Filomat 31:18 (2017), 5769–5781
https://doi.org/10.2298/FIL1718769L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximate Controllability of Semilinear Fractional Differential Systems of Order $1 < q < 2$ via Resolvent Operators

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Abstract. Under a compactness assumption on the resolvent, some properties on relevant operators generated by resolvent are given. Existence results of fractional control systems are obtained by Schauder's fixed point theorem and approximation techniques. Furthermore, the approximately controllable result is acquired under the assumption that the corresponding linear system is approximately controllable, which improves and extends some results on this topic.

1. Introduction

In this paper, we investigate the approximate controllability of the following fractional evolution control system:

$$\begin{cases} {}^C D^q x(t) = Ax(t) + f(t, x(t)) + Bu(t), \ t \in J = [0, b], \\ x(0) + g(x) = x_0 \in X, \qquad x'(0) + h(x) = y_0 \in X, \end{cases} \tag{1}$$

where $1 < q < 2$; the state x takes values in a Hilbert space X; ${}^C D^q$ is the Caputo fractional derivative operator of order q; $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a resolvent family $\{C_q(t)\}_{t\geq 0}$; the control function u is given in $L^2(J, U)$, U is a Hilbert space; B is a bounded linear operator from U to X; f, g and h are appropriate continuous functions to be specified later.

In the past two decades, fractional calculus provided great challenging interest for mathematicians and physicists in fractional theory. Fractional differential equations are considered as valuable models of many phenomena in various fields, such as electrochemistry, physics, porous media, control theory, etc. For more details on fractional calculus theory, one can see the monographs of Kilbas et al. [12], and Podlubny [22]. We also refer the reader to some other works [1, 21] on the solvability of nonlinear fractional differential problems. On the other hand, controllability plays an important role both in mathematical and control theory. The concept of exact controllability is usually too strong for the infinite dimensional space. Then,

2010 *Mathematics Subject Classification*. Primary 34K37, 47A10; Secondary 93B05.
Keywords. Fractional differential systems, approximate controllability, resolvent operators, approximate technique, mild solution.
Received: 16 May 2016; Accepted: 19 July 2016
Communicated by Maria Alessandra Ragusa
Research supported by the National Natural Science Foundation of China (11571300 and 11271316), the Qing Lan Project of Jiangsu Province of China and High-Level Personnel Support Program of Yangzhou University.
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approximate controllability governed by fractional derivatives has been studied extensively. We refer the readers to the recent papers [4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 20, 24–26].

However, to the best of our knowledge, most of the existing results on approximate controllability are obtained for fractional orders $0 < \alpha < 1$. Limited work has been done for the order $1 < \alpha < 2$. Recently, Li Kexue et. al [15] studied the controllability of nonlocal fractional differential systems of order $\alpha \in (1, 2]$ in infinite dimensional Banach spaces. The mild solution is given through the α-order strongly continuous fractional cosine family (or resolvent) for the first time. Shukla et. al ([27], [28]) studied the approximate controllability of semilinear fractional control system of order $\alpha \in (1, 2]$ with finite and infinite delay in Banach spaces respectively by using the theory of strongly continuous α-order cosine family and sequential method. Since there is no further studies on the properties of resolvent operators, the results obtained in [15, 26–28] are all under the same condition that the nonlinear function is Lipschitz continuous, which is also needed in some papers for $0 < \alpha < 1$.

So, one purpose of this article is first to investigate the characters of resolvent operators for the order $1 < \alpha < 2$. The main difficulty is that the semigroup property does not hold for the resolvent. Recently, Fan [6] obtained some good results which is similar to the semigroup property when the resolvent is compact and continuous in the uniform operator topology. Based on these results, we prove that the operators $S_q(t)$ and $P_q(t)$ (see Definition 2.6) generated by resolvent are also compact and continuous in the uniform operator topology. Hence, the Lipschtz continuity of the nonlinear function is no longer needed in this paper, and the Schauder's fixed point theorem can be successfully used in solving system (1). On the other hand, the compactness or Lipschitz continuity of nonlocal terms g and h is not needed in our results by using the approximate technique developed in the article [3]. In this paper, g and h are only assumed to be continuous. Finally, approximate controllability is obtained under the conditions that the corresponding linear system is approximately controllable and the resolvent family (see Definition 2.4) is compact. Therefore, our results improve and generalize some existing conclusions on this topic.

The rest of the paper is organized as follows. In section 2, we recall some definitions of Caputo fractional derivatives, resolvent and the mild solution of system (1). We also obtain some basic properties of the operators $C_q(t)$, $S_q(t)$ and $P_q(t)$. Section 3 is devoted to studying the approximate controllability of system (1) provided that the corresponding linear system is approximately controllable. Finally, we will conclude with our main results.

2. Preliminaries

Throughout this paper, we assume that X, U are two Hilbert space with norm $\|\cdot\|$ and $\|\cdot\|_U$, respectively. Let $b > 0$ be fixed. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of positive integer, real number, and nonnegative real number, respectively. We denote by $C(J, X)$ the space of X-valued continuous functions on J with the norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, and denote by $L^P(J, X)$ the space of X-valued Bochner integrable functions on J with the norm $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p dt)^{1/p}$, where $1 \le p < \infty$. We also denote by $\mathfrak{B}(X)$ the space of all bounded linear operators from X to X endowed with the operator norm $\|\cdot\|$. In this paper, we always suppose that A is a closed and densely defined linear operator on X and $1 < q < 2$.

First let us recall the following basic definitions and results about fractional derivative and resolvent.

