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Tauberian Theorems for Lambert and Zeta Summability Methods in Fuzzy Number Space

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Abstract. We extend Lambert and zeta summability methods to space of fuzzy numbers and prove Tauberian theorems for Lambert and zeta summability methods of fuzzy numbers, the one for zeta summability method providing a new proof when the sequence is of real numbers.

1. Introduction

Lambert summability method and Dirichlet series $\sum a_k k^{-s}$, in particular the Riemann zeta function $\zeta(s) = \sum k^{-s}$, have significant place in the advancement of Tauberian theory due to the applications to analytic number theory. Realizing the close relation with the asymptotic distribution of primes, authors used them as an analytic tool to handle problems in analytic theory of primes. Historically, Hardy and Littlewood[1] proved a Tauberian theorem for Lambert summability method by use of results from number theory and showed that their Tauberian theorem is equivalent to prime number theorem(PNT) which describes the asymptotic density of primes. In the sequel, Wiener[2] made an autonomous proof for Lambert Tauberian theorem by means of his general theory of Tauberians. His proof was independent from number theory and exploited only the property of Riemann zeta function that $\zeta(x + iy)$ does not vanish on the line $x = 1$, and in this way Wiener also gave a new proof of PNT. Besides Ikehara[3] proved a Tauberian theorem known as Wiener-Ikehara theorem for Dirichlet series, which provides another proof for PNT with use of non-vanishing property of Riemann zeta function. Following the early applications to PNT above, many studies have been done dealing with Lambert series and Dirichlet series in number theory and in other branches of mathematics[4–12]. Furthermore, motivated by wide-range usage of zeta and related functions authors have recently introduced new families of generalized Riemann zeta functions and investigated corresponding properties. They have given series-integral representations and expansion formulas for generalized Riemann zeta functions and discussed potential applications of generalized Riemann zeta functions in different fields of mathematics. For some of these results and applications we refer to[13–17].

Since its introduction, fuzzy set theory has fascinated many researchers from different branches of science and engineering and become a powerful tool for modeling problems involving uncertainty and vagueness. In mathematics, many concepts in classical analysis have been extended to fuzzy analysis. Of these concepts, the concept of convergence of sequences and series of fuzzy numbers is also of interest recently. Authors have examined convergence characters of sequences of fuzzy numbers from various

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aspects and come up with different types of convergence. Besides, they have used summability methods to recover the sequences of fuzzy numbers which fails to converge in the space of fuzzy numbers and given various Tauberian conditions under which summability of a sequence of fuzzy numbers by a certain method implies its convergence(see [18–27]). In this paper, we extend Lambert summability method and zeta summability method, also known as Dirichlet density, to fuzzy analysis and mainly prove two Tauberian theorems for these methods. The one for zeta summability method provides a new proof under weaker conditions when the sequence is of real numbers. Corollaries concerning summability of series of fuzzy numbers have also been obtained.

2. Preliminaries

A *fuzzy number* is a fuzzy set on the real axis, i.e. u is normal, fuzzy convex, upper semi-continuous and $\operatorname{supp} u = \overline{\{t \in \mathbb{R} : u(t) > 0\}}$ is compact [28]. E^1 denotes the space of fuzzy numbers. *α-level set* $[u]_\alpha$ is defined by

$$[u]_\alpha := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \alpha\} \quad , & if \quad 0 < \alpha \leq 1, \\ \overline{\{t \in \mathbb{R} : u(t) > \alpha\}} \quad , & if \quad \alpha = 0. \end{cases}$$

$r \in \mathbb{R}$ may be seen as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(t) := \begin{cases} 1 \quad , & if \quad t = r, \\ 0 \quad , & if \quad t \neq r. \end{cases}$$

Let $u, v \in E^1$ and $k \in \mathbb{R}$. The addition and scalar multiplication are defined by

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \quad , \quad [ku]_\alpha = k[u]_\alpha$$

where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, for all $\alpha \in [0,1]$.

