, 3K_2)$-multidecomposition and the deletion of the edges of $P_m \square P_4$ from $P_m \square K_4$ results in mP_4. Clearly, $(3K_2)|(2P_4)$ and $(3K_2)|(3P_4)$, by lemma 2.8.. Using this one can find a decomposition of mP_4 by $3K_2$.
Case 2. For $n \equiv 1 \, mod \, 6$.
Proof follows from Theorem 5.1.

Theorem 5.3. *For integers $m \geq 2$, $n \geq 3$ and $p \geq 3$, $P_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition if $n \equiv 0 \, mod \, p$.*

Proof. Consider two cases.
Case 1. If n is odd.
As n is odd, there is a decomposition of K_n into $\frac{n-1}{2}$ Hamilton cycles. Note that each Hamilton cycle is of length $n \equiv 0 \, mod \, p$. First decompose each of the m disjoint K_n's in $P_m \square K_n$ into Hamilton cycles and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into pK_2's, by lemma 3.2.. The deletion of the edges of these pK_2's results in $P_m \square C_n$, and by Theorem 3.1., it clearly admits a (C_{2p}, pK_2)-multidecomposition.
Case 2. If n is even.
As n is even, there is a decomposition of K_n into $\frac{n-2}{2}$ Hamilton cycles and a 1-factor; Note that each Hamilton cycle is of length $n \equiv 0 \, mod \, p$ and the 1-factor is a matching of cardinality $\frac{n}{2} \equiv 0 \, mod \, p$. First decompose each of the m disjoint K_n's in $P_m \square K_n$ into Hamilton cycles and a 1-factor and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into pK_2's, also in each layer decompose the 1-factor into pK_2's. The deletion of the edges of these pK_2's results in $P_m \square C_n$, and by Theorem 3.1, it clearly admits a (C_{2p}, pK_2)-multidecomposition.

6. Cartesian Product of a Cycle and a Clique

In this section, we have proved that $C_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition, for $p = 3$.

If $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition, then 3 divides $|E(C_m \square K_n)| = \frac{mn(n+1)}{2}$ and hence neither $m \equiv 1 \, mod \, 3 \equiv n$ nor $m \equiv 2 \, mod \, 3$ and $n \equiv 1 \, mod \, 3$.

Lemma 6.1. *For integers $m, n \geq 2$, $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $n \equiv 0 \, mod \, 3$.*

Proof. Consider two cases.
Case 1. For $n \equiv 3 \, mod \, 6$.
As n is odd, there is a decomposition of K_n into $\frac{n-1}{2}$ Hamilton cycles. Note that each Hamilton cycle is of length $n \equiv 0 \, mod \, 3$. Decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton cycles and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $C_m \square C_n$, and by Theorem 4.1., it clearly admits a $(C_6, 3K_2)$-multidecomposition.
Case 2. $n \equiv 0 \, mod \, 6$.
As n is even, there is a decomposition of K_n into $\frac{n-2}{2}$ Hamilton cycles and a 1-factor; Note that each Hamilton cycle is of length $n \equiv 0 \, mod \, 6$ and the 1-factor is a matching of cardinality $\frac{n}{2} \equiv 0 \, mod \, 3$. First decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton cycles and a 1-factor and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into $3K_2$'s, also in each layer decompose the 1-factor into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $C_m \square C_n$, and by Theorem 4.1, it clearly admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 6.2. *For integers $m, n \geq 2$, $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 0 \, mod \, 3$ and $n \equiv 1 \, mod \, 3$.*

Proof. Consider two cases.
Case 1. For $n \equiv 4 \, mod \, 6$.

As n is even, there is a decomposition of K_n into $\frac{n}{2}$ Hamilton paths. Note that each Hamilton path is of length $n - 1 \equiv 3 \, mod \, 6$. First decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton paths and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $C_m \square P_n$ and, by Theorem 3.3., it clearly admits a $(C_6, 3K_2)$-multidecomposition.
Case 2. $n \equiv 1 \, mod \, 6$.