Definition 2.1. *[22] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0, \infty), \mathbb{R})$ is defined as*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\,\mathrm{d}s, \ t > 0,$$

provided the right side is point-wisely defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[22] The Riemann-Liouville fractional derivative of order $q > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0, \infty), \mathbb{R})$ is defined as*

$$^{L}D^q f(t) = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-q-1} f(s)\mathrm{d}s, \ t > 0, \quad n-1 < q < n, \quad n \in \mathbb{N}.$$

Definition 2.3. *[22] The Caputo fractional derivative of order $q > 0$ with the lower limit zero for a function $f(\cdot) \in L^1([0,\infty),\mathbb{R})$ is defined as*

$$^{C}D^{q} f(t) = {}^{L}D^{q}\Big(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{k}(0)\Big), \ \ t > 0, \quad n-1 < q < n, \quad n \in \mathbb{N}.$$

If $f(\cdot) \in C^n[0,\infty)$, then

$$^{C}D^{q} f(t) = I^{n-q} f^{(n)}(s) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) \mathrm{d}s, \ \ t > 0, \quad n-1 < q < n, \quad n \in \mathbb{N}.$$

If f is an abstract function with values in X, the integrals which appear in the above three definitions are taken in Bochner's sense.

Definition 2.4. *[23] A family $\{C_q(t)\}_{t\geq 0} \subseteq \mathfrak{B}(X)$ of bounded linear operators in X is called resolvent (or solution operator) generated by A if the following conditions are satisfied:*
(a) $C_q(t)$ is strongly continuous for $t \geq 0$ and $C_q(0) = I$;
(b) $C_q(t)D(A) \subset D(A)$ and $AC_q(t)x = C_q(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;
(c) the resolvent equation holds

$$C_q(t)x = x + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} AC_q(s)x \mathrm{d}s,$$

for all $x \in D(A), t \geq 0$.

Since A is a closed and densely defined operator on X, it is easy to show that the resolvent equation holds for all $x \in X$ (see[23]).

Definition 2.5. *A resolvent $C_q(t)$ is called compact for $t > 0$, if for every $t > 0$, $C_q(t)$ is a compact operator.*

Using the Laplace transformation, similar to proof in [15], we can give the following definition of mild solution to the system (1).

Definition 2.6. *A function $x \in C(J,X)$ is called a mild solution of* (1) *if x satisfies*

$$x(t) = C_q(t)(x_0 - g(x)) + S_q(t)(y_0 - h(x)) + \int_0^t P_q(t-s)(f(s,x(s)) + Bu(s))\mathrm{d}s, \quad t \in J,$$

where $S_q(t) = \int_0^t C_q(s)\mathrm{d}s$, $\quad P_q(t) = \frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} C_q(s)\mathrm{d}s$.

In the remaining of this paper, we always suppose that

(HA) $C_q(t)$ generated by A is compact and continuous in the uniform operator topology for all $t > 0$, and $M = \sup_{t\in J} \|C_q(t)\| < +\infty$.

Remark 2.7. *From [6], we know that an analytic resolvent of analyticity type (ω_0, θ_0) is continuous in the uniform operator topology for all $t > 0$ when the fractional order $0 < q < 1$.*

Now, we can formulate some basic properties of operators $C_q(t)$, $S_q(t)$ and $P_q(t)$.

Lemma 2.8. *Let $q \in (1,2)$, and (HA) is satisfied. Then*
(1) $\lim_{h\to 0^+} \|C_q(t+h) - C_q(t)C_q(h)\| = 0$ for $t > 0$;
(2) $\lim_{h\to 0^+} \|C_q(t) - C_q(h)C_q(t-h)\| = 0$ for $t > 0$.

Proof. The proof is similar to Lemma 3.4, Lemma 3.5 in [6], so we omit it. □

Lemma 2.9. *For fixed $t \geq 0$, $S_q(t)$ and $P_q(t)$ are linear and bounded operators on X.*

Proof. For any fixed $t \geq 0$, it is easy to check that $S_q(t)$ and $P_q(t)$ are also linear operators since $C_q(t)$ is a linear operator. For any $x \in X$, we have

$$\|S_q(t)x\| = \| \int_0^t C_q(s)x\mathrm{d}s\| \leq Mb\|x\|,$$

and

$$\|P_q(t)x\| = \|\frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} C_q(s)x\mathrm{d}s\| \leq \frac{Mb^{q-1}}{\Gamma(q)}\|x\|. \quad \square$$

Lemma 2.10. *Assume that (HA) is satisfied, then $S_q(t)$ and $P_q(t)$ are compact operators on X for every $t \geq 0$.*

Proof. If $t = 0$, $S_q(t)$ and $P_q(t)$ are null operator, which are obviously compact. If $t > 0$, For each $r > 0$, set $B_r = \{x \in X : \|x\| \leq r\}$, and B_r is clearly a bounded subset in X. Let $t > 0$ be fixed. We first prove that the set

$$\Omega_1(t) = \{P_q(t)x : x \in B_r\} = \{\frac{1}{\Gamma(q-1)} \int_0^t (t-s)^{q-2} C_q(s)x\mathrm{d}s : x \in B_r\},$$

is relatively compact in X. For any $0 < 2\varepsilon < t$, define the subset in X by

$$\Omega_1^\varepsilon(t) = \{C_q(\varepsilon)\frac{1}{\Gamma(q-1)} \int_\varepsilon^t (t-s)^{q-2} C_q(s-\varepsilon)x\mathrm{d}s : x \in B_r\}.$$

From the compactness of $C_q(\varepsilon)(\varepsilon > 0)$, we obtain that $\Omega_1^\varepsilon(t)$ is relatively compact in X. Moreover, for every $x \in B_r$, we have

$$\begin{aligned}
& \|C_q(\varepsilon) \int_\varepsilon^t (t-s)^{q-2} C_q(s-\varepsilon)x\mathrm{d}s - \int_0^t (t-s)^{q-2} C_q(s)x\mathrm{d}s\| \\
\leq\ & \int_{2\varepsilon}^t (t-s)^{q-2} \|C_q(\varepsilon)C_q(s-\varepsilon) - C_q(s)\|\|x\|\mathrm{d}s + \int_\varepsilon^{2\varepsilon} (t-s)^{q-2}\|C_q(\varepsilon)C_q(s-\varepsilon)x\|\mathrm{d}s \\
& + \int_0^{2\varepsilon} (t-s)^{q-2}\|C_q(s)x\|\mathrm{d}s \\
\leq\ & r\int_{2\varepsilon}^t (t-s)^{q-2}\|C_q(\varepsilon)C_q(s-\varepsilon) - C_q(s)\|\mathrm{d}s + \frac{M^2 r}{q-1}[(t-\varepsilon)^{q-1} - (t-2\varepsilon)^{q-1}] \\
& + \frac{Mr}{q-1}[t^{q-1} - (t-2\varepsilon)^{q-1}].
\end{aligned}$$