Fuzzy number $\overline{0}$ is identity element in $(E^1, +)$ and none of $u \neq \overline{r}$ has inverse in $(E^1, +)$. For any $k_1, k_2 \in \mathbb{R}$ with $k_1 k_2 \geq 0$, distribution property $(k_1 + k_2)u = k_1 u + k_2 u$ holds but for general $k_1, k_2 \in \mathbb{R}$ it fails to hold. On the other hand properties $k(u+v) = ku + kv$ and $k_1(k_2 u) = (k_1 k_2)u$ holds for any $k, k_1, k_2 \in \mathbb{R}$[29]. It should be noted that E^1 with addition and scalar multiplication defined above is not a linear space over $\mathbb{R}$.

The metric D on E^1 is defined as

$$D(u,v) := \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\},$$

and it has the following properties[29]

$$D(ku, kv) = |k| D(u,v) \quad , \quad D(u+v, w+z) \leq D(u,w) + D(v,z)$$

where $u, v, w, z \in E^1$ and $k \in \mathbb{R}$.

Definition 2.1. *[30] Let (u_k) be a sequence of fuzzy numbers. Denote $s_n = \sum_{k=0}^n u_k$ for all $n \in \mathbb{N}$, if the sequence (s_n) converges to a fuzzy number u then we say that the series $\sum u_k$ of fuzzy numbers converges to u and write $\sum u_k = u$ which implies that*

$$\sum_{k=0}^{n} u_k^-(\alpha) \to u^-(\alpha) \;\; and \;\; \sum_{k=0}^{n} u_k^+(\alpha) \to u^+(\alpha) \qquad (n \to \infty)$$

uniformly in $\alpha \in [0,1]$. Conversely, for the sequence (u_k) of fuzzy numbers if $\sum_k u_k^-(\alpha) = \beta(\alpha)$ and $\sum_k u_k^+(\alpha) = \gamma(\alpha)$ converge uniformly in α, then $\{(\beta(\alpha), \gamma(\alpha)) : \alpha \in [0,1]\}$ defines a fuzzy number u represented by $[u]_\alpha = [\beta(\alpha), \gamma(\alpha)]$ and $\sum u_k = u$.

Besides, we say that the series $\sum u_k$ is bounded if the sequence (s_n) is bounded. The set of bounded series of fuzzy numbers is denoted by $bs(F)$ and the set of bounded sequences of fuzzy numbers is denoted by $\ell_\infty(F)$.

Theorem 2.2. *[31] If $\sum u_k$ and $\sum v_k$ converge, then $D\left(\sum u_k, \sum v_k\right) \leq \sum D(u_k, v_k)$.*

Theorem 2.3. *[31] If $\sum D(u_k, \bar{0}) < \infty$, then the series $\sum u_k$ is convergent.*

3. Lambert summability of series of fuzzy numbers

Definition 3.1. *A series $\sum u_k$ of fuzzy numbers is said to be Lambert summable to fuzzy number v if the series $\sum_{k=1}^{\infty}\frac{kx^k u_k}{1-x^k}$ converges for all $x \in [0,1)$ and*

$$\lim_{x\to 1^-}(1-x)\sum_{k=1}^{\infty}\frac{kx^k u_k}{1-x^k}=v.$$

Theorem 3.2. *If series $\sum u_k$ of fuzzy numbers converges to $v \in E^1$, then $\sum u_k$ is Lambert summable to v.*

Proof. Suppose that series $\sum u_k$ is convergent to v. Then we have $u_k \to \bar{0}$, which implies the existence of an $M > 0$ such that $D(u_k,\bar{0}) < M$ for all k. Since

$$\sum_{k=1}^{\infty} D\left(\frac{kx^k u_k}{1-x^k},\bar{0}\right)=\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k}D(u_k,\bar{0})\le M\sum_{k=1}^{\infty}\frac{kx^k}{1-x^k}<\infty, \qquad (0\le x<1) \tag{1}$$

series $\sum_{k=1}^{\infty}\frac{kx^k u_k}{1-x^k}$ of fuzzy numbers is convergent whenever $x \in [0,1)$ by Theorem 2.3. Our aim is to prove that $\lim_{x\to 1^-}(1-x)\sum_{k=1}^{\infty}\frac{kx^k u_k}{1-x^k}=v$. It is sufficient to show that

