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Solvability Conditions for Mixed Sylvester Equations in Rings

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Abstract. This paper has been motivated by Wang and He [Q.W. Wang and Z.H. He, Solvability conditions and general solution for mixed Sylvester equations. Automatica, 49 (2013) 2713-2719] in which the authors consider some solvability conditions for mixed Sylvester matrix equations. The paper also considers the same problem in the setting of a regular ring. Using the purely algebraic technique, we present some necessary and sufficient conditions for the solvability to mixed Sylvester equations in rings.

1. Introduction

Linear matrix equations such as the Sylvester equation, Lyapunov equation, Stein equation arise frequently from a variety of important applications, including control theory, completely integrable systems, vibration system, Lie algebra, signal processing, finite element models of PDEs, invariant subspace computation, and many others disciplines. Many papers have presented different approaches for several matrix equations [7–9, 12–14, 17, 19, 20]. Especially, many problems in control theory can be transformed into the Sylvester matrix equations, such as singular system control [4, 21], robust control [3, 26], neural network [25, 36]. The solvability of linear equations is a fundamental problem, and various results are developed, such as solvability conditions of linear equations for matrices over the complex field [1, 2, 10, 11, 18, 22, 23, 29–34, 37], solvability conditions of linear equations over algebras or rings [5, 6, 24, 27, 28, 35].

Recently, Lee and Vu [16] proved that the mixed Sylvester matrix

$$A_1X - YB_1 = C_1 \quad \text{and} \quad A_2Z - YB_2 = C_2, \tag{1}$$

is consistent if and only if there exist invertible matrices R_1, R_2 and S such that

$$\begin{pmatrix} A_1 & C_1 \\ O & B_1 \end{pmatrix} R_1 = S \begin{pmatrix} A_1 & O \\ O & B_1 \end{pmatrix},$$

$$\begin{pmatrix} A_2 & C_2 \\ O & B_2 \end{pmatrix} R_2 = S \begin{pmatrix} A_2 & O \\ O & B_2 \end{pmatrix},$$

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where A_i, B_i and C_i ($i = 1, 2$) are given complex matrices, X, Y and Z are variable matrices. Liu [18] also gave a solvability condition to (1). Wang and He [31] presented new necessary and sufficient solvability conditions for the system (1), and gave an expression of the general solution when it is solvable. When $X = Z$, the system (1) becomes pairs of generalized Sylvester equations

$$A_1X - YB_1 = C_1 \quad \text{and} \quad A_2X - YB_2 = C_2. \tag{2}$$

Wimmer [33] gave a necessary and sufficient condition for the existence of a simultaneous solution of (2). Kågström [15] obtained a solution of (2) by using generalized Schur methods.

Motivated by the wide applications on the system of Sylvester matrix equation, it is interesting to consider the mixed Sylvester matrix equations in a ring

$$a_1x - yb_1 = c_1 \text{ and } a_2z - yb_2 = c_2, \tag{3}$$

where a_i, b_i and c_i are given elements in a ring $\mathcal{R}$, x, y, and z are arbitrary elements of $\mathcal{R}$. The paper is organized as follows. In Section 2, we give some known results and lemmas. In Section 3, we consider the solvability conditions of the mixed Sylvester matrix equations in $\mathcal{R}$, which extends the results of [31, Theorem 3.1] to the ring case.

2. Preliminaries

Let $\mathcal{R}$ represent an associative ring with unity 1. For $x \in \mathcal{R}$, an inner inverse of x is an element y such that $xyx = x$, we denote any inner inverse of x by x^-. An element is said regular if it possesses an inner inverse. If $a \in \mathcal{R}$ is regular, let L_a and R_a stand for the two idempotents $L_a = 1 - a^-a$ and $R_a = 1 - aa^-$ induced by a, respectively. We first review some lemmas which are used in the further development of this paper.

Lemma 2.1. *([6, Theorem 3.1]) Let $a, b, c \in \mathcal{R}$ with a, b regular. Then the equation*

$$axb = c \tag{4}$$

is consistent in $\mathcal{R}$ if and only if $c = aa^-cb^-b$. If $c = aa^-cbb^-b$, then the general solution of (2.1) is given by

$$x = a^-cb^- + u - a^-aubb^-, \tag{5}$$

where $u \in \mathcal{R}$ is arbitrary.