From Lemma 2.8, we have that $\|C_q(\varepsilon)C_q(s-\varepsilon) - C_q(s)\| \to 0$ as $\varepsilon \to 0$ for $s \in [2\varepsilon, t]$. Then it follows from the Lebesgue dominated convergence theorem that

$$\lim_{\varepsilon \to 0} \|C_q(\varepsilon) \int_\varepsilon^t (t-s)^{q-2} C_q(s-\varepsilon)x\mathrm{d}s - \int_0^t (t-s)^{q-2} C_q(s)x\mathrm{d}s\| = 0.$$

Therefore, there is a relatively compact set arbitrarily close to the set $\Omega_1(t)$ for $t > 0$, which implies that $\Omega_1(t)$ is also relatively compact in X. Similarly, we can prove that $\Omega_2(t) = \{S_q(t)x : x \in B_r\} = \{\int_0^t C_q(s)x\mathrm{d}s : x \in B_r\}$ is also relatively compact in X. □

Lemma 2.11. *Let $q \in (1, 2)$. Operators $S_q(t)$ and $P_q(t)$ are continuous in the uniform operator topology for $t \in [0, b]$, and then they are also equicontinuous for $t \in [0, b]$.*

Proof. For any $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned}
& \|P_q(t_2) - P_q(t_1)\| \\
\leq\ & \frac{1}{\Gamma(q-1)}\| \int_0^{t_1} [(t_2-s)^{q-2} - (t_1-s)^{q-2}] C_q(s)\mathrm{d}s\| + \| \int_{t_1}^{t_2} (t_2-s)^{q-2} C_q(s)\mathrm{d}s\| \\
\leq\ & \frac{M}{\Gamma(q)}(t_2^{q-1} - t_1^{q-1}) \\
\leq\ & \frac{M}{\Gamma(q)}(t_2 - t_1)^{q-1},
\end{aligned}$$

which implies that $\lim_{t_1\to t_2} \|P_q(t_2) - P_q(t_1)\| = 0$. Then, we can conclude that $P_q(t)$ is equicontinuous for $t \in [0, b]$. Similarly, we can prove that the conclusions also hold for the operator $S_q(t)$ for $t \in [0, b]$. $\square$

3. Approximate Controllability

In this section, we first give the expression of the control function u for the approximate control system (1). Then, we establish the sufficient conditions for the approximate controllability of system (1).

Definition 3.1. *The system* (1) *is said to be approximately controllable on J if for any $(x_0, y_0) \in X \times X$, there exists a control $u \in L^2(J, U)$, such that the closure of the reachable set $K_b(x_0, y_0)$ is dense in X, where*
$K_b(x_0, y_0) = \{x(b) : x$ *is a mild solution of* (1) *for some* $u \in L^2(J, U)\}$.

Consider the following linear fractional control system:

$$\begin{cases} {}^C D^q x(t) = Ax(t) + Bu(t),\ t \in J = [0, b],\ 1 < q < 2, \\ x(0) = x_0, \qquad x'(0) = x_1. \end{cases} \tag{2}$$

We introduce two relevant operators defined on X associated with (2) as
$\Lambda_0^b = \int_0^b P_q(b-s)BB^*P_q^*(b-s)\mathrm{d}s : X \to X,$
$R(\alpha, \Lambda_0^b) = (\alpha I + \Lambda_0^b)^{-1} : X \to X,$
where $B^*, P_q^*(t)$ are the adjoint of B and $P_q(t)$, respectively. It is easy to verify that the operator Λ_0^b is a linear bounded symmetric operator. Hence, we have the following equivalent result.

Theorem 3.2. *[[19], Theorem 2.3] The following conditions are equivalent:*
(1) The linear fractional differential system (2) *is approximately controllable.*
(2) The controllability operator Λ_0^b is positive, that is, $\langle x^*, \Lambda_0^b x^*\rangle > 0$ for all nonzero $x^* \in X^*$.
(3) For all $x \in X$, $\alpha R(\alpha, \Lambda_0^b)x$ converges to zero as $\alpha \to 0^+$ in the strong topology.

The conditions of Theorem 3.2 ensure, via Banach-Steinhaus, $\|R(\alpha, \Lambda_0^b)\| \le \frac{C}{\alpha}$ for some constant $0 < C \le 1$ and $\alpha > 0$ small enough. So, without loss of generality, we hereafter suppose that $\|R(\alpha, \Lambda_0^b)\| \le \frac{1}{\alpha}$ for all $\alpha > 0$.

Next, we will give the express of control u in the approximate control system (1). Define the linear regulator problem: to minimize

$$\Im(u) = \|x(b) - x_b\|^2 + \alpha \int_0^b \|u(t)\|^2 \mathrm{d}t, \tag{3}$$

over all $u \in L^2(J, U)$, where x is the mild solution of the system (1) with the control u, $x_b \in X$ and $\alpha > 0$.

It is well known the control u of differential equation in the sense of integer order can be chosen as the unique solution of the above linear regulator problem (3). We refer the readers to the paper [18]. The same idea can be used to the fractional sense, and we get the expression of u formally from the following Lemma.