As $n + 1$ is even, there is a decomposition of K_{n+1} into $\frac{n-1}{2}$ Hamilton cycles and a 1-factor; consequently, there is a decomposition of K_n into $\frac{n-1}{2}$ Hamilton paths and a near 1-factor. Note that each Hamilton path is of length $n-1 \equiv 0 \, mod \, 6$ and the near 1-factor is a matching of cardinality $\frac{n-1}{2} \equiv 0 \, mod \, 3$. First decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton paths and a near 1-factor and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s, also in each layer decompose the near 1-factor into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $C_m \square P_n$ and, by Theorem 3.3., it clearly admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 6.3. *For integers $m, n \geq 2$, $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 0 \, mod \, 3$ and $n \equiv 2 \, mod \, 3$.*

Proof. Consider two cases.
Case 1. For $n \equiv 2 \, mod \, 6$.

As n is even, there is a decomposition of K_n into $\frac{n}{2}$ Hamilton paths. First decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton paths and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s, by choosing one edge from each Hamilton path, with cardinality m, $m \equiv 0 \, mod \, 3$. The deletion of the edges of these $3K_2$'s results in $C_m \square P_n$ and, by Theorem 3.1., it clearly admits a $(C_6, 3K_2)$-multidecomposition.
Case 2. $n \equiv 5 \, mod \, 6$.

As $n + 1$ is even, there is a decomposition of K_{n+1} into $\frac{n-1}{2}$ Hamilton cycles and a 1-factor; consequently, there is a decomposition of K_n into $\frac{n-1}{2}$ Hamilton paths and a near 1-factor. Note that the near 1-factor is a matching of cardinality $\frac{n-1}{2} \equiv 0 \, mod \, 3$. First decompose each of the m disjoint K_n's in $C_m \square K_n$ into Hamilton paths and a near 1-factor and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s, also in each layer decompose the near 1-factor into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $C_m \square P_n$ and, by Theorem 3.3., it clearly admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 6.4. *If $n \equiv 2 \, mod \, 3$, and if $n \neq 6$, then $(3K_2) | K_n - \mathscr{P}$.*

Proof. For even n, first decompose K_n into $\frac{n}{2}$ Hamilton paths, let one of the Hamilton path be $\mathscr{P} = \{0, 1, n-1, 2, n-2, 3, n-3, \cdots, \frac{n}{2} - 2, \frac{n}{2} + 2, \frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2}\}$ after removing this Hamilton path, from the remaining Hamilton path deleting the following edges $\{(\frac{3n}{4} + i, \frac{n}{4} + i) : i \in \{1, 2, \ldots, \frac{n-2}{2}\}\}$ if $\frac{n}{2}$ is even and $\{(\frac{3n+2}{4} + i, \frac{n+2}{4} + i) : i \in \{1, 2, \ldots, \frac{n-2}{2}\}\}$ if $\frac{n}{2}$ is odd from each Hamilton path. Which is a matching of cardinality $\frac{n-2}{2}$, leaves $2P_{\frac{n}{2}}$, each of the length $\frac{n-2}{2}$, as n is even, $n \equiv 2 \, mod \, 3$, implies $\frac{n}{2} \equiv 1 \, mod \, 3$, by the Lemma 2.8., $3K_2 | P_{\frac{n}{2}}$, hence $(3K_2) | K_n - \mathscr{P}$.
For odd n first decompose K_n into $\frac{n-1}{2}$ Hamilton paths and a near one factor,let one of the Hamilton path be $\mathscr{P} = \{0, 1, n-1, 2, n-2, 3, n-3, \cdots, \frac{n-1}{2} - 2, \frac{n-1}{2} + 3, \frac{n-1}{2} - 1, \frac{n-1}{2} + 2, \frac{n-1}{2}, \frac{n-1}{2} + 1\}$ after removing this Hamilton path, from the remaining Hamilton path deleting the following edges $\{(\frac{3(n-1)}{4} + i, \frac{n-5}{4} + i) : i \in \{2, 3, \ldots, \frac{n-1}{2}\}\}$ if $\frac{n-1}{2}$ is even, $\{(\frac{3(n+1)}{4} + i, \frac{n+1}{4} + i) : i \in \{1, 2, \ldots, \frac{n-3}{2}\}\}$ if $\frac{n-1}{2}$ is odd, gives two disjoint paths. For each $i \in \{2, 3, \ldots, \frac{n-1}{2}\}$ if $\frac{n-1}{2}$ is even, on combining the vertices $\{\frac{3(n-1)}{4} + i\}$ and $\{\frac{n-5}{4} + i\}$ gives the path $P_{n-1}(i)$ each of order $n - 1 \equiv 1 \, mod \, 3$. For each $i \in \{1, 2, \ldots, \frac{n-3}{2}\}$ if $\frac{n-1}{2}$ is odd, on combining the vertices $\{\frac{3(n+1)}{4} + i\}$