Lemma 2.2. *([5, Theorem 3.2]) Let $a_i, b_i, c_i \in \mathcal{R}$ with a_i, b_i regular, $i = 1, 2$. If $a_1a_1^-c_1b_1^-b_1 = c_1$ and $a_2a_2^-c_2b_2^-b_2 = c_2$. Then the following equations*

$$a_1xb_1 = c_1 \text{ and } a_2xb_2 = c_2 \tag{6}$$

have a common solution if and only if

$$(1 - ss^-)(c_2 - gc_1f)(1 - t^-t) = 0,$$

where $s = a_2L_{a_1}$, $t = R_{b_1}b_2$, $g = (1 - ss^-)a_2a_1^-$ and $f = b_1^-b_2(1 - t^-t)$.

3. Solvability Conditions of the Mixed Sylvester Matrix Equations

Using algebra methods, in this section, we give some necessary and sufficient conditions for the consistence to Eq.(3) in a ring. Let $a_i, b_i \in \mathcal{R}$ ($i = 1, 2$) in Eq. (3) be regular, and write $\tilde{s} = R_{a_2}a_1a_1^-$ and $\tilde{t} = R_{b_1}b_2$.

Theorem 3.1. *Let a_i, b_i and c_i $(i = 1, 2)$ be given in $\mathcal{R}$ and set $\tilde{s}$ and $\tilde{t}$ be regular. Write*

$$a = \tilde{s}a_1, \quad b = b_2(1 - \tilde{t}^-\tilde{t}) \quad and \quad c = R_{a_2}R_{a_1}c_1b_1^-b_2(1 - \tilde{t}^-\tilde{t}) - R_{a_2}c_2(1 - \tilde{t}^-\tilde{t}).$$

Then the following statements are equivalent:

(1) The mixed Sylvester matrix equations (3) is solvable;

(2) $R_{a_1}c_1L_{b_1} = 0$, $R_{a_2}c_2L_{b_2} = 0$ and $\tilde{s}\tilde{s}^-c = c$;

(3) $R_{a_1}c_1L_{b_1} = 0$, $c = aa^-c = cb^-b$.

Proof. (1) ⇔ (2). Let $d_1 = c_1 + yb_1$ for some $y \in \mathcal{R}$. Consider the equation $a_1x = d_1$, by Lemma 2.1, it is solvable if and only if $a_1a_1^-d_1 = d_1$. Substituting $d_1 = c_1 + yb_1$ into $a_1a_1^-d_1 = d_1$, which can reduce to $(1 - a_1a_1^-)yb_1 = a_1a_1^-c_1 - c_1$.

Similarly, let $d_2 = c_2 + yb_2$ for some $y \in R$. According to Lemma 2.1, the equation $a_2z = d_2$ is solvable if and only if $a_2a_2^-d_2 = d_2$, i.e., $(1 - a_2a_2^-)yb_2 = a_2a_2^-c_2 - c_2$.

Therefore, Eq.(3) is solvable if and only if the following pair of equations have a common solution y:

$$\begin{cases} (1 - a_1a_1^-)yb_1 = -(1 - a_1a_1^-)c_1. \\ (1 - a_2a_2^-)yb_2 = -(1 - a_2a_2^-)c_2. \end{cases} \tag{7}$$

Using Lemma 2.1, the first equation in Eq.(7) is solvable if and only if $(1 - a_1a_1^-)c_1b_1^-b_1 = (1 - a_1a_1^-)c_1$, that is,

$$R_{a_1}c_1L_{b_1} = 0. \tag{8}$$

Similarly, the second equation in Eq.(7) is solvable if and only if

$$R_{a_2}c_2L_{b_2} = 0. \tag{9}$$

Combining (8) and (9), applying Lemma 2.2, it follows that the system (7) have a common solution if and only if

$$(1 - \tilde{s}\tilde{s}^-)(-R_{a_2}c_2 + \tilde{g}R_{a_1}c_1\tilde{f})(1 - \tilde{t}^-\tilde{t}) = 0,$$

where $\tilde{f} = b_1^-b_2(1 - \tilde{t}^-\tilde{t})$ and $\tilde{g} = (1 - \tilde{s}\tilde{s}^-)R_{a_2}R_{a_1}$.

