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Existence of Positive Periodic Solutions of Fourth-order Singular p-Laplacian Neutral Functional Differential Equations

Fanchao Kong[a], Shiping Lu[b]

[a]*Department of Mathematics, Hunan Normal University, Changsha, Hunan 410081, P. R. China*
[b]*College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, P. R. China*

Abstract. This work deals with the existence of positive periodic solutions for the fourth-order p-Laplacian neutral functional differential equations with a time-varying delay and a singularity. The results are established using the continuation theorem of coincidence degree theory and some analysis methods. A numerical example is presented to illustrate the effectiveness and feasibility of the proposed criterion.

1. Introduction

During the past several years, neutral functional differential equations have received more and more attention because of its widely applied backgrounds, for example population ecology, heat exchanges, mechanics and economics, see [6], [9], [10], [27]. In 1995, Zhang [30] studied the following linear and quasilinear neutral functional differential equations:

$$(x(t) - bx(t - \tau))' = -ax(t - r + \gamma h(t, x(t + \cdot))) + e(t),$$

where a, τ, r are nonzero constants and $\gamma \in \mathbb{R}$ is a small parameter, $e \in C_{2\pi}$, $h : \mathbb{R} \times C_{2\pi}$(real functions) $\to \mathbb{R}$ is continuous such that $h(t + 2\pi, \varphi) \equiv h(t, \varphi)$ on $\mathbb{R} \times C_{2\pi}$, $C_{2\pi} := \{x|x \in C(\mathbb{R}, \mathbb{R}), x(t + 2\pi) \equiv x(t)\}$. Using some a priori estimation and the Leray-Schauder degree theory, the author obtained some existence theorems of periodic solutions.

On the basis of work of Zhang in [30], Lu in [12] discussed the the following first-order neutral functional differential equation

$$\frac{d}{dt}(u(t) - ku(t - \tau)) = g_1(u(t)) + g_2(u(t - \tau_1)) + p(t),$$

where $g_1, g_2 \in C(\mathbb{R}, \mathbb{R})$, $p(t) \in C(\mathbb{R}, \mathbb{R})$ and $p(t + T) \equiv P(t)$, τ, τ_1, k are constants such that $|k| \neq 1$. By means of Mawhin's continuation theorem, existence criteria are established for the periodic solutions. Moreover, Lu in [13] gave some inequalities for A:

If $|c| < 1$ then A has continuous inverse on C_T and

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Email addresses: fanchaokong88@sohu.com (Fanchao Kong), lushiping@sohu.com (Shiping Lu)

(1) $\| A^{-1}x \| \leq \frac{\|x\|_0}{|1-|c||}$, $\forall x \in C_T$;
(2) $\int_0^T |(A^{-1}f)(t)|dt \leq \frac{1}{|1-|c||}\int_0^T |f(t)|dt$, $\forall f \in C_T$;
(3) $\int_0^T |A^{-1}f|^2(t)dt \leq \frac{1}{(1-|c|)^2}\int_0^T f^2(t)dt$, $\forall f \in C_T$.

After that, based on the work of Zhang and Lu, many authors further established the existence results of periodic solutions to different kinds of neutral functional differential equations, see [2], [8], [13], [14], [15], [16], [17], [21], [23], [24], [28] and the references therein. For example, in [24], Wang and Zhu studied a kind of fourth-order p-Laplacian neutral functional differential equation with a deviating argument in the form:

$$(\varphi_p(x(t) - cx(t-\delta))'')'' = f(x(t))x'(t) + g(t, x(t - \tau(t, |x|_\infty))) + e(t).$$

By means of Mawhin's continuation theorem, the existence results of periodic solutions are obtained.

In recent years, singular equations appear in a lot of physical models, see [18], [19], [20], [29] and the references therein. Different kinds of singular equations have been proposed by many authors, see for example [3], [4], [7], [22], [25], [26], [32] and the references therein.

However, to the best of our knowledge, there are few papers about the positive periodic solutions for the neutral functional differential equations with a singularity.

Recently, Kong and Lu in [11] study the existence of positive periodic for the following neutral Liénard differential equation with a singularity and a deviating argument

$$\Big((x(t) - cx(t-\sigma))\Big)'' + f(x(t))x'(t) + g(t, x(t-\delta)) = e(t),$$

where c is a constant with $|c| < 1$, $0 \leq \sigma, \delta < T$, $f : \mathbb{R} \to \mathbb{R}$ is continuous, $g : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function and can be singular at $u = 0$. $e(t)$ is T-periodic with $\int_0^T e(t)dt = 0$.

Inspired by the works mentioned above, in this paper, we consider the following fourth-order p-Laplacian neutral functional differential equation with a time-varying delay and a singularity

$$\Big(\varphi_p\Big(x(t) - cx(t-\delta)\Big)''\Big)'' + f(x(t))x'(t) + g(t, x(t - \delta(t))) = e(t), \tag{1.1}$$

where $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(u) = |u|^{p-2}u$, $p > 1$; c is a constant with $|c| < 1$, δ is a continuous function; $f : (0, +\infty) \to \mathbb{R}$ is continuous; $g : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function and can be singular at $u = 0$; $e(t)$ is T-periodic with $\int_0^T e(t)dt = 0$. By applying the continuation theorem of coincidence degree theory, we prove that Eq.(1.1) has at least one positive T-periodic solution.