$$\lim_{x\to 1^-}\sum_{k=1}^{\infty}\frac{kx^k u_k}{\sum_{r=0}^{k-1}x^r}=v.$$

Let define the sequence of continuous fuzzy-number-valued functions $(S_n(x))$ with $S_n:[0,1]\to E^1$ such that $S_n(x)=\sum_{k=1}^{n}\frac{kx^k u_k}{\sum_{r=0}^{k-1}x^r}\cdot$ By a similar argument that in the expression in (1) and by the fact that $\sum u_n = v$, sequence (S_n) converges for $x \in [0,1]$ and thus there exists a fuzzy-number-valued function $S(x)=\sum_{k=1}^{\infty}\frac{kx^k u_k}{\sum_{r=0}^{k-1}x^r}$ such that $S_n(x)\to S(x)$ pointwise on $[0,1]$. In view of Theorem 3.3 and Theorem 3.5 in [32], function $S(x)$ is continuous on [0,1] if and only if sequences $([S_n(x)]^{\mp}(\alpha))$ uniformly converge to $([S(x)]^{\mp}(\alpha))$ on $[0,1]\times[0,1]$. Then we investigate the uniform convergence of the series $\sum_{k=1}^{\infty}\frac{kx^k u_k^{\mp}(\alpha)}{\sum_{r=0}^{k-1}x^r}$ on $[0,1]\times[0,1]$ by applying Abel's uniform convergence test with $f_k(x,\alpha)=u_k^{\mp}(\alpha)$ and $g_k(x,\alpha)=\frac{kx^k}{\sum_{r=0}^{k-1}x^r}\cdot$ Series $\sum f_k=\sum u_k^{\mp}(\alpha)$ converge uniformly on $[0,1]\times[0,1]$ by the convergence of series $\sum u_k$ of fuzzy numbers in view of Definition 2.1. Besides for all $(x,\alpha)\in[0,1]\times[0,1]$ condition $0<g_k(x,\alpha)\le 1$ holds and (g_k) is monotone decreasing since

$$\begin{aligned}
g_k-g_{k+1} &= \frac{kx^k}{\sum_{r=0}^{k-1}x^r}-\frac{(k+1)x^{k+1}}{\sum_{r=0}^{k}x^r}=x^k\left(\frac{1}{\frac{1}{k}\sum_{r=0}^{k-1}x^r}-\frac{x}{\frac{1}{k+1}\sum_{r=0}^{k}x^r}\right)\\
&= \frac{k(k+1)x^k}{\left(\sum_{r=0}^{k-1}x^r\right)\left(\sum_{r=0}^{k}x^r\right)}\left\{\frac{1}{k+1}\sum_{r=0}^{k}x^r-\frac{1}{k}\sum_{r=0}^{k-1}x^{r+1}\right\}\\
&= \frac{k(k+1)x^k}{\left(\sum_{r=0}^{k-1}x^r\right)\left(\sum_{r=0}^{k}x^r\right)}\left\{\frac{1}{k+1}-\frac{1}{k(k+1)}\sum_{r=1}^{k}x^r\right\}\\
&= \frac{kx^k}{\left(\sum_{r=0}^{k-1}x^r\right)\left(\sum_{r=0}^{k}x^r\right)}\underbrace{\left\{1-\frac{1}{k}\sum_{r=1}^{k}x^r\right\}}_{\ge 0}\ge 0.
\end{aligned}$$

So by Abel's uniform convergence test series $\sum_{k=1}^{\infty}\frac{kx^k u_k^{\mp}(\alpha)}{\sum_{r=0}^{k-1}x^r}$ converge uniformly on $[0,1]\times[0,1]$ which implies that sequences $([S_n(x)]^{\mp}(\alpha))$ uniformly converge to $([S(x)]^{\mp}(\alpha))$ on $[0,1]\times[0,1]$. Then, fuzzy-number-valued function $S(x)$ is continuous on $[0,1]$ and we have $\lim_{x\to 1^-}S(x)=S(1)=v$, which completes the proof. □