By direct computation, one can see

$$(1 - \tilde{s}\tilde{s}^-)(-R_{a_2}c_2 + \tilde{g}R_{a_1}c_1\tilde{f})(1 - \tilde{t}^-\tilde{t}) = 0 \text{ if and only if } c = \tilde{s}\tilde{s}^-c.$$

(2) ⇔ (3). Since $\tilde{s} = R_{a_2}a_1a_1^-$, note that $\tilde{s} = \tilde{s}a_1a_1^-$, for the choice $(\tilde{s}a_1)^- = a_1^-\tilde{s}^-$, we obtain

$$aa^-\tilde{s}\tilde{s}^- = (\tilde{s}a_1)(\tilde{s}a_1)^-\tilde{s}\tilde{s}^- = (\tilde{s}a_1)(\tilde{s}a_1)^-(\tilde{s}a_1a_1^-)\tilde{s}^- = \tilde{s}a_1a_1^-\tilde{s}^- = \tilde{s}\tilde{s}^-.$$

It gives that $aa^-\tilde{s}\tilde{s}^-c = \tilde{s}\tilde{s}^-c$. If $\tilde{s}\tilde{s}^-c = c$, we get at once $aa^-c = c$. Conversely, assume that $aa^-c = c$, then $(\tilde{s}a_1)(\tilde{s}a_1)^-c = c$. For the choice $(\tilde{s}a_1)^- = a_1^-\tilde{s}^-$, by $\tilde{s} = \tilde{s}a_1a_1^-$. Then we obtain that $c = (\tilde{s}a_1)(\tilde{s}a_1)^-c = \tilde{s}a_1a_1^-\tilde{s}^-c = \tilde{s}\tilde{s}^-c$. Thus, one can see that $aa^-c = c$ is equivalent to $\tilde{s}\tilde{s}^-c = c$.

Now we show that $R_{a_2}c_2L_{b_2} = 0$ is equivalent to $cb^-b = c$. Indeed, if $R_{a_2}c_2L_{b_2} = 0$, it gives that $R_{a_2}c_2 = R_{a_2}c_2b_2^-b_2$. By $b = b_2(1 - \tilde{t}^-\tilde{t})$, one can obtain that

$$\begin{aligned} & R_{a_2}c_2(1 - \tilde{t}^-\tilde{t})b^-b \\ = \;& (R_{a_2}c_2b_2^-b_2)(1 - \tilde{t}^-\tilde{t})b^-b \\ = \;& R_{a_2}c_2b_2^-bb^-b = R_{a_2}c_2b_2^-b \\ = \;& R_{a_2}c_2b_2^-b_2(1 - \tilde{t}^-\tilde{t}) \\ = \;& R_{a_2}c_2(1 - \tilde{t}^-\tilde{t}). \end{aligned} \tag{10}$$

And as $b = b_2(1 - \tilde{t}^-\tilde{t})$, we also have

$$\begin{aligned} & R_{a_2}R_{a_1}c_1b_1^-b_2(1 - \tilde{t}^-\tilde{t})b^-b \\ = \;& R_{a_2}R_{a_1}c_1b_1^-b \\ = \;& R_{a_2}R_{a_1}c_1b_1^-b_2(1 - \tilde{t}^-\tilde{t}). \end{aligned} \tag{11}$$

In view of (10) and (11), we have at once $cb^{-}b = c$.

Conversely, if $cb^{-}b = c$, by (11) $R_{a_2}R_{a_1}c_1b_1^{-}b_2(1-\tilde{t}^{-}\tilde{t})b^{-}b = R_{a_2}R_{a_1}c_1b_1^{-}b_2(1-\tilde{t}^{-}\tilde{t})$, it means that

$$R_{a_2}c_2(1-\tilde{t}^{-}\tilde{t})b^{-}b = R_{a_2}c_2(1-\tilde{t}^{-}\tilde{t}). \tag{12}$$