Lemma 3.3. *For any $\alpha > 0$, assume that u is the optimal control of (3). Then*

$$u(t) = B^*P_q^*(b-t)R(\alpha, \Lambda_0^b)p(x), \ \ a.e.\ \ t \in [0, b],$$

where

$$p(x) = x_b - C_q(b)(x_0 - g(x)) - S_q(b)(y_0 - h(x)) - \int_0^b P_q(b-s)f(s, x(s))\mathrm{d}s.$$

Proof. Let u be the optimal control of (3). Denote by

$$\mathcal{F}(\lambda) := \mathfrak{J}(u + \lambda v)$$

with $v \in L^2(J, U)$. It is obviously that $\lambda = 0$ is the critical point of $\mathfrak{J}$. Then, we have

$$\mathcal{F}'(\lambda)|_{\lambda=0} = 0,$$

that is

$$\Big[2\langle y(b) - x_b, \frac{d}{d\lambda}(y(b) - x_b)\rangle + 2\alpha \int_0^b \langle u(t) + \lambda v(t), v(t)\rangle_U \mathrm{d}t\Big]\Big|_{\lambda=0} = 0,$$

where y is the mild solution of system (1) with the control $u + \lambda v$. Hence

$$\langle x(b) - x_b, \int_0^b P_q(b-t)Bv(t)\mathrm{d}t\rangle + \alpha \int_0^b \langle u(t), v(t)\rangle_U \mathrm{d}t = 0,$$

$$\int_0^b \langle B^* P_q^*(b-t)(x(b) - x_b) + \alpha u(t), v(t)\rangle_U \mathrm{d}t = 0.$$

Consequently, from the arbitrariness of v in $L^2(J, U)$, we have

$$B^* P_q^*(b-t)(x(b) - x_b) + \alpha u(t) = 0,$$

which implies that

$$u(t) = -\alpha^{-1} B^* P_q^*(b-t)(x(b) - x_b), \tag{4}$$

for all most all $t \in J$. Therefore, the state of system (1) at the point b with the above control is given by

$$\begin{aligned} x(b) &= C_q(b)(x_0 - g(x)) + S_q(b)(y_0 - h(x)) + \int_0^b P_q(b-s)f(s, x(s))\mathrm{d}s \\ &\quad -\alpha^{-1} \int_0^b P_q(b-s)BB^* P_q^*(b-s)(x(b) - x_b)\mathrm{d}s, \\ &= C_q(b)(x_0 - g(x)) + S_q(b)(y_0 - h(x)) + \int_0^b P_q(b-s)f(s, x(s))\mathrm{d}s \\ &\quad -\alpha^{-1}\Lambda_0^b(x(b) - x_b), \end{aligned}$$

which implies that

$$\begin{aligned} -\alpha^{-1}(\alpha I + \Lambda_0^b)(x(b) - x_b) &= x_b - C_q(b)(x_0 - g(x)) - S_q(b)(y_0 - h(x)) \\ &\quad - \int_0^b P_q(b-s)f(s, x(s))\mathrm{d}s. \end{aligned}$$

Let

$$p(x) = x_b - C_q(b)(x_0 - g(x)) - S_q(b)(y_0 - h(x)) - \int_0^b P_q(b-s)f(s, x(s))\mathrm{d}s.$$

Thus

$$x(b) - x_b = -\alpha(\alpha I + \Lambda_0^b)^{-1} p(x) = -\alpha R(\alpha, \Lambda_0^b)p(x). \tag{5}$$

Substituting (5) in (4), we obtain

$$u(t) = B^* P_q^*(b-t)R(\alpha, \Lambda_0^b)p(x), \quad a.e. \;\; t \in J.$$

□

According to Lemma 3.3, for every $\alpha > 0$, $x_b \in X$, we now construct the following integral systems

$$\begin{cases} x(t) = C_q(t)(x_0 - g(x)) + S_q(t)(y_0 - h(x)) + \int_0^t P_q(t-s)(f(s,x(s)) + Bu(s))\mathrm{d}s, \\ u(t) = B^* P_q^*(b-t)R(\alpha, \Lambda_0^b)p(x), \\ p(x) = x_b - C_q(b)(x_0 - g(x)) - S_q(b)(y_0 - h(x)) - \int_0^b P_q(b-s)f(s,x(s))\mathrm{d}s. \end{cases} \tag{6}$$

We will prove the existence and approximate controllability of the fractional system (1) using this integral system (6).

Set $W_r = \{x \in C(J,X) : \|x\| \leq r\}$. We assume the following conditions:

(Hf) $f : J \times X \to X$ is continuous and there exists a constant $N_1 > 0$ such that $\|f(t,x)\| \leq N_1$, for all $(t,x) \in J \times X$.

(Hg) $g : C(J,X) \to X$ is continuous and there exists a constant $N_2 > 0$ such that $\|g(x)\| \leq N_2$, for all $x \in C(J,X)$. In addition, for each $r > 0$, there is a $\delta = \delta(r) > 0$ such that $g(x) = g(y)$ for any $x, y \in W_r$ with $x(s) = y(s)$, $s \in [\delta, b]$.

(Hh) $h : C(J,X) \to X$ is continuous, and there exists a constant $N_3 > 0$ such that $\|h(x)\| \leq N_3$, for all $x \in C(J,X)$.

(HB) $B : U \to X$ is a linear bounded operator, and put $\|B\| = M_B$.

First, we give the following result.

Lemma 3.4. *If the hypotheses* (HA), (Hf), (Hg), (Hh) *and* (HB) *are satisfied, then for each* $r > 0$, *the mapping* $Q : W_r \to C(J,X)$ *defined by*

$$(Qx)(t) = \int_0^t P_q(t-s)(f(s,x(s)) + Bu(s))\mathrm{d}s, \ t \in J$$

is compact, where u *comes from Lemma 3.3.*

Proof. For each $x \in C(J,X)$, from (Hf), (Hg), (Hh) and (HB), we have

$$\|p(x)\| \leq \|x_b\| + M(\|x_0\| + N_2) + Mb(\|y_0\| + N_3) + \frac{MN_1 b^q}{\Gamma(q)} := L$$

and

$$\|f(s,x(s)) + Bu(s)\| \leq N_1 + \frac{1}{\alpha}\frac{MM_B^2 b^{q-1}}{\Gamma(q)}L. \tag{7}$$