A Lambert summable series is not necessarily convergent which can be seen by series $\sum u_k$ of fuzzy numbers whose general term u_k is defined by

$$u_k(t)=\begin{cases} k^4(t-(-1)^k), & (-1)^k\le t\le(-1)^k+k^{-4}\\ 1, & (-1)^k+k^{-4}\le t\le(-1)^k+k^{-2}\\ k^2\left((-1)^k-t\right)+2, & (-1)^k+k^{-2}\le t\le(-1)^k+2k^{-2}\\ 0, & (otherwise).\end{cases}$$

Series $\sum_{k=1}^{\infty}u_k$ of fuzzy numbers is Lambert summable to fuzzy number

$$v(t)=\begin{cases} \frac{90(t-1/2)}{\pi^4}, & 1/2\le t\le 1/2+\frac{\pi^4}{90}\\ 1, & 1/2+\frac{\pi^4}{90}\le t\le 1/2+\frac{\pi^2}{6}\\ \frac{6((1/2)-t)}{\pi^2}+2, & 1/2+\frac{\pi^2}{6}\le t\le 1/2+\frac{\pi^2}{3}\\ 0, & (otherwise),\end{cases}$$

but it is not convergent.

Theorem 3.3. *If series $\sum u_k$ is Lambert summable to fuzzy number v and $kD(u_k,\bar{0})=o(1)$ then $\sum u_k=v$.*

Proof. Suppose that series $\sum u_k$ is Lambert summable to v. Since $\sum u_k$ is Lambert summable, series $\sum\frac{kx^ku_k}{1-x^k}$ exists in E^1 for $x\in[0,1)$ and $\lim_{x\to1^-}(1-x)\sum\frac{kx^ku_k}{1-x^k}=v$. So we obtain

$$\begin{aligned} D\left(\sum_{k=1}^{n}u_k,v\right) &\le D\left(\sum_{k=1}^{n}u_k,(1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k}\right)+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right)\\ &\le D\left(\sum_{k=1}^{n}u_k,(1-x)\sum_{k=1}^{n}\frac{kx^ku_k}{1-x^k}\right)+D\left((1-x)\sum_{k=n+1}^{\infty}\frac{kx^ku_k}{1-x^k},\bar{0}\right)+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right)\\ &\le (1-x)\sum_{k=1}^{n}D(u_k,\bar{0})\left|\frac{1}{1-x}-\frac{kx^k}{1-x^k}\right|+(1-x)\sum_{k=n+1}^{\infty}kD(u_k,\bar{0})\frac{x^k}{1-x^k}+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right).\\ &\le (1-x)\sum_{k=1}^{n}kD(u_k,\bar{0})+\sum_{k=n+1}^{\infty}kD(u_k,\bar{0})\left\{\frac{x^k}{1-x^k}-\frac{x^{k+1}}{1-x^k}\right\}+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right). \end{aligned}$$

By the assumption $kD(u_k,\bar{0})=o(1)$, for given $\varepsilon>0$ we have an $n_1\in\mathbb{N}$ such that $kD(u_k,\bar{0})<\varepsilon/4$ for $k>n_1$ and an $M>0$ such that $kD(u_k,\bar{0})<M$ for all k. Then we get

$$\begin{aligned} D\left(\sum_{k=1}^{n}u_k,v\right) &\le (1-x)\sum_{k=1}^{n_1}kD(u_k,\bar{0})+(1-x)\sum_{k=n_1+1}^{n}kD(u_k,\bar{0})\\ &\quad+\sum_{k=n+1}^{\infty}kD(u_k,\bar{0})\left\{\frac{x^k}{1-x^k}-\frac{x^{k+1}}{1-x^{k+1}}\right\}+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right)\\ &< (1-x)n_1M+\frac{\varepsilon}{4}(1-x)(n-n_1)+\frac{\varepsilon}{4}\frac{x^{n+1}}{(1-x^{n+1})}+D\left((1-x)\sum_{k=1}^{\infty}\frac{kx^ku_k}{1-x^k},v\right). \end{aligned}$$

