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Minimal Extremal Graphs for Addition of Algebraic Connectivity and Independence Number of Connected Graphs

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Abstract. Let $G = (V, E)$ be a simple connected graph. Denote by $D(G)$ the diagonal matrix of its vertex degrees and by $A(G)$ its adjacency matrix. Then the Laplacian matrix of graph G is $L(G) = D(G) - A(G)$. Let $a(G)$ and $\alpha(G)$, respectively, be the second smallest Laplacian eigenvalue and the independence number of graph G. In this paper, we characterize the extremal graph with second minimum value for addition of algebraic connectivity and independence number among all connected graphs with $n \geq 6$ vertices (Actually, we can determine the p-th minimum value of $a(G)+\alpha(G)$ under certain condition when p is small). Moreover, we present a lower bound to the addition of algebraic connectivity and radius of connected graphs.

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, and let $\overline{G}$ denote its complement graph. Denote by $d_G(v_i)$ the degree of vertex v_i, and denote by $N_G(v_i)$ the neighbor set of the vertex v_i, where $i = 1, 2, \ldots, n$. Hereafter, if there is no confusion, we always simply $d_G(v_i)$ and $N_G(v_i)$ as $d(v_i)$ and $N(v_i)$, respectively. Let $N[v_i] = N(v_i) \cup \{v_i\}$ and let $|X|$ be the number of elements in the set X. Then $|N(v_i)| = d(v_i)$ and $|N[v_i]| = d(v_i) + 1$, where $i = 1, 2, \ldots, n$. If vertices v_i and v_j are adjacent, we denote that by $v_iv_j \in E(G)$. The adjacency matrix $A(G)$ of G is defined by its entries $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise, and the degree matrix $D(G)$ of G is the diagonal matrix whose entries are the degrees of the vertices of G. The Laplacian matrix of graph G is $L(G) = D(G) - A(G)$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ denote the Laplacian eigenvalues of G. Recently, the Laplacian eigenvalues of graphs attract more and more attentions [6, 7]. Among all eigenvalues of the Laplacian matrix of a graph G, the most studied is the second smallest, which is denoted by $a(G)$ hereafter. It is well known that a graph G is connected if and only if $a(G) > 0$, and hence Fiedler called $a(G)$ the algebraic connectivity of G [5, 8]. From the definition, it easily follows that $a(G) = \mu_{n-1}(G)$.

The independent set S of G is a set of vertices such that any two vertices of S are not adjacent. An independent set of G is called maximum if G contains no larger independent set. Hereafter, the cardinality

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of a maximum independent set of G is called the *independent number* of G and denoted by $\alpha(G)$. As usual, K_n and $K_{p,q}$ ($p + q = n$) denote respectively the complete graph and the complete bipartite graph with n vertices.

In [1, p. 408], it is conjectured that (See Conjecture A. 368 (SO,T)):

Conjecture 1.1. [1] *If G is a connected graph with n vertices, then $a(G) + \alpha(G)$ is minimum for the graph composed of 2 cliques with $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ vertices, respectively, and linked with a single edge.*

One of the present authors proved Conjecture 1.1 [4]. Motivated from these results, we characterize the extremal graph with second minimum value for addition of algebraic connectivity and independence number among all connected graphs (Actually, our method is effective to determine the p-th minimum value of $a(G) + \alpha(G)$ under certain condition when p is small). Moreover, we present a lower bound to the addition of algebraic connectivity and radius of connected graphs.

2. Lower Bound for Algebraic Connectivity and Independence Number of Connected Graphs

Suppose that $X \subseteq V(G)$ and $v \in V(G)$. Then, we use the symbol $N_X(v)$ to denote the neighbor set of v, which is belonged to X. Let F be a semiregular bipartite graph with bipartition $\{U, W\}$. Denote by F^+ the supergraph of F with the following property: if $uv \in E(F^+)$, then either $uv \in E(F)$ or $u,\ v \in U$ (respectively W) with $N_W(u) = N_W(v)$ (respectively $N_U(u) = N_U(v)$). Set

$$\mathfrak{F}^+ = \{F^+ :\ F \text{ is a semiregular bipartite graph}\}.$$

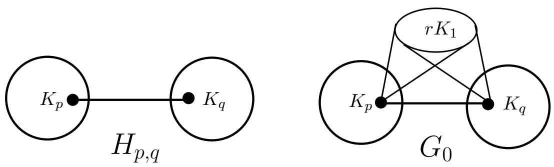

Figure 1: Graphs $H_{p,q}$ and G_0.

Lemma 2.1. [3] *Let G ($\ncong K_n$) be a graph on vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then*

$$a(G) \geq \min_{v_iv_j \notin E(G)} \Big\{ d(v_i) + d(v_j) + 2 - |N[v_i] \cup N[v_j]| \Big\}. \qquad (1)$$

Moreover, if $\overline{G}$ is connected, then the equality holds in (1) *if and only if $\overline{G} \in \mathfrak{F}^+$.*

Let $H_{p,q}$ ($q \geq p \geq 1$) be a graph of order $p + q$ obtain from K_p and K_q by adding an edge between one vertex from K_p to any one vertex in K_q (see, Figure 1). That is, $\overline{H}_{p,q} = K_{p,q} \backslash \{e\}$, e is any edge in $K_{p,q}$, where $q \geq p \geq 1$.