We firstly prove that QW_r is equicontinuous on $[0,b]$. For every $x \in W_r$, $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned} \|(Qx)(t_2) - (Qx)(t_1)\| &\leq \|\int_{t_1}^{t_2} P_q(t_2-s)(f(s,x(s)) + Bu(s))\mathrm{d}s\| \\ &\quad + \|\int_0^{t_1} (P_q(t_2-s) - P_q(t_1-s))(f(s,x(s)) + Bu(s))\mathrm{d}s\| \\ &\leq (t_2-t_1)\Big(\frac{MN_1 b^{q-1}}{\Gamma(q)} + \frac{1}{\alpha}(\frac{MM_B}{\Gamma(q)})^2 b^{2q-2} L\Big) \\ &\quad + b(N_1 + \frac{1}{\alpha}\frac{MM_B^2 b^{q-1}L}{\Gamma(q)}) \sup_{0 \leq s \leq t_1} \|P_q(t_2-s) - P_q(t_1-s)\|. \end{aligned}$$

It follows from Lemma 2.11 that $\lim_{t_1 \to t_2} \|(Qx)(t_2) - (Qx)(t_1)\| = 0$, uniformly for $x \in W_r$.

Next, we verify that $QW_r(t)$ is relatively compact in X. Obviously, $QW_r(0)$ is relatively compact in X. Let $0 < t \leq b$. For any $0 < \varepsilon < t$, every $x \in W_r$, we have

$$\begin{aligned}
& \|C_q(\varepsilon)\int_0^{t-\varepsilon} P_q(t-s-\varepsilon)(f(s,x(s))+Bu(s))ds - \int_0^t P_q(t-s)(f(s,x(s))+Bu(s))ds\| \\
\leq & \int_0^{t-\varepsilon} \|(C_q(\varepsilon)-I)P_q(t-s-\varepsilon)(f(s,x(s))+Bu(s))\|ds \\
& + \int_0^{t-\varepsilon} \|P_q(t-s-\varepsilon)-P_q(t-s)\|\|(f(s,x(s))+Bu(s))\|ds \\
& + \int_{t-\varepsilon}^t \|P_q(t-s)(f(s,x(s))+Bu(s))\|ds \\
\leq & \int_0^{t-\varepsilon} \|(C_q(\varepsilon)-I)P_q(t-s-\varepsilon)(f(s,x(s))+Bu(s))\|ds \\
& +(t-\varepsilon)(N_1+\frac{1}{\alpha}\frac{MM_B^2 b^{q-1}}{\Gamma(q)}L)\sup_{0\leq s\leq t-\varepsilon}\|P_q(t-s-\varepsilon)-P_q(t-s)\| \\
& +\varepsilon\Big(\frac{MN_1 b^{q-1}}{\Gamma(q)}+\frac{1}{\alpha}(\frac{MM_B}{\Gamma(q)})^2 b^{2q-2}L\Big).
\end{aligned}$$

It follows from the strong continuity of $C_q(t)$, the compactness of $P_q(t)$ for $t \geq 0$, and Lemma 2.11 that $\lim\limits_{\varepsilon\to 0}\|C_q(\varepsilon)\int_0^{t-\varepsilon} P_q(t-s-\varepsilon)(f(s,x(s))+Bu(s))ds - \int_0^t P_q(t-s)(f(s,x(s))+Bu(s))ds\| = 0$. From the compactness $C_q(t)$, we have that $QW_r(t)$ is relatively compact in X for every $t \in [0,b]$ since there is a family of relatively compact sets arbitrarily close to it. Therefore, $Q : W_r \to C(J,X)$ is a compact mapping by the Ascoli-Arzela theorem. □

To prove the existence of system (6), we now apply a good approximation scheme developed in our article [3]. For fixed $n \geq 1$, all $\alpha > 0$ and $x_b \in X$, we consider the following integral system

$$\begin{cases} x(t) = C_q(t)(x_0 - C_q(\frac{1}{n})g(x)) + S_q(t)(y_0 - h(x)) + \int_0^t P_q(t-s)(f(s,x(s))+Bu(s))ds, \\ u(t) = B^*P_q^*(b-t)R(\alpha,\Lambda_0^b)p(x), \\ p(x) = x_b - C_q(b)(x_0 - C_q(\frac{1}{n})g(x)) - S_q(b)(y_0-h(x)) - \int_0^b P_q(b-s)f(s,x(s))ds, \end{cases} \tag{8}$$

where A is a infinitesimal generator of resolvent $C_q(t)$. By applying $C_q(\frac{1}{n})$, the compactness of g is not required.

Lemma 3.5. *If the hypotheses* (HA), (Hf), (Hg), (Hh) *and* (HB) *are satisfied, then for fixed* $n \geq 1$, *the fractional integral system* (8) *has a mild solution on* $[0,b]$.

Proof. For fixed $n \geq 1$, all $\alpha > 0$ and $x_b \in X$, we define the operator $Q_n^\alpha : C(J,X) \to C(J,X)$ as

$$(Q_n^\alpha x)(t) = C_q(t)\Big(x_0 - C_q(\frac{1}{n})g(x)\Big) + S_q(t)(y_0-h(x)) + \int_0^t P_q(t-s)[f(s,x(s))+Bu(s)]ds,$$

where

$$u(s) = B^*P_q^*(b-s)R(\alpha,\Lambda_0^b)p(x),$$

$$p(x) = x_b - C_q(b)\Big(x_0 - C_q(\frac{1}{n})g(x)\Big) - S_q(b)(y_0-h(x)) - \int_0^b P_q(b-s)f(s,x(s))ds.$$

It is easy to see that the fixed point of Q_n^α is a mild solution of fractional control system (8). Subsequently, we will show that for all $\alpha > 0$, the operator Q_n^α has a fixed point by using the Schauder's fixed point theorem. For the sake of convenience, we subdivide the proof into the following steps.