and $\{\frac{n+1}{4}+i\}$ gives the path $P_{n-1}(i)$, each of order $n-1 \equiv 1\, mod\, 3$. Hence by lemma 2.8., $3K_2|P_{n-1}(i)$. After removing one $3K_2$: $\{(\frac{n-1}{2},n)(\frac{3(n-1)}{4}+1,\frac{n-1}{4}-1)(\frac{3(n-1)}{4},\frac{n-1}{4})\}$ from $\{(\frac{3(n-1)}{4}+i,\frac{n-5}{4}+i) : i \in \{2,3,\ldots,\frac{n-1}{2}\}\}$ if $\frac{n-1}{2}$ is even, union the near one factor $\{((n-1)-i,n+i) : i \in \{0,1,\ldots,\frac{n-3}{2}\}\}$ gives the path P_{n-4} of order $n-4 \equiv 1\, mod\, 3$. Hence by lemma 2.8., $3K_2|P_{n-4}(i)$. Similarly after removing two disjoint paths $\{\frac{n-1}{2},n,n-1,\frac{n-3}{2}\} \cup \{\frac{3n-1}{4},\frac{n-3}{4},\frac{3n-1}{4}-1,\frac{n-3}{4}+1\}$ from $\{(\frac{3(n+1)}{4}+i,\frac{n+1}{4}+i) : i \in \{1,2,\ldots,\frac{n-3}{2}\}\}$ if $\frac{n-1}{2}$ is odd, union the near one factor $\{((n-1)-i,n+i) : i \in \{0,1,\ldots,\frac{n-3}{2}\}\}$ gives the path P_{n-7} of order $n-7 \equiv 1\, mod\, 3$. Hence by lemma 2.8., $3K_2|P_{n-7}(i)$. Now by choosing the edges $\{(n,\frac{n-1}{2})(n-1,\frac{n-3}{2})(\frac{n-3}{4},\frac{3n-1}{4}-1)\}$ and $\{(n,n-1)(\frac{3n-1}{4},\frac{n-3}{4})(\frac{3n-1}{4}-1,\frac{n-3}{4}+1)\}$ are matching and isomorphic to $3K_2$. Thus $(3K_2)|K_n-\mathscr{P}$.

Lemma 6.5. *For integers $m,n \geq 2$, $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 1\, mod\, 3$ and $n \equiv 2\, mod\, 3$.*

Proof. After removing $m-$times $K_n-\mathscr{P}$ from $C_m \square K_n$, One have $C_m \square P_n$, by the Lemma 6.4.,$(3K_2)|K_n-\mathscr{P}$ and by the Theorem 3.2., $C_m \square P_n$ admits a $(C_6, 3K_2)$-multidecomposition. Hence $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 6.6. *For integers $m,n \geq 2$, $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 2\, mod\, 3 \equiv n$.*

Proof. After removing $m-$times $K_n-\mathscr{P}$ from $C_m \square K_n$, one have $C_m \square P_n$, by the Lemma 6.4., $(3K_2)|K_n-\mathscr{P}$ and by the Theorem 3.2., $C_m \square P_n$ admits a $(C_6, 3K_2)$-multidecomposition. Hence $C_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition.