Remark 1.1. *The theorem and methods used to obtain the periodic solutions in [24] can be applied to the Eq.*(1.1) *if there is no singularity in Eq.*(1.1)*. So, we extend the neutral functional differential equation to the singular case.*

The rest of the paper is organized as follows. In Section 2, we state some necessary definitions and lemmas. In Section 3, we prove the main result. Finally, an example is given to support the effectiveness of our result in Section 4.

2. Preliminaries

Thought the paper, let

$$C_T = \Big\{\phi \in C(\mathbb{R}, \mathbb{R}), \phi(t+T) \equiv \phi(t)\Big\}$$

with the norm $| \phi |_0 = \max_{t \in [0,T]} |\phi(t)|$,

$$C_T^1 = \Big\{\phi \in C^1(\mathbb{R}, \mathbb{R}), \phi(t+T) \equiv \phi(t)\Big\}$$

with the norm $\| \phi \| = \max\{| \phi |_0, | \phi' |_0\}$.

Denote the operator A by

$$\mathrm{A} : C_T \to C_T,\ (\mathrm{A}\,x)(t) = x(t) - cx(t-\sigma),\ \forall t \in \mathbb{R}.$$

In order to use coincidence degree theory to study the existence of positive T-periodic solutions for (1.1), we rewrite (1.1) in the following form:

$$\begin{cases} (Au)''(t) = \varphi_q(v(t)) \\ v''(t) = -f(u(t))u'(t) - g(t, u(t-\delta(t))) + e(t). \end{cases} \tag{2.1}$$

where $q > 1$ is a constant with $\frac{1}{p} + \frac{1}{q} = 1$. Clearly, if $x(t) = (u(t), v(t))^\top$ is a T-periodic solution to system (2.1), then $u(t)$ must be a T-periodic solution of equation (1.1). Thus, the problem of finding a positive T-periodic solution for (1.1) reduces to finding one for (2.1).

Let

$$X = \left\{x(t) = (u(t), v(t))^\top \in C^2(\mathbb{R}, \mathbb{R}^2),\ x(t) \equiv x(t+T)\right\},$$

$$Y = \left\{x(t) = (u(t), v(t))^\top \in C^2(\mathbb{R}, \mathbb{R}^2),\ x(t) \equiv x(t+T)\right\},$$

The normal $\|x\| = \max\{| u |_0, | v |_0\}$, and $| u |_0 = \max_{t\in[0,T]} |u|$, $| v |_0 = \max_{t\in[0,T]} |v|$. It is obviously that X and Y are Banach spaces.

Define the operator

$$L : D(L) \subset X \to Y,\ \ Lx = x'' = \left((\mathrm{A}\,u)'', v''\right)^\top, \tag{2.2}$$

where $D(L) = \left\{x(t) = (u(t), v(t))^\top \in C^2(\mathbb{R}, \mathbb{R}^2), x(t) \equiv x(t+T)\right\}$.

Define a nonlinear operator $N : D(N) \subset X \to Y$ as follows:

$$(Nx)(t) = \begin{pmatrix} \varphi_q(v(t)) \\ -f(u(t))u'(t) - g(t, u(t-\delta(t))) + e(t) \end{pmatrix},\ \forall t \in \mathbb{R}. \tag{2.3}$$

where $D(N) = \left\{x = (u, v)^\top \in X : u(t) > 0, t \in [0, T]\right\}$. Then (2.1) can be converted to the abstract equation $Lx = Nx$.

From the definition of L, we can easily see that

$$\ker L \cong \mathbb{R}^2,\ \ \mathrm{Im}L = \left\{y \in Y, \int_0^T y(s)ds = 0\right\}.$$

Thus L is a Fredholm operator with index zero. Let the projections P and Q be

$$P : X \to \ker L,\ \ Px = \frac{1}{T}\int_0^T x(s)ds,$$

$$Q : Y \to \mathrm{Im}Q,\ \ Qy = \frac{1}{T}\int_0^T y(s)ds.$$

Then we can see that $\mathrm{Im}P = \ker L$ and $\ker Q = \mathrm{Im}L$. Let $L_p = L|_{D(L)\cap\ker P}$. We can easily prove that L_p is invertible, $L_p^{-1} : \mathrm{Im}L \to D(L) \cap \ker P$, and

$$(L_p^{-1}y)(t) = \int_0^T G(t, s)y(s)ds,$$

where $G(t, s) = \begin{cases} \frac{-s(T-t)}{T}, & 0 \le s \le t \le T; \\ \frac{-t(T-s)}{T}, & s \le t \le s \le T. \end{cases}$

Lemma 2.1. *[14] If $|c| \neq 1$, then operator* A *has a unique continuous bounded inverse and satisfies the following conditions:*

(1) $\int_0^T |[A^{-1}f](t)|dt \leq \frac{\int_0^T |f(t)|dt}{|1-|c||}, \forall f \in C_T$;

(2) $(Ax)'' = A''x,\ \forall x \in C_T^2 := \{x \in C^2(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$.