Step 1. We show the mapping Q_n^α is continuous on $C(J,X)$ for arbitrary $\alpha > 0$. For this purpose, let $\{x_k\}_{k=1}^\infty$ be a sequence in $C(J,X)$ with $\lim\limits_{k\to\infty} x_k = x$ in $C(J,X)$. By the continuity of f, we obtained $f(s,x_k(s))$ converges to $f(s,x(s))$ uniformly for $s \in [0,b]$. Then

$$\begin{aligned}\|p(x_k)-p(x)\| \leq\ & M^2\|g(x_k)-g(x)\| + Mb\|h(x_k)-h(x)\| \\ & +\frac{Mb^q}{\Gamma(q)}\sup_{s\in[0,b]}\|f(s,x_k(s))-f(s,x(s))\|\end{aligned}$$

and

$$\begin{aligned}& \|(Q_n^\alpha x_k)(t)-(Q_n^\alpha x)(t)\| \\ \leq\ & M^2\|g(x_k)-g(x)\| + Mb\|h(x_k)-h(x)\| + \frac{Mb^q}{\Gamma(q)}\sup_{s\in[0,b]}\|f(s,x_k(s))-f(s,x(s))\| \\ & +\frac{1}{\alpha}\left(\frac{MM_B}{\Gamma(q)}\right)^2 b^{2q-1}\|p(x_k)-p(x)\|.\end{aligned}$$

By the continuity of g and f, we get $\|p(x_k)-p(x)\| \to 0$ as $k\to\infty$. Furthermore, $\|Q_n^\alpha x_k - Q_n^\alpha x\| \to 0$ as $k\to\infty$, which implies that Q_n^α is continuous on $C(J,X)$.

Step 2. We will show that for arbitrary $\alpha > 0$ there exists a positive constant $r := r(\alpha)$ such that $Q_n^\alpha(W_{r(\alpha)}) \subseteq W_{r(\alpha)}$. In fact, for all $x \in C(J,X)$, from (7) we have

$$\begin{aligned}\|(Q_n^\alpha x)(t)\| \leq\ & \|C_q(t)(x_0 - C_q(\tfrac{1}{n})g(x))\| + \|S_q(t)(y_0-h(x))\| \\ & +\left\|\int_0^t P_q(t-s)(f(s,x(s))+Bu(s))\mathrm{d}s\right\| \\ \leq\ & M(\|x_0\|+MN_2)+Mb(\|y_0\|+N_3)+\frac{Mb^q}{\Gamma(q)}\left(N_1+\frac{1}{\alpha}\frac{MM_B^2 b^{q-1}}{\Gamma(q)}L\right).\end{aligned}$$

We obtain that for large enough $r(\alpha) > 0$, the inequality $\|(Q_n^\alpha x)(t)\| \leq r(\alpha)$ holds for all $x \in C(J,X)$. Then, $\|Q_n^\alpha x\| = \sup\limits_{t\in J}\|(Q_n^\alpha x)(t)\| \leq r(\alpha)$, which implies that Q_n^α maps $W_{r(\alpha)}$ into itself.

Step 3. We will show that Q_n^α is a compact operator. For $t \in J$, let $(Q_n^\alpha x)(t) = (Q_{n1}^\alpha x)(t) + (Q_{n2}^\alpha x)(t)$, where $(Q_{n1}^\alpha x)(t) = C_q(t)(x_0 - C_q(\frac{1}{n})g(x)) + S_q(t)(y_0 - h(x))$, and $(Q_{n2}^\alpha x)(t) = \int_0^t P_q(t-s)[f(s,x(s))+Bu(s)]\mathrm{d}s$. From Lemma 3.4, it remains to prove Q_{n1}^α is a compact operator. For any $0 \leq t_1 < t_2 \leq b$ and $x \in W_{r(\alpha)}$, we have

$$\begin{aligned}& \|(Q_{n1}^\alpha x)(t_2)-(Q_{n1}^\alpha x)(t_1)\| \\ \leq\ & \|(C_q(t_2)-C_q(t_1))(x_0-C_q(\tfrac{1}{n})g(x))\| + \|(S_q(t_2)-S_q(t_1))(y_0-h(x))\| \\ \leq\ & \|(C_q(t_2)-C_q(t_1))(x_0-C_q(\tfrac{1}{n})g(x))\| + (t_2-t_1)M(\|y_0\|+N_3).\end{aligned}$$

The combination of the fact that $C_q(t)$ is strongly continuous for $t \geq 0$ with $x \to x_0 - C_q(\frac{1}{n})g(x)$ is compact implies that $\lim\limits_{t_1\to t_2}\|(Q_{n1}^\alpha x)(t_2)-(Q_{n1}^\alpha x)(t_1)\| = 0$, uniformly for $x \in W_{r(\alpha)}$, which means that the set $Q_{n1}^\alpha(W_{r(\alpha)})$ is equicontinuous on J. On the other hand, if $t = 0$, it is easy to see that the set $\{(Q_{n1}^\alpha x)(0) : x \in W_{r(\alpha)}\} = \{x_0 - C_q(\frac{1}{n})g(x) : x \in W_{r(\alpha)}\}$ is relatively compact in X for $C_q(\frac{1}{n})$ is compact. If $0 < t \leq b$, by the compactness of $C_q(t)$ and $S_q(t)$ at $t > 0$, and the continuity of g and h, we have $\{(Q_{n1}^\alpha x)(t) : x \in W_{r(\alpha)}\}$ is relatively compact in X. Hence, Q_{n1}^α is compact by the Arzela-Ascoli theorem, which implies that Q_n^α is compact. Then, applying Schauder's fixed point theorem, the operator Q_n^α has a fixed point in $W_{r(\alpha)}$ for all $\alpha > 0$, which gives a mild solution of fractional control system (8) on $[0,b]$. □

Now, define the solution set D and $D(t)$ by

$D = \{x_n \in C(J,X) : x_n = Q_n^\alpha x_n, n \geq 1\}$,

$D(t) = \{x_n(t) : x_n \in D, n \geq 1, t \in J\}$.