Combining $\tilde{t} = R_{b_1}b_2$ and $b = b_2(1-\tilde{t}^{-}\tilde{t})$, it gives that

$$\begin{aligned} bb_2^{-}b_2 &= b_2(1-\tilde{t}^{-}\tilde{t})b_2^{-}b_2 \\ &= b_2(1-\tilde{t}^{-}R_{b_1}b_2)b_2^{-}b_2 \\ &= (1-b_2t^{-}R_{b_1})b_2 \\ &= b_2(1-\tilde{t}^{-}R_{b_1}b_2) \\ &= b_2(1-\tilde{t}^{-}\tilde{t}) \\ &= b, \end{aligned}$$

that is, $b(1-b_2^{-}b_2) = 0$. So post-multiply (12) by $1-b_2^{-}b_2$ gives that

$$R_{a_2}c_2(1-\tilde{t}^{-}\tilde{t})(1-b_2^{-}b_2) = 0.$$

Thus, we obtain

$$R_{a_2}c_2(1-b_2^{-}b_2) = R_{a_2}c_2\tilde{t}^{-}\tilde{t}(1-b_2^{-}b_2) = R_{a_2}c_2\tilde{t}^{-}R_{b_1}b_2(1-b_2^{-}b_2) = 0.$$

It gives that $R_{a_2}c_2L_{b_2} = 0$. This proof is completed. □

Corollary 3.2. *([31, Theorem 3.1]) Let A_i, B_i, and $C_i (i = 1, 2)$ be given. Set*

$$D_1 = R_{B_1}B_2,\ \ A = R_{A_2}A_1,\ \ B = B_2L_{D_1},$$
$$C = R_{A_2}(R_{A_1}C_1B_1^{\dagger}B_2 - C_2)L_{D_1}$$

Then the following statements are equivalent:

(1) The mixed Sylvester matrix equations (1) is consistent.

(2) $R_{A_1}C_1L_{B_1} = 0,\ \ R_AC = 0,\ \ CL_B = 0.$

(3) $R_{A_1}C_1 = R_{A_1}C_1B_1^{\dagger}B_1,\ \ C = AA^{\dagger}C = CB^{\dagger}B.$

References

[1] J.K. Baksalary and R. Kala, The matrix equation $AXB + CYD = E$, Linear Algebra Appl., 30 (1980), 141-147.

[2] J.K. Baksalary and R. Kala, The matrix equation $AX - YB = C$, Linear Algebra Appl., 25 (1979), 41-43.

[3] R.K. Cavinlii and S.P. Bhattacharyya, Robust and well-conditioned eigenstructure assignment via Sylvester's equation, Optimal Control Appl. Methods, 4 (1983), 205-212.

[4] E.B. Castelan and V. Gomes da Silva, On the solution of a Sylvester matrix equation appearing in descriptor systems control theory, Systems Control Lett., 54 (2005), 109-117.

[5] A. Dajić, Common solutions of linear equations in a ring, with applications, Electron. J. Linear Algebra, 30 (2015) 66-79.

[6] A. Dajić and J.J. Koliha, Equations $ax = c$ and $xb = d$ in rings and rings with involution with applications to Hilbert space operators, Linear Algebra Appl., 429 (2008), 1779-1809.

[7] M. Dehghan and M. Hajarian, Finite iterative algorithms for the reflexive and anti-reflexive solutions of the matrix equation $A_1XB_1 + A_2XB_2 = C$, Math. Comput. Model, 49 (2009), 1937-1959.

[8] M. Dehghan and M. Hajarian, The general coupled matrix equations over generalized bisymmetric matrices, Linear Algebra Appl., 432 (2010), 1531-1552.

[9] M. Dehghan and M. Hajarian, Two algorithms for finding the Hermitian reflexive and skew-Hermitian solutions of Sylvester matrix equations, Appl. Math. Lett., 24 (2011), 444-449.

[10] M. Dehghan and M. Hajarian, On the generalized reflexive and anti-reflexive solutions to a system of matrix equations, Linear Algebra Appl., 437 (2012), 2793-2812.

[11] F. Ding and T. Chen, On iterative solutions of general coupled matrix equations, SIAM. J. Control Optim., 44 (2006), 2269-2284.

[12] A. El Guennouni, K. Jbilou and J. Riquet, Block Krylov subspace methods for solving large Sylvester equations, Numer. Algorithms, 29 (2002), 75-96.