Lemma 2.2. *[5] Let X and Y be two real Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \subset X \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L),\ \forall \lambda \in (0,1)$;

(2) $QNx \neq 0, \forall x \in \partial\Omega \cap \ker L$;

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \ker L$ *is an homeomorphism map. Then the equation* $Lx = Nx$ *has at least one solution on* $D(L) \cap \bar{\Omega}$.

Lemma 2.3. *[31] If $x \in C^1(\mathbb{R}, \mathbb{R})$ and $x(0) = x(T) = 0$, then*

$$\int_0^T |x'(t)|^p dt \leq \left(\frac{T}{\pi_p}\right)^p \int_0^T |x''(t)|^p dt,$$

where $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{[1-s^p/(p-1)]^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

For the sake of convenience, we list the following assumptions:

$[H_1]$ There exist positive constants D_1 and D_2 with $D_1 < D_2$ such that

(1) for each positive continuous T-periodic function $x(t)$ satisfying $\int_0^T g(t, x(t))dt = 0$, then there exists a positive point $t_0 \in [0, T]$ such that

$$D_1 \leq x(t_0) \leq D_2;$$

(2) $\overline{g}(x) < 0$ for all $x \in (0, D_1)$ and $\overline{g}(x) > 0$ for all $x > D_2$, where $\overline{g}(x) = \frac{1}{T}\int_0^T g(t,x)dt,\ x > 0$.

$[H_2]$ $g(t, x(t-\delta(t))) = g_1(t, x(t-\delta(t)) + g_0(x(t))$, where $g_0 : (0, +\infty) \to \mathbb{R}$ is a continuous function, $g_1 : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function and

(1) there exist positive constants m_0 and m_1 such that

$$g(t, x) \leq m_0 x^{p-1} + m_1,\ \text{for all } (t, x) \in [0, T] \times (0, +\infty);$$

(2) $\int_0^1 g_0(x)dx = -\infty$.

$[H_3]$ There exist positive constants α and β such that

$$F(x) \leq \alpha x^{p-1} + \beta, \quad \text{for all } x \in (0, +\infty),$$

where $F(x) = \int_0^x f(s)ds$.

3. Main Results

Theorem 3.1. *Suppose that conditions $[H_1]$-$[H_3]$ hold and*

$$\frac{[2(1+|c|)m_0+\alpha|c|]T^{2p-1}}{\pi_p^p|1-|c||^p}<1,$$

then there exist positive constants A_1, A_2, A_3 and ρ, which are independent of λ such that

$$A_1 \le u(t) \le A_2,\ \ |u'|_0 \le A_3,\ \ |v|_0 \le \rho,$$

where $u(t)$ is any solution to the equation $Lx = \lambda Nx, \lambda \in (0,1]$.

Proof Consider the following operator equation

$$Lx = \lambda Nx,\ \lambda \in (0,1),$$

where L and N are defined by (2.2) and (2.3), respectively. Let

$$\Omega_1 = \left\{(u,v)^\top \in X : \min_{t\in[0,T]} u(t) > 0, Lx = \lambda Nx, \lambda \in (0,1)\right\}.$$

If $x = (u,v)^\top \in \Omega_1$, then (u,v) satisfies

$$\begin{cases}(\mathrm{A}\,u)''(t) = \lambda\varphi_q(v(t))\\ v''(t) = -\lambda f(u(t))u'(t) - \lambda g(t,u(t-\delta(t))) + \lambda e(t).\end{cases} \tag{3.1}$$

From the first equation of (3.1), we can get $v(t) = \lambda^{-1}(\mathrm{A}\,u)''(t)$, which combining with the second equation of (3.1) yields

$$\left(\varphi_p((Au)''(t))\right)'' + \lambda^p f(u(t))u'(t) + \lambda^p g(t,u(t-\delta(t))) = \lambda^p e(t). \tag{3.2}$$

Integrating the equation (3.2) on the interval $[0,T]$, we have

$$\int_0^T \left(\varphi_p((Au)''(t))\right)'' dt + \lambda^p \int_0^T f(u(t))u'(t)dt + \lambda^p \int_0^T g(t,u(t-\delta(t)))dt = \lambda^p \int_0^T e(t)dt,$$

then we can have

$$\int_0^T g(t,u(t-\delta(t)))dt = 0. \tag{3.3}$$

It follows from $[H_1](1)$ that there exist positive constants D_1, D_2 and $t_0 \in [0,T]$ such that

$$D_1 \le u(t_0) \le D_2. \tag{3.4}$$

From this, we obtain

$$|u|_0 = \max_{t\in[0,T]}|u(t)| \le \max_{t\in[0,T]}\left|u(t_0) + \int_{t_0}^t u'(s)ds\right| \le D_2 + \int_0^T |u'(s)|ds, \tag{3.5}$$