Taking $x=1-1/n$, it gives

$$\begin{aligned} D\left(\sum_{k=1}^{n}u_k,v\right) &\le \frac{n_0M}{n}+\frac{\varepsilon}{4}\frac{(n-n_0)}{n}+\frac{\varepsilon}{4}\frac{(1-1/n)^{n+1}}{(1-(1-1/n)^{n+1})}+D\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{k(1-1/n)^ku_k}{1-(1-1/n)^k},v\right)\\ &< \frac{\varepsilon}{2}+\frac{n_0M}{n}+D\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{k(1-1/n)^ku_k}{1-(1-1/n)^k},v\right). \end{aligned}$$

In view of the facts that $\frac{n_0 M}{n} \to 0 \ \wedge \ D\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{k(1-1/n)^k u_k}{1-(1-1/n)^k}, \nu\right) \to 0$,

- *there exists* $n_2 \in \mathbb{N}$ *such that* $\frac{n_0 M}{n} < \frac{\varepsilon}{4}$ *whenever* $n > n_2$,
- *there exists* $n_3 \in \mathbb{N}$ *such that* $D\left(\frac{1}{n}\sum_{k=1}^{\infty}\frac{k(1-1/n)^k u_k}{1-(1-1/n)^k}, \nu\right) < \varepsilon/4$ *whenever* $n > n_3$.

So we conclude that $D(\sum_{k=1}^{n} u_k, \nu) < \varepsilon$ whenever $n > \max\{n_1, n_2, n_3\}$, which completes the proof. □

Corollary 3.4. *If series* $\sum u_k$ *of fuzzy numbers is Lambert summable to fuzzy number* ν *and* $kD(u_k, \bar{0}) = O(1)$, *then* $(u_k) \in bs(F)$.

4. Zeta Summability of Sequences of Fuzzy Numbers

Definition 4.1. *A sequence* (u_k) *of fuzzy numbers is said to be zeta summable to a fuzzy number* μ *if series* $\sum_{k=1}^{\infty}\frac{u_k}{k^s}$ *of fuzzy numbers converges for all* $s > 1$ *and*

$$\lim_{s\to 1^+}\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s} = \mu.$$

Remark 4.2. *It should be noted that a sequence* (u_k) *of fuzzy numbers is zeta summable to* μ *iff*

$$\lim_{s\to 1^+}(s-1)\sum_{k=1}^{\infty}\frac{u_k}{k^s} = \mu$$

in view of the fact that $\lim_{s\to 1^+}\zeta(s)(s-1) = 1$.

Theorem 4.3. *If sequence* (u_k) *of fuzzy numbers converges to a fuzzy number* μ, *then* (u_k) *is zeta summable to* μ.

Proof. Suppose that $u_k \to \mu$. Then (u_k) is bounded, which implies the existence of an $M > 0$ such that $D(u_k, \bar{0}) \le M$ for all k. So, for $s > 1$ we obtain

$$\sum_{k=1}^{\infty} D\left(\frac{u_k}{k^s}, \bar{0}\right) = \sum_{k=1}^{\infty}\frac{D(u_k, \bar{0})}{k^s} \le M\sum_{k=1}^{\infty}\frac{1}{k^s} < \infty$$

and series $\sum_{k=1}^{\infty}\frac{u_k}{k^s}$ converges for $s > 1$ by Theorem 2.3. Besides, given $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $D(u_k, \mu) < \frac{\varepsilon}{2}$ for $k > n_0$. By properties of metric D and by Theorem 2.2 we get