7. Cartesian Product of Cliques

In this section, we have proved that $K_m \square K_n$ admits a (C_{2p}, pK_2)-multidecomposition, for $p = 3$.

If $K_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition, then 3 divides $|E(K_m \square K_n)| = \frac{mn(m+n-2)}{2}$ and hence either $m \equiv 0\, mod\, 3$ or $n \equiv 0\, mod\, 3$ or $m \equiv 1\, mod\, 3 \equiv n$.

Lemma 7.1. *For integers $m,n \geq 2$, $K_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 0\, mod\, 3$.*

Proof. Consider two cases.
Case 1. For $m \equiv 0\, mod\, 6$.

As m is even, there is a decomposition of K_m into $\frac{m-2}{2}$ Hamilton cycles and a 1-factor. Note that each Hamilton cycle is of length $m \equiv 0\, mod\, 6$. First decompose each of the n disjoint K_m's in $K_m \square K_n$ into Hamilton cycles and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into $3K_2$'s by Lemma 3.2 and the 1-factor is a matching of cardinality $\frac{m}{2} \equiv 0\, mod\, 3$. The deletion of the edges of these $3K_2$'s results in $C_m \square K_n$ and, by Lemma 6.1,6.2 and 6.3., it clearly admits a $(C_6, 3K_2)$-multidecomposition.
Case 2. For $m \equiv 3\, mod\, 6$.

As m is odd, there is a decomposition of K_m into $\frac{m-1}{2}$ Hamilton cycles. Note that each Hamilton cycle is of length $m \equiv 0\, mod\, 3$. Decompose each of the n disjoint K_m's in $K_m \square K_n$ into Hamilton cycles and in each layer except one Hamilton cycle decompose each of the remaining Hamilton cycles into $3K_2$'s by Lemma 3.2.. The deletion of the edges of these $3K_2$'s results in $C_m \square K_n$ and, by Lemma 6.1,6.2 and 6.3., it clearly admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 7.2. *For integers $m,n \geq 2$, $K_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $n \equiv 0\, mod\, 3$.*

Proof. Since $K_m \square K_n = K_n \square K_m$ and $n \equiv 0\, mod\, 3$, by Lemma 7.1., $K_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition.

Lemma 7.3. *For integers $m,n \geq 2$, $K_m \square K_n$ admits a $(C_6, 3K_2)$-multidecomposition if $m \equiv 1\, mod\, 3 \equiv n$.*

Proof. Consider two cases.
Case 1. For $m \equiv 4 \, mod \, 6$.

As m is even, there is a decomposition of K_m into $\frac{m}{2}$ Hamilton paths. Note that each Hamilton path is of length $m - 1 \equiv 3 \, mod \, 6$. First decompose each of the n disjoint K_m's in $K_m \Box K_n$ into Hamilton paths and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $P_m \Box K_n$ and, by Lemma 5.2., it clearly admits a $(C_6, 3K_2)$-multidecomposition.

Case 2. For $m \equiv 1 \, mod \, 6$.

As $m+1$ is even, there is a decomposition of K_{m+1} into $\frac{m-1}{2}$ Hamilton cycles and a 1-factor; consequently, there is a decomposition of K_m into $\frac{m-1}{2}$ Hamilton paths and a near 1-factor. Note that each Hamilton path is of length $m - 1 \equiv 0 \, mod \, 6$ and the near 1-factor is a matching of cardinality $\frac{m-1}{2} \equiv 0 \, mod \, 3$. First decompose each of the n disjoint K_m's in $K_m \Box K_n$ into Hamilton paths and a near 1-factor and in each layer except one Hamilton path decompose each of the remaining Hamilton paths into $3K_2$'s, also in each layer decompose the near 1-factor into $3K_2$'s. The deletion of the edges of these $3K_2$'s results in $P_m \Box K_n$ and, by Lemma 5.2., it clearly admits a $(C_6, 3K_2)$-multidecomposition.

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