Multiplying the both sides of (3.2) by $(Au)(t)$ and integrating on the interval $[0, T]$, we get

$$\begin{aligned}\int_0^T |(A\,u)''(t)|^p dt &= \lambda^p \int_0^T f(u(t))u'(t)(A\,u)(t)dt + \lambda^p \int_0^T g(t, u(t-\delta(t)))(A\,u)(t)dt \\ &\quad - \lambda^p \int_0^T e(t)(A\,u)(t)dt \\ &\leq \left| \int_0^T f(u(t))u'(t)[u(t) - cu(t-\sigma)]dt \right| \\ &\quad + (1+|c|) \mid u \mid_0 \int_0^T |g(t, u(t-\delta(t)))|dt + (1+|c|) \mid u \mid_0 \int_0^T |e(t)|dt \\ &= \left| \int_0^T \Big[f(u(t))u'(t)u(t) - cf(u(t))u'(t)u(t-\sigma) \Big] dt \right| \\ &\quad + (1+|c|) \mid u \mid_0 \int_0^T |g(t, u(t-\delta(t)))|dt + (1+|c|) \mid u \mid_0 \int_0^T |e(t)|dt \\ &= |c| \left| \int_0^T F(u(t))u'(t-\sigma)dt \right| + (1+|c|) \mid u \mid_0 \int_0^T |g(t, u(t-\delta(t)))|dt \\ &\quad + (1+|c|) \mid u \mid_0 \int_0^T |e(t)|dt \\ &\leq |c| \int_0^T |F(u(t))||u'(t-\sigma)|dt + (1+|c|) \mid u \mid_0 \int_0^T |g(t, u(t-\delta(t)))|dt \\ &\quad + (1+|c|) \mid u \mid_0 \int_0^T |e(t)|dt,\end{aligned}$$

which combining with $[H_3]$ yields

$$\begin{aligned}\int_0^T |(A\,u)''(t)|^p dt &\leq \alpha|c| \mid u \mid_0^{p-1} \int_0^T |u'(t-\sigma)|dt + \beta|c| \int_0^T |u'(t-\sigma)|dt \\ &\quad + (1+|c|) \mid u \mid_0 \int_0^T |g(t, u(t-\delta(t)))|dt \\ &\quad + (1+|c|) \mid u \mid_0 |e|_0 T.\end{aligned} \tag{3.6}$$

Write

$$E_+ = \{t \in [0, T] : g(t, u(t-\delta(t))) \geq 0\};$$

$$E_- = \{t \in [0, T] : g(t, u(t-\delta(t))) \leq 0\}.$$

Then it follows from (3.3) and $[H_2](1)$ that

$$\begin{aligned}\int_0^T |g(t, u(t-\delta(t)))|dt &= \int_{E_+} g(t, u(t-\delta(t)))dt - \int_{E_-} g(t, u(t-\delta(t)))dt \\ &= 2\int_{E_+} g(t, u(t-\delta(t)))dt \\ &\leq 2m_0 \int_0^T u^{p-1}(t-\delta(t))dt + 2\int_0^T m_1 dt \\ &\leq 2m_0 T \mid u \mid_0^{p-1} + 2Tm_1.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), combining with (3.5), we can have

$$\begin{aligned}\int_0^T |(A\,u)''(t)|^p dt &\le 2(1+|c|)m_0T \mid u \mid_0^p + \alpha|c| \mid u \mid_0^{p-1} \int_0^T |u'(t-\sigma)|dt \\ &\quad + \beta|c| \int_0^T |u'(t-\sigma)|dt + 2(1+|c|)m_1 T \mid u \mid_0 + (1+|c|)|e|_0 T \mid u \mid_0 \\ &\le 2(1+|c|)m_0T\left(D_2 + \int_0^T |u'(s)|ds\right)^p \\ &\quad + \alpha|c|\left(D_2 + \int_0^T |u'(s)|ds\right)^{p-1} \int_0^T |u'(t)|dt \\ &\quad + \beta|c|\int_0^T |u'(t)|dt + 2(1+|c|)m_1T\left(D_2 + \int_0^T |u'(s)|ds\right) \\ &\quad + (1+|c|)|e|_0T\left(D_2 + \int_0^T |u'(s)|ds\right). \end{aligned} \tag{3.8}$$

Moreover,

$$\left(D_2 + \int_0^T |u'(s)|ds\right)^p = \left(\int_0^T |u'(t)|dt\right)^p \left(1 + \frac{D_2}{\int_0^T |u'(t)|dt}\right)^p. \tag{3.9}$$

By classical elementary inequalities, we see that there exists a $l(p) > 0$ which is dependent on p only, such that

$$(1+x)^p < 1 + (1+p)x, \quad x \in (0, l(p)]. \tag{3.10}$$

Then, from (3.9) we should consider the following two cases:

Case 1. $\frac{D_2}{\int_0^T |u'(t)|dt} > l(p)$, then $\int_0^T |u'(t)|dt < \frac{D_2}{l(p)}$ and from (3.5), we have

$$\mid u \mid_0 < D_2 + \frac{D_2}{l(p)} := M_0. \tag{3.11}$$

Case 2. $\frac{D_2}{\int_0^T |u'(t)|dt} \le l(p)$, then it follows from (3.9) and (3.10) that

$$\begin{aligned}\left(D_2 + \int_0^T |u'(s)|ds\right)^p &= \left(\int_0^T |u'(t)|dt\right)^p \left(1 + \frac{D_2}{\int_0^T |u'(t)|dt}\right)^p \\ &\le \left(\int_0^T |u'(t)|dt\right)^p \left(1 + \frac{D_2(p+1)}{\int_0^T |u'(t)|dt}\right) \\ &= \left(\int_0^T |u'(t)|dt\right)^p + D_2(p+1)\left(\int_0^T |u'(t)|dt\right)^{p-1}. \end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.8), we see that

$$
\begin{aligned}
\int_0^T |(\mathrm{A}\, u)''(t)|^p dt \le\ & 2(1+|c|)m_0 T\left[\Big(\int_0^T |u'(t)|dt\Big)^p + D_2(p+1)\Big(\int_0^T |u'(t)|dt\Big)^{p-1}\right] \\
& + \alpha|c|\left[\Big(\int_0^T |u'(t)|dt\Big)^{p-1} + D_2(p+1)\Big(\int_0^T |u'(t)|dt\Big)^{p-2}\right]\cdot \int_0^T |u'(t)|dt \\
& + \beta|c|\int_0^T |u'(t)|dt + 2(1+|c|)m_1 T\left(D_2 + \int_0^T |u'(s)|ds\right) \\
& + (1+|c|)|e|_0 T\left(D_2 + \int_0^T |u'(s)|ds\right),
\end{aligned}
\tag{3.13}
$$

which together with the Hölder inequality and Lemma 2.3 yields

$$
\begin{aligned}
\int_0^T |(\mathrm{A}\, u)''(t)|^p dt \le\ & \frac{[2(1+|c|)m_0 + \alpha|c|]T^{2p-1}}{\pi_p^p}\int_0^T |x''(t)|^p dt \\
& + [2(1+|c|)m_0 T + \alpha|c|]D_2(p+1)T^{\frac{(p-1)^2}{p}}\left[\Big(\frac{T}{\pi_p}\Big)^p \int_0^T |x''(t)|^p dt\right]^{\frac{p-1}{p}} \\
& + [\beta|c| + 2(1+|c|)m_1 T + (1+|c|)|e|_0 T]T^{\frac{p-1}{p}}\left[\Big(\frac{T}{\pi_p}\Big)^p \int_0^T |x''(t)|^p dt\right]^{\frac{1}{p}} \\
& + [2m_1 + |e|_0](1+|c|)TD_2.
\end{aligned}
$$

Furthermore, by applying Lemma 2.1, we can obtain

$$
\begin{aligned}
\int_0^T |(\mathrm{A}\, u)''(t)|^p dt \le\ & \frac{[2(1+|c|)m_0 + \alpha|c|]T^{2p-1}}{\pi_p^p|1-|c||^p}\int_0^T |(\mathrm{A}\, u)''(t)|^p dt \\
& + [2(1+|c|)m_0 T + \alpha|c|]D_2(p+1)T^{\frac{(p-1)^2}{p}}\Big[\Big(\frac{T}{\pi_p}\Big)^p \int_0^T |x''(t)|^p dt\Big]^{\frac{p-1}{p}} \\
& + [\beta|c| + 2(1+|c|)m_1 T + (1+|c|)|e|_0 T]T^{\frac{p-1}{p}}\Big[\Big(\frac{T}{\pi_p}\Big)^p \int_0^T |x''(t)|^p dt\Big]^{\frac{1}{p}} \\
& + [2m_1 + |e|_0](1+|c|)TD_2.
\end{aligned}
$$

It follows from $\frac{[2(1+|c|)m_0+\alpha|c|]T^{2p-1}}{\pi_p^p|1-|c||^p} < 1$ that there exists a positive constant $M_1 > 0$ such that

$$
\int_0^T |(\mathrm{A}\, u)''(t)|^p dt \le M_1.
$$

Then by Lemma 2.1, we can see that

$$
\int_0^T |u''(t)|^p dt \le \frac{M_1}{|1-|c||^p} := M_2,
\tag{3.14}
$$

which together with (3.5) and Lemma 2.3 yields

$$\begin{aligned}
|u|_0 &\leq D_2 + T^{\frac{p-1}{p}} \left(\int_0^T |u'(t)|^p dt \right)^{\frac{1}{p}} \\
&\leq D_2 + T^{\frac{p-1}{p}} \left[\left(\frac{T}{\pi_p} \right)^p \int_0^T |u''(t)|^p dt \right]^{\frac{1}{p}} \\
&\leq D_2 + \frac{T^{2p-1} M_2^{\frac{1}{p}}}{\pi_p} := M_3.
\end{aligned}$$