$$\begin{aligned} D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s}, \mu\right) &= D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s}, \frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{\mu}{k^s}\right) \\ &\le \frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{D(u_k,\mu)}{k^s} \\ &= \frac{1}{\zeta(s)}\sum_{k=1}^{n_0}\frac{D(u_k,\mu)}{k^s} + \frac{1}{\zeta(s)}\sum_{k=n_0+1}^{\infty}\frac{D(u_k,\mu)}{k^s} \\ &< \frac{1}{\zeta(s)}\sum_{k=1}^{n_0}\frac{D(u_k,\mu)}{k^s} + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\lim_{s\to 1+}\frac{1}{\zeta(s)}\sum_{k=1}^{n_0}\frac{D(u_k,\mu)}{k^s} = 0$, we have a $\delta > 0$ such that $\frac{1}{\zeta(s)}\sum_{k=1}^{n_0}\frac{D(u_k,\mu)}{k^s} < \frac{\varepsilon}{2}$ whenever $s \in (1, 1+\delta)$. So we have

$$D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s}, \mu\right) < \varepsilon$$

whenever $s \in (1, 1+\delta)$ and the proof is completed. □

A zeta summable sequence is not necessarily convergent which can be seen by sequence (u_k) of fuzzy numbers defined by

$$u_k(t)=\begin{cases} t+(-1)^k, & (-1)^{k-1}\le t\le(-1)^{k-1}+1\\ -t+(-1)^{k-1}+2, & (-1)^{k-1}+1\le t\le(-1)^{k-1}+2\\ 0, & (otherwise).\end{cases}$$

Sequence (u_k) of fuzzy numbers is zeta summable to fuzzy number

$$\mu(t)=\begin{cases} t, & 0\le t\le 1\\ 2-t, & 1\le t\le 2\\ 0, & (otherwise),\end{cases}$$

but it is not convergent.

We now give a Tauberian condition for zeta summability method of sequences of fuzzy numbers. In real case, this condition is obtained by means of the equivalence theorem of zeta summability method and logarithmic summability method proved by Persi Diaconis[5, Theorem 1] on bounded sequences. We obtain the Tauberian condition for zeta summability method of sequences of fuzzy numbers directly and use no further condition on the sequence, thus providing also an alternative proof with weakened condition in the special case that sequence is of real numbers.

Theorem 4.4. *If sequence (u_k) of fuzzy numbers is zeta summable to a fuzzy number μ and $(k\ln k)D(u_{k-1},u_k)=o(1)$, then (u_k) converges to μ.*

Proof. Suppose that (u_n) is zeta summable to μ and $(n\ln n)D(u_{n-1},u_n)=o(1)$. Since $D(u_{n-1},u_n)=o(1/(n\ln n))$, we have $D(u_k,u_n)=o(1)|\ln(\ln n)-\ln(\ln k)|$. Then we conclude