Lemma 3.6. *If the hypotheses* (HA), (Hf), (Hg), (Hh) *and* (HB) *are satisfied, then the solution set* D *is equicontinuous on* $[0, b]$ *and* $D(t)$ *is relatively compact in* X *for every* $t \in (0, b]$.

Proof. For $x_n \in D$, $n \geq 1$, we have

$$x_n(t) = C_q(t)\Big(x_0 - C_q(\frac{1}{n})g(x_n)\Big) + S_q(t)(y_0 - h(x_n)) + \int_0^t P_q(t-s)[f(s, x_n(s)) + Bu(s)]\mathrm{d}s,\ t \in [0, b].$$

Firstly, we prove that $D(t) = \{x_n(t) \in X : n \geq 1\}$ is relatively compact on $(0, b]$. For $t \in (0, b]$, by the compactness of $C_q(t)$ and $S_q(t)$, we get that the set $\{C_q(t)\big(x_0 - C_q(\frac{1}{n})g(x_n)\big) + S_q(t)(y_0 - h(x_n)) : n \geq 1\}$ is relatively compact in X for $t \in (0, b]$ since $\{x_0 - C_q(\frac{1}{n})g(x_n) : n \geq 1\}$ and $\{y_0 - h(x_n) : n \geq 1\}$ are total bounded. Moreover, similar with the proof in Lemma 3.4, we can show that $\{\int_0^t P_q(t-s)[f(s, x_n(s)) + Bu(s)]\mathrm{d}s : n \geq 1\}$ is relatively compact in X for $t \in (0, b]$. Therefore, $D(t)$ is relatively compact for $t \in (0, b]$.

Next, we show that D is equicontinuous on $[0, b]$. Let $0 \leq t_1 < t_2 \leq b$, for each $n \geq 1$, we have

$$\begin{aligned}
\|x_n(t_2) - x_n(t_1)\| \leq\ & \|(C_q(t_2) - C_q(t_1))(x_0 - C_q(\frac{1}{n})g(x_n))\| + \|(S_q(t_2) - S_q(t_1))(y_0 - h(x_n))\| \\
& + \|\int_{t_1}^{t_2} P_q(t_2 - s)(f(s, x_n(s) + Bu(s))\mathrm{d}s\| \\
& + \|\int_0^{t_1} (P_q(t_2 - s) - P_q(t_1 - s))(f(s, x_n(s) + Bu(s)))\mathrm{d}s\| \\
\leq\ & \|(C_q(t_2) - C_q(t_1))(x_0 - C_q(\frac{1}{n})g(x_n))\| + (t_2 - t_1)M(\|y_0\| + N_3) \\
& + (t_2 - t_1)\Big(\frac{MN_1 b^{q-1}}{\Gamma(q)} + \frac{1}{\alpha}(\frac{MM_B}{\Gamma(q)})^2 b^{2q-2} L\Big) \\
& + b(N_1 + \frac{1}{\alpha}\frac{MM_B^2 b^{q-1} L}{\Gamma(q)}) \sup_{0 \leq s \leq t_1} \|P_q(t_2 - s) - P_q(t_1 - s)\|.
\end{aligned}$$

From the strong continuity and the compactness of $C_q(t)$, and Lemma 2.11, it follows that $\|x_n(t_2) - x_n(t_1)\| \to 0$ as $t_1 \to t_2$, which implies that D is equicontinuous on $[0, b]$. □

Theorem 3.7. *If the hypotheses* (HA), (Hf), (Hg), (Hh) *and* (HB) *are satisfied, then the fractional integral system* (6) *has a mild solution on* $[0, b]$.

Proof. To prove the solution set D is relatively compact in $C(J, X)$, we only need to prove that $D(0)$ is relatively compact due to Lemma 3.6.

For $x_n \in D$, $n \geq 1$, set

$$\widehat{x_n}(t) = \begin{cases} x_n(t), & t \in [\delta, b], \\ x_n(\delta), & t \in [0, \delta], \end{cases} \tag{9}$$

where δ comes from the condition (Hg), and $g(\widehat{x_n}) = g(x_n)$ due to (Hg). Meanwhile, it is easy to check that $\{\widehat{x_n} : n \geq 1\}$ is relatively compact in $C(J, X)$. Without loss of generality, we may suppose that $\widehat{x_n} \to x^*$ as $n \to \infty$. Then

$$\begin{aligned}
\|x_n(0) - (x_0 - g(x^*))\| &= \|(x_0 - C_q(\frac{1}{n})g(x_n)) - (x_0 - g(x^*))\| \\
&\leq \|C_q(\frac{1}{n})g(x_n) - C_q(\frac{1}{n})g(x^*)\| + \|C_q(\frac{1}{n})g(x^*) - g(x^*)\| \\
&\leq M\|g(\widehat{x_n}) - g(x^*)\| + \|C_q(\frac{1}{n})g(x^*) - g(x^*)\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$, i.e., $D(0) = \{x_n(0) = x_0 - C_q(\frac{1}{n})g(x_n) : n \geq 1\}$ is relatively compact. From Lemma 3.6 we have D is relatively compact in $C(J, X)$ for $t \in [0, b]$. Then, there exists a subsequence of $\{x_n : n \geq 1\}$, not relabeled, converging to some $x_* \in C(J, X)$ as $n \to \infty$. System (8) implies that

$$x_n(t) = C_q(t)\Big(x_0 - C_q(\tfrac{1}{n})g(x_n)\Big) + S_q(t)(y_0 - h(x_n)) + \int_0^t P_q(t-s)[f(s, x_n(s)) + Bu_n(s)]\mathrm{d}s$$

with

$$u_n = B^* P_q^*(b - \cdot)R(\alpha, \Lambda_0^b)p(x_n),$$

$$p(x_n) = x_b - C_q(b)(x_0 - C_q(\tfrac{1}{n})g(x_n)) - S_q(b)(y_0 - h(x_n)) - \int_0^b P_q(b-s)f(s, x_n(s))\mathrm{d}s.$$

Taking the limit $n \to \infty$ to both side of above identities, and using the Lebesgue dominated convergence theorem, one has

$$p(x_n) \to p(x_*),$$

$$u_n \to u_* := B^* P_q^*(b - \cdot)R(\alpha, \Lambda_0^b)p(x_*)$$

and

$$x_*(t) = C_q(t)\Big(x_0 - g(x_*)\Big) + S_q(t)\Big(y_0 - h(x_*)\Big) + \int_0^t P_q(t-s)[f(s, x_*(s)) + Bu_*(s)]\mathrm{d}s,$$

which implies that x_* is a mild solution of fractional integral system (6). □

Now, we can give our main result on approximate controllability.