Therefore, in both Case 1 and Case 2, we can conclude that

$$|u|_0 \leq M_3. \tag{3.15}$$

Moreover, since $u(t)$ is T-periodic, there exists a point $t_0 \in [0, T]$ such that $u'(t_0) = 0$. Then by applying Hölder's inequality and in view of (3.14), we have

$$\begin{aligned}
|u'|_0 &\leq \int_0^T |u''(t)| dt \\
&\leq T^{\frac{1}{p}} \left(\int_0^T |u''(t)|^p dt \right)^{1/p} \\
&\leq T^{\frac{1}{p}} M_2^{\frac{1}{p}} := A_3.
\end{aligned} \tag{3.16}$$

From the second equation of (3.1), we can get

$$\int_0^T |v''(t)| dt \leq \int_0^T |f(u(t))||u'(t)| dt + \lambda \int_0^T |g(t, u(t-\delta(t)))| dt + \lambda \int_0^T |e(t)| dt.$$

Set $f_{M_3} = \max_{|u|<M_3} |f(u(t))|$, then by (3.7) we have

$$\int_0^T |v''(t)| dt \leq \lambda \Big[f_{M_3} T \, |u'|_0 + 2m_0 T \, |u|_0^{p-1} + 2T m_1 + |e|_0 T \Big],$$

in view of (3.15) and (3.16), we get

$$\int_0^T |v''(t)| dt \leq \lambda \left[f_{M_3} T A_3 + 2m_0 T M_3^{p-1} + 2T m_1 + |e|_0 T \right]. \tag{3.17}$$

Moreover, integrating the first equation of (3.1) on the interval $[0, T]$, we have

$$\int_0^T u''(t) dt = \int_0^T (A^{-1} v)(t) dt = 0,$$

which implies that there exists $\eta \in [0, T]$ such that $v(\eta) = 0$. Thus,

$$|v(t)| = \left| \int_\eta^t v'(s) ds + v(\eta) \right| \leq \int_0^T |v'(s)| ds \leq \sqrt{T} \left(\int_0^T |v'(s)|^2 ds \right)^{1/2},$$

which together with Lemma 2.3 and (3.14) gives

$$\begin{aligned}
|v|_0 &\leq \sqrt{T} \left(\int_0^T |v'(s)|^2 ds \right)^{1/2} \cdot \frac{T\sqrt{T}}{\pi} \cdot \left(\int_0^T |v''(s)|^2 ds \right)^{1/2} \\
&\leq \frac{T^2}{\pi} \left(f_{M_3} T A_3 + 2m_0 T M_3^{p-1} + 2T m_1 + |e|_0 T \right)^{1/2} \\
&:= \rho.
\end{aligned} \tag{3.18}$$

On the other hand, it follows from the second equation of (3.1) and $[H_2]$ that

$$v''(t) = -\lambda f(u(t))u'(t) - \lambda[g_1(t, u(t-\delta(t))) + g_0(u(t))] + \lambda e(t). \tag{3.19}$$

Multiplying both sides of Eq.(3.19) by $u'(t)$, we have

$$v''(t)u'(t) = -\lambda f(u(t))u'(t)u'(t) - \lambda[g_1(t, u(t-\delta(t))) + g_0(u(t))]u'(t) + \lambda e(t)u'(t). \tag{3.20}$$

Let $t_0 \in [0, T]$ be as in (3.4). For any $t \in [t_0, T]$, integrating Eq.(3.20) on the interval $[t_0, T]$, we can get

$$\begin{aligned}
\lambda \int_{u(t_0)}^{u(t)} g_0(u)du &= \lambda \int_{t_0}^{t} g_0(u(t))u'(t)dt \\
&= -\int_{t_0}^{t} v''(t)u'(t)dt - \lambda \int_{t_0}^{t} f(u(t))u'(t)u'(t)dt \\
&\quad - \lambda \int_{t_0}^{t} g_1(t, u(t-\delta(t)))u'(t)dt + \lambda \int_{t_0}^{t} e(t)u'(t)dt,
\end{aligned}$$

which together with (3.17) yields

$$\begin{aligned}
\lambda \left| \int_{u(t_0)}^{u(t)} g_0(u)du \right| &= \lambda \left| \int_{t_0}^{t} g_0(u(t))u'(t)dt \right| \\
&\leq \int_0^T |v''(t)||u'(t)|dt + \lambda \int_0^T |f(u(t))||u'(t)||u'(t)|dt \\
&\quad + \lambda \int_0^T |g_1(t, u(t-\delta(t)))||u'(t)|dt + \lambda \int_0^T |e(t)||u'(t)|dt \\
&\leq \lambda \mid u' \mid_0 \left[f_{M_3} T A_3 + 2m_0 T M_3^{p-1} + 2Tm_1 + |e|_0 T \right] \\
&\quad + \lambda \mid u' \mid_0^2 f_{M_3} T + \lambda \mid u' \mid_0 \int_0^T |g_1(t+\delta(t), u(t))|dt \\
&\quad + \lambda \mid u' \mid_0 \int_0^T |e(t+\delta(t))|dt.
\end{aligned}$$