[13] D.Y. Hu and L. Reichel, Krylov-subspace methods for the Sylvester equation, Linear Algebra Appl., 172 (1992), 283-313.

[14] L.P. Huang and Q. Zeng, The solvability of matrix equation $AXB + CYD = E$ over a simple Arinian ring, Linear Multilinear Algebra, 38 (1995), 225-232.

[15] B. Kägström, A perturbation analysis of the generalized Sylvester equation $(AR - LB, DR - LE) = (C, F)$, SIAM. J. Matrix Anal. Appl., 15 (1994), 1045-1060.
[16] S.G. Lee and Q.P. Vu, Simultaneous solutions of matrix equations and simultaneous equivalence of matrices, Linear Algebra Appl., 437 (2012), 2325-2339.
[17] A.P. Liao and Y. Lei, Optimal approximate solution of the matrix equation $AXB = C$ over symmetric matrices, J. Comput. Math., 25 (2007), 543-252.
[18] Y.H. Liu, Ranks of solutions of the linear matrix equation $AX + YB = C$, Comput. Math. Appl., 52 (2006), 861-872.
[19] S.K. Mitra, Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, Proc. Camb. Philos. Soc., 74 (1973), 213-216.
[20] A.B. Özgüler, The matrix equation $AXB + CYD = E$ over a principal ideal domain, SIAM. J. Matrix Anal. Appl., 12 (1991), 581-591.
[21] A. Shahzad, B.L. Jones, E.C. Kerrigan and G.A. Constantinides, An efficient algorithm for the solution of a coupled Sylvester equation appearing in descriptor systems, Automatica, 47 (2011), 244-248.
[22] C.Q. Song and G.L. Chen, On solutions of matrix equations $XF - AX = C$ and $XF - A\widetilde{X} = C$ over quaternion field, J. Math. Anal. Appl., 37 (2011), 57-68.
[23] Y.G. Tian, The solvability of two linear matrix equations, Linear Multilinear Algebra, 48 (2000), 123-147.
[24] Y.G. Tian, Solving some linear equations over alternative rings, Results in Mathematics, 58 (2010), 355-364.
[25] J.W. Van der Woude, Almost noninteracting control by measurement feedback, IEEE Trans. Automat. Control, 9 (1987),7-16.
[26] A. Varga, Robust pole assignment via Sylvester equation based state feedback parametrization. In Computer-aided control system design, 2000. CACSD 2000. IEEE international symposium, 57: 13-18, 2000.
[27] Q.W. Wang, A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. Linear Algebra Appl., 384 (2004), 43-54.
[28] Q.W. Wang, J.H. Sun and S.Z. Li, Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra, Linear Algebra Appl., 353 (2002), 169-182.
[29] Q.W. Wang, J.W. van der Woude and H.X. Chang, A system of real quaternion matrix equations with applications, Linear Algebra Appl., 431 (2009), 2291-2303.
[30] Q.W. Wang and C.L. Yang, The Re-nonnegative definite solutions to the matrix equation $AXB = C$, Comment. Math. Univ. Carolin, 39 (1998), 7-13.
[31] Q.W. Wang and Z.H. He, Solvability conditions and general solution for the mixed Sylvester equations, Automatica, 49 (2013), 2713-2719.
[32] Q.W. Wang and Z.H. He, Systems of coupled generalized Sylvester matrix equations, Automatica, 50 (2014), 2840-2844.
[33] H.K. Wimmer, Consistency of a pair of generalized Sylvester equations, IEEE Trans. Automat. Control, 39 (1994), 1014-1016.
[34] A.G. Wu, G.R. Duan and B. Zhou, Solution to generalized sylvester matrix equations, IEEE Trans. Automat. Control, 53 (2008), 811-815.
[35] Q.X. Xu, Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$, Linear Algebra Appl., 429 (2008), 1-11.
[36] Y.N. Zhang, D.C. Jiang and J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, IEEE Trans. Neural Networks, 13 (2002), 1053-1063.
[37] B. Zhou and G.R. Duan, On the generalized Sylvester mapping and matrix equations, Systems Control Lett., 57 (2008), 200-208.