Theorem 3.8. *Assume that* (HA), (Hf), (Hg), (Hh) *and* (HB) *are satisfied, and in addition the condition (1) in Theorem 3.2 holds. Then the semilinear fractional control system* (1) *is approximately controllable on* J.

Proof. According to Theorem 3.7, for every $\alpha > 0$ and $x_b \in X$, there exists a mild solution $x_\alpha \in C(J, X)$ such that

$$x_\alpha(t) = C_q(t)(x_0 - g(x_\alpha)) + S_q(t)(y_0 - h(x_\alpha)) + \int_0^t P_q(t-s)[f(s, x_\alpha(s)) + Bu_\alpha(s)]\mathrm{d}s, \; t \in J,$$

where

$$u_\alpha(s) = B^* P_q^*(b-s)R(\alpha, \Lambda_0^b)p(x_\alpha),$$

$$p(x_\alpha) = x_b - C_q(b)(x_0 - g(x_\alpha)) - S_q(b)(y_0 - h(x_\alpha)) - \int_0^b P_q(b-s)f(s, x_\alpha(s))\mathrm{d}s.$$

Then

$$x_\alpha(b) = x_b - \alpha R(\alpha, \Lambda_0^b)p(x_\alpha). \tag{10}$$

Recall that $\{C_q(t)\}_{t>0}$ and $\{S_q(t)\}_{t\geq 0}$ are compact. This fact together with the uniform boundedness of g and h, we obtain that there are subsequences of $\{C_q(b)(x_0 - g(x_\alpha)) : \alpha > 0\}$ and $\{S_q(b)(y_0 - h(x_\alpha)) : \alpha > 0\}$, not relabeled, converging to some x' and y' in X, respectively, as $\alpha \to 0$. By assumption (Hf), we have

$$\int_0^b \|f(s, x_\alpha(s))\|^2 \leq N_1^2 b,$$

which implies that the sequence $\{f(\cdot, x_\alpha(\cdot)) : \alpha > 0\}$ is bounded in the Hilbert space $L^2(J, X)$. Then, there exists a subsequence of $\{f(\cdot, x_\alpha(\cdot)) : \alpha > 0\}$, not relabeled, converging weakly to some $\varphi \in L^2(J, X)$ as $\alpha \to 0$. Let

$$\omega = x_b - x' - y' - \int_0^b P_q(b-s)\varphi(s)\mathrm{d}s.$$

We have

$$\begin{aligned} \|p(x_\alpha) - \omega\| \;\; &\leq \;\; \|C_q(b)(x_0 - g(x_\alpha)) - x'\| + \|S_q(b)(y_0 - h(x_\alpha)) - y'\| \\ &\quad + \Big\|\int_0^b P_q(b-s)(f(s, x_\alpha(s)) - \varphi(s))\mathrm{d}s\Big\|. \end{aligned}$$

According to the compactness of $P_q(t)$, similar to the proof of Lemma 3.4, we can verify that the mapping

$$x(t) \to \int_0^t P_q(t-s)x(s)\mathrm{d}s$$

from $L^2(J, X)$ to $C(J, X)$ is compact. So, we get

$\int_0^b P_q(b-s)(f(s,x_\alpha(s))-\varphi(s))\mathrm{d}s \to 0$
as $\alpha \to 0$ since $f(\cdot,x_\alpha(\cdot)) \to \varphi(\cdot)$ weakly in $L^2(J,X)$. Hence,

$$\|p(x_\alpha)-\omega\| \to 0 \tag{11}$$

as $\alpha \to 0$. In view of (10), (11) and the condition (1) in Theorem 3.2 we get

$$\begin{aligned} \|x_\alpha(b)-x_b\| &= \|\alpha R(\alpha,\Lambda_0^b)p(x_\alpha)\| \\ &\le \|\alpha R(\alpha,\Lambda_0^b)\|\|p(x_\alpha)-\omega\| + \|\alpha R(\alpha,\Lambda_0^b)\omega\| \\ &\to 0 \end{aligned}$$

as $\alpha \to 0$, which means that the semilinear fractional system (1) is approximately controllable on J. □

4. Conclusion

The fractional differential systems whose order is between 1 and 2 have wide applications in various problems in science and engineering. For example, they are able to model processes intermediate between exponential decay ($q=1$) and pure sinusoidal oscillation ($q=2$) (see [9]), and the fractional wave equation ($q \in (1,2)$) governs the propagation of mechanical diffusive waves in viscoelastic media which exhibit a simple power-law creep (see [2]). In our paper, starting from the classical definitions of Riemann-Liouville fractional integral and derivatives, the Caputo fractional derivatives, resolvent, and using the powerful tool of the Laplace transform method, Schauder's fixed point theorem and approximate technique, we have discussed the approximate controllability of semilinear fractional evolution system of order $1<q<2$ with nonlocal conditions.

If the closed and densely defined operator A generates a strongly continuous cosine family $C(t)$, from the subordinate principle (see Theorem 3.1, [2]), we have that A generates a strongly continuous exponentially bounded fractional resolvent $C_q(t)$ for $1<q<2$. Under the assumption that $C_q(t)$ is compact and continuous in the uniform operator topology, we have investigated in depth the properties of the resolvent and the operators generated by resolvent. Thus, Schauder's fixed theorem can be successfully used in our framework and the existence and approximate controllability can be finally derived.

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