Set $g_{M_3} = \max_{t\in[0,T],|u|\leq M_3} |g_1(t, u)|$, then we have

$$\begin{aligned}
\left| \int_{u(t_0)}^{u(t)} g_0(u)du \right| &\leq \mid u' \mid_0 \left[f_{M_3} T A_3 + 2m_0 T M_3^{p-1} + 2Tm_1 + |e|_0 T \right] \\
&\quad + \mid u' \mid_0^2 T f_{M_3} + \mid u' \mid_0 T g_{M_3} + \mid u' \mid_0 T |e|_0,
\end{aligned}$$

which combining with (3.16) gives

$$\begin{aligned}
\left| \int_{u(t_0)}^{u(t)} g_0(u)du \right| &\leq A_3 \left[f_{M_3} T A_3 + 2m_0 T M_3^{p-1} + 2Tm_1 + |e|_0 T \right] \\
&\quad + A_3^2 T f_{M_3} + T A_3 g_{M_3} + T A_3 |e|_0 \\
&< +\infty.
\end{aligned}$$

According to condition (2) in $[H_2]$, we can see that there exists a constant $M_4 > 0$ such that, for $t \in [t_0, T]$,

$$u(t) \geq M_4. \tag{3.21}$$

For the case $t \in [0, t_0]$, we can handle similarly.
Let us define

$$0 < A_1 = \min\{D_1, M_3\},$$

and

$$A_2 = \max\{D_2, M_4\}.$$

Then by (3.4), (3.15) and (3.21), we can obtain

$$A_1 \le u(t) \le A_2. \tag{3.22}$$

Clearly, A_1 and A_2 are independent of λ. Therefore, from (3.16), (3.18) and (3.22), we can see that the proof of Theorem 3.1 is now complete.

Theorem 3.2. *Suppose that all the conditions in Theorem 3.1 hold, then system* (2.1) *has at least one positive T-periodic solution $(u, v)^\top \subset \Omega_1$ such that*

$$A_1 \le u(t) \le A_2, \ \mid u' \mid_0 \le A_3, \ \mid v \mid_0 \le \rho.$$

Proof Set

$$\Omega = \left\{ x = (u, v)^\top \in X : \frac{A_1}{2} < u(t) < A_2 + 1, \mid u' \mid_0 < A_3 + 1, \mid v \mid_0 < \rho + 1 \right\},$$

then condition (1) of Lemma 2.2 is satisfied.

Suppose that there exists $x = (u, v)^\top \in \partial\Omega \cap \ker L$ such that

$$QNx = \frac{1}{T}\int_0^T Nx(s)ds = (0, 0)^\top,$$

then $u \in \mathbb{R}$ and $v \in \mathbb{R}$ are constant valued functions and satisfy

$$\begin{cases} \dfrac{1}{T}\displaystyle\int_0^T [\mathrm{A}^{-1}(v)]dt = 0, \\ \dfrac{1}{T}\displaystyle\int_0^T [-f(u)u' - g(t, u) + e(t)]dt = 0. \end{cases} \tag{3.23}$$

It follows from condition (1) in $[H_1]$ and the second part of Lemma 2.2 that

$$\frac{A_1}{2} < D_1 \le u \le D_2 < A_2 + 1, v = 0,$$

which contradicts the assumption $x \in \partial\Omega$. This contradiction implies that condition (2) of Lemma 2.2 is satisfied.

Finally, we will show that condition (3) of Lemma 2.2 is also satisfied.
Let

$$z = Kx = K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \frac{A_1 + A_2}{2} \\ v \end{pmatrix},$$

then, we have

$$x = z + \begin{pmatrix} \frac{A_1 + A_2}{2} \\ 0 \end{pmatrix}.$$

Define $J : \mathrm{Im}Q \to \ker L$ is a linear isomorphism of the form

$$J(u, v) = \begin{pmatrix} -v \\ u \end{pmatrix}.$$

Moreover, define

$$H(\mu, x) = \mu Kx + (1-\mu)JQNx, \quad \forall (x,\mu) \in (\Omega \cap \ker L) \times [0,1].$$

Then,

$$H(\mu, x) = \begin{pmatrix} \mu u - \frac{\mu(A_1+A_2)}{2} \\ \mu v \end{pmatrix} + \frac{1-\mu}{T} \begin{pmatrix} \int_0^T [f(u)u' + g(t,u)]dt \\ \int_0^T \varphi_q(v)dt \end{pmatrix}. \tag{3.24}$$

Now we claim that $H(\mu, x)$ is a homotopic mapping. Assume, by way of contradiction, *i.e.*, there exists $\mu_0 \in [0,1]$ and $x_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \partial(\Omega \cap \ker L)$ such that $H(\mu_0, x_0) = 0$.
Substituting μ_0 and x_0 into (3.21), we have

$$H(\mu_0, x_0) = \begin{pmatrix} \mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)f(u_0)u_0' + (1-\mu_0)\overline{g}(u_0) \\ \mu_0 v_0 + (1-\mu_0)\varphi_q(v_0) \end{pmatrix}. \tag{3.25}$$