$$\begin{aligned}
D(u_n,\mu) &\le D\left(u_n,\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s}\right)+D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s},\mu\right)\\
&\le \frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{D(u_n,u_k)}{k^s}+D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s},\mu\right)\\
&\le (s-1)\sum_{k=1}^{5}\frac{D(u_n,u_k)}{k^s}+(s-1)\sum_{k=6}^{\infty}\frac{D(u_n,u_k)}{k^s}+D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s},\mu\right)\qquad\left(\text{taking } s=1+\frac{1}{\ln n}\right)\\
&= o(1)\frac{1}{\ln n}\left\{\sum_{k=6}^{\infty}\frac{|\ln(\ln n)-\ln(\ln k)|}{k^{(1+1/\ln n)}}\right\}+\underbrace{\frac{1}{\ln n}\sum_{k=1}^{5}\frac{D(u_n,u_k)}{k^{(1+1/\ln n)}}+D\left(\frac{1}{\zeta(1+\frac{1}{\ln n})}\sum_{k=1}^{\infty}\frac{u_k}{k^{(1+1/\ln n)}},\mu\right)}_{=o(1)\ as\ n\to\infty.}\\
&= o(1)\frac{1}{\ln n}\left\{\ln(\ln n)\sum_{k=6}^{n}\frac{1}{k^{(1+1/\ln n)}}-\sum_{k=6}^{n}\frac{\ln(\ln k)}{k^{(1+1/\ln n)}}+\sum_{k=n+1}^{\infty}\frac{\ln(\ln k)}{k^{(1+1/\ln n)}}-\ln(\ln n)\sum_{k=n+1}^{\infty}\frac{1}{k^{(1+1/\ln n)}}\right\}\\
&= o(1)\frac{1}{\ln n}\left\{\ln(\ln n)\left(\frac{1}{6^{(1+1/\ln n)}}+\int_6^n\frac{dx}{x^{(1+1/\ln n)}}\right)-\int_6^n\frac{\ln(\ln x)}{x^{(1+1/\ln n)}}dx\right.\\
&\qquad\left.+\frac{ln(ln(n+1))}{(n+1)^{(1+1/\ln n)}}+\int_{n+1}^{\infty}\frac{\ln(\ln x)}{x^{(1+1/\ln n)}}dx-\ln(\ln n)\int_{n+1}^{\infty}\frac{dx}{x^{(1+1/\ln n)}}\right\}\\
&= o(1)\left\{\frac{\ln(\ln n)}{(\ln n)6^{(1+1/\ln n)}}+\frac{\ln(\ln n)}{6^{(1/\ln n)}}-\frac{\ln(\ln n)}{e}+\frac{\ln(\ln n)}{e}+\mathrm{Ei}\left(-\frac{\ln 6}{\ln n}\right)-\mathrm{Ei}(-1)-\frac{\ln(\ln 6)}{6^{(1/\ln n)}}\right.\\
&\qquad\left.+\frac{\ln(\ln(n+1))}{(\ln n)(n+1)^{(1+1/\ln n)}}+\frac{\ln(\ln(n+1))}{(n+1)^{(1/\ln n)}}-\mathrm{Ei}\left(-\frac{\ln(n+1)}{\ln n}\right)-\frac{\ln(\ln n)}{(n+1)^{(1/\ln n)}}\right\}\\
&= o(1)\qquad as\quad n\to\infty,
\end{aligned}$$

which completes the proof. □

Theorem 4.5. *If sequence (u_k) of fuzzy numbers is zeta summable to a fuzzy number μ and $(k\ln k)D(u_{k-1},u_k)=O(1)$, then $(u_k)\in\ell_\infty(F)$.*

Proof. Suppose that sequence (u_n) of fuzzy numbers is zeta summable to μ and $(n\ln n)D(u_{n-1},u_n)=O(1)$. Then, as in the proof of Theorem 4.4, we get

$$\begin{aligned} D(u_n,\bar{0}) &\leq D\left(u_n,\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s}\right)+D\left(\frac{1}{\zeta(s)}\sum_{k=1}^{\infty}\frac{u_k}{k^s},\mu\right)+D(\mu,\bar{0}) \\ &= \underbrace{\frac{O(1)}{\ln n}\left\{\sum_{k=6}^{\infty}\frac{|\ln(\ln n)-\ln(\ln k)|}{k^{(1+1/\ln n)}}\right\}}_{=O(1)}+\underbrace{\frac{1}{\ln n}\sum_{k=1}^{5}\frac{D(u_n,u_k)}{k^{(1+1/\ln n)}}+D\left(\frac{1}{\zeta(1+\frac{1}{\ln n})}\sum_{k=1}^{\infty}\frac{u_k}{k^{(1+1/\ln n)}},\mu\right)+D(\mu,\bar{0})}_{=o(1)} \\ &= O(1) \quad as \quad n\to\infty \end{aligned}$$

and the proof is completed. □

Definition 4.6. *A series $\sum u_k$ of fuzzy numbers is said to be zeta summable to fuzzy number ν if the sequence of partial sums of the series $\sum u_k$ is zeta summable to ν.*

Corollary 4.7. *If series $\sum u_k$ of fuzzy numbers converges to fuzzy number ν, then it is zeta summable to ν.*

Corollary 4.8. *If series $\sum u_k$ of fuzzy numbers is zeta summable to fuzzy number ν and $(k\ln k)D(u_k,\bar{0})=o(1)$, then $\sum u_k=\nu$.*

Corollary 4.9. *If series $\sum u_k$ of fuzzy numbers is zeta summable to fuzzy number ν and $(k\ln k)D(u_k,\bar{0})=O(1)$, then $(u_k)\in bs(F)$.*

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