It follows $H(\mu_0, x_0) = 0$ that

$$\mu_0 v_0 + (1-\mu_0)\varphi_q(v_0) = 0,$$

which together with $\mu_0 \in [0,1]$ yields $v_0 = 0$. Thus, $u_0 = \frac{A_1}{2}$ or $u_0 = A_2 + 1$.
Furthermore,
If $u_0 = \frac{A_1}{2}$, it follows from $[H_1](2)$ that $\overline{g}(u_0) < 0$, then substituting $u_0 = \frac{A_1}{2}$ and $v_0 = 0$ into (3.25), we have

$$\begin{aligned} &\frac{\mu_0 A_1}{2} - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\overline{g}\left(\frac{A_1}{2}\right) \\ &= -\frac{\mu_0 A_2}{2} + (1-\mu_0)\overline{g}\left(\frac{A_1}{2}\right) \\ &< 0. \end{aligned} \tag{3.26}$$

If $u_0 = A_2 + 1$, it follows from $[H_1](2)$ that $\overline{g}(u_0) > 0$, then substituting $u_0 = A_2 + 1$ and $v_0 = 0$ into (3.22), we have

$$\begin{aligned} &\mu_0(A_2+1) - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\overline{g}(A_2+1) \\ &= \frac{\mu_0(A_2-A_1+2)}{2} + (1-\mu_0)\overline{g}(A_2+1) \\ &> 0. \end{aligned} \tag{3.27}$$

From (3.26) and (3.27), we can see that $H(\mu_0, x_0) \neq 0$, which contradicts the assumption. Therefore $H(\mu, x)$ is a homotopic mapping and $x^\top H(\mu, x) \neq 0$. Moreover, for all $(x, \mu) \in (\partial\Omega \cap \ker L) \times [0,1]$, we have

$$\begin{aligned} \deg(JQN, \Omega \cap \ker L, 0) &= \deg(H(0,x), \Omega \cap \ker L, 0) \\ &= \deg(H(1,x), \Omega \cap \ker L, 0) \\ &= \deg(Kx, \Omega \cap \ker L, 0) \\ &= \sum_{x \in K^{-1}(0)} \operatorname{sgn}(\det K'(x)) \\ &= 1 \neq 0. \end{aligned}$$

Thus, the condition (3) of Lemma 2.2 is also satisfied. Therefore, by applying Lemma 2.2, we can see that (2.1) has a positive T-periodic solution $(u,v)^\top \subset \overline{\Omega}$. Clearly, u is a positive T-periodic solution to (1.1), and $(u,v)^\top$ must be in Ω_1 for the case of $\lambda = 1$. Thus, by using Theorem 3.1, we have

$$A_1 \leq u(t) \leq A_2, \quad |\, u' \,|_0 \leq A_3, \quad |\, v \,|_0 \leq \rho.$$

Hence, we can conclude that Eq.(1.1) has at least one positive T-periodic solution.

4. Example

In this section, we provide an example to illustrate our main result.

Example 4.1. *Consider the following fourth-order p-Laplacian neutral functional differential equation with a time-varying delay and a singularity:*

$$\left(\varphi_4\left(x(t)-\frac{1}{2}x\left(t-\frac{\pi}{8}\right)\right)''\right)''+\left(2\sin u(t)+\frac{1}{4}\right)u'(t)+\frac{1}{64}(1+\sin 6t)u(t-\cos 6t)-\frac{1}{u(t-\cos 6t)}=\sin 6t. \qquad (4.1)$$

Conclusion: The Problem (4.1) *has at least one positive* $\pi/3$*-periodic solution.*

Proof. Corresponding to (1.1), we have

$$f(u(t))=2\sin u(t)+\frac{1}{4},\ e(t)=\sin 6t,\ \delta(t)=\cos 6t,$$

$$g(t,u(t-\delta(t)))=\frac{1}{64}(1+\sin 6t)u(t-\cos 6t)-\frac{1}{u(t-\cos 6t)}.$$

Then, we can have and choose

$$p=4,\ T=\frac{\pi}{3},\ c=\frac{1}{2},\ m_0=\frac{1}{32},\ \alpha=\frac{1}{4},\ D_1=1,\ D_2=9.$$

It is easy to see that $[H_1]$-$[H_3]$ hold. Moreover, we also have $\pi_4=\frac{2\pi 3^{1/4}}{4\sin(\pi/4)}=\frac{3^{1/4}\pi}{\sqrt{2}}$,

$$\frac{[2(1+|c|)m_0+\alpha|c|]T^{2p-1}}{\pi_p^p|1-|c||^p}\approx 0.0662<1,$$

which implies that Theorem 3.1 is satisfied. Therefore, by Theorem 3.2 we can see that Eq.(4.1) has at least one positive $\pi/3$-periodic solution. □

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