Lemma 2.2. *If $n = p + q$, $q \geq p + 2$ and $p \geq 1$, then*

$$a(H_{p,q}) > a(H_{p+1,q-1}) > \cdots > a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right) > a\left(H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right).$$

Proof. One can easily see that $a\left(H_{p,q}\right)$ satisfies the following system of equations:

$$\begin{cases} (x - p)\, x_1 = -(p - 1)\, x_2 - x_3, \\ (x - 1)\, x_2 = -x_1, \\ (x - q)\, x_3 = -x_1 - (q - 1) x_4, \\ (x - 1)\, x_4 = -x_3. \end{cases}$$

Thus, $a\left(H_{p,q}\right)$ satisfies $f(x;\, p,\, q) = 0$, where

$$f(x;\, p,\, q) = x^3 - (p+q+2)\,x^2 + (pq+p+q+2)\,x - (p+q).$$

Similarly, $a\left(H_{p+1,q-1}\right)$ satisfies $f(x;\, p+1,\, q-1) = 0$, where

$$f(x;\, p+1,\, q-1) = x^3 - (p+q+2)\,x^2 + (pq+2q+1)\,x - (p+q).$$

Let $t = a\left(H_{p,q}\right) > 0$. Then

$$t^3 - (p+q+2)\,t^2 + (pq+p+q+2)\,t - (p+q) = 0, \text{ i.e., } t^3 - (p+q+2)\,t^2 = -(pq+p+q+2)\,t + (p+q).$$

Using the above result, we have

$$f(0;\, p+1,\, q-1) = -(p+q) < 0$$

and

$$f(t;\, p+1,\, q-1) = t^3 - (p+q+2)\,t^2 + (pq+2q+1)\,t - (p+q) = (q-p-1)\,t > 0.$$

Hence $0 < a\left(H_{p+1,q-1}\right) < t = a\left(H_{p,q}\right)$. This completes the proof. □

Corollary 2.3. *If $n \geq 7$, then*

$$a\left(H_{\lfloor \frac{n}{2} \rfloor -1, \lceil \frac{n}{2} \rceil +1}\right) < \frac{18}{4n+9}. \tag{2}$$

Proof. From the proof of Lemma 2.2, we can conclude that $a\left(H_{\lfloor \frac{n}{2} \rfloor -1, \lceil \frac{n}{2} \rceil +1}\right)$ satisfies $f(x;\, \lfloor \frac{n}{2} \rfloor - 1,\, \lceil \frac{n}{2} \rceil + 1) = 0$, where

$$f\left(x;\, \left\lfloor \frac{n}{2} \right\rfloor - 1,\, \left\lceil \frac{n}{2} \right\rceil + 1\right) = x^3 - (n+2)\,x^2 + \left(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right) x - n.$$

Let $s = \dfrac{18}{4n+9}$. Since $n \geq 7$, we have $f(0;\, \lfloor \frac{n}{2} \rfloor - 1,\, \lceil \frac{n}{2} \rceil + 1) = -n < 0$ and

$$\begin{aligned}
& f\left(s;\, \left\lfloor \frac{n}{2} \right\rfloor - 1,\, \left\lceil \frac{n}{2} \right\rceil + 1\right) \\
&= \frac{5832}{(4n+9)^3} - \frac{324(n+2)}{(4n+9)^2} + \frac{18}{4n+9} - n + \frac{18}{4n+9} \left\lfloor \frac{n}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil + 2\right) \\
&\geq \frac{5832}{(4n+9)^3} - \frac{324(n+2)}{(4n+9)^2} + \frac{18}{4n+9} - n + \frac{18(n-1)}{4n+9} + \frac{18(n^2-1)}{4(4n+9)} \\
&= \frac{16n^4 + 360n^3 - 1359n^2 - 10206n - 729}{2(4n+9)^3} > 0,
\end{aligned}$$

which implies inequality (2). □

Let $H_{p,q}$ be the graph as defined before, and let v and w be two vertices in $H_{p,q}$ such that $d_{H_{p,q}}(v) = p$ and $d_{H_{p,q}}(w) = q$. Let G_0 (see, Figure 1) be a graph with n vertices obtained from $H_{p,q}$ ($q \geq p \geq 2$) such that $V(G_0) = V(H_{p,q}) \cup U$ and $E(G_0) = E(H_{p,q}) \cup \{vv_i : v_i \in U\} \cup \{wv_i : v_i \in U\}$, where $U = \{v_1, v_2, \ldots, v_n\} \backslash V(H_{p,q})$ and $|U| = r \geq 1$.

Lemma 2.4. *Let G_0 be a graph as defined above. If $n \geq 6$, then*

$$a(G_0) > a\left(H_{\lfloor \frac{n}{2} \rfloor -1, \lceil \frac{n}{2} \rceil +1}\right).$$

Proof. When $n = 6$, since $n = p + q + r$ and $q \geq p \geq 2$, there are exactly two graphs for G_0. By 'Sage', we can easily check that

$$a(G_0) > 0.63 > 0.49 > a(H_{2,4}) \tag{3}$$

Thus, we may suppose that $n \geq 7$ in the sequel. By Corollary 2.3, it suffices to show that

$$a(G_0) > \frac{18}{4n+9} \,. \tag{4}$$

To prove inequality (4), we first prove the following claim:

Claim 1.

$$n(4n+9)(r+2) > 36(n-p)(n-q) + 18(3n-10)\,.$$

Proof of Claim 1: Since $n = p + q + r$, we have

$$9(n+r)^2 = 9(n-p+n-q)^2 \geq 36(n-p)(n-q).$$

Thus, it suffices to show that

$$n(4n+9)(r+2) > 9(n+r)^2 + 18(3n-10). \tag{5}$$

For this let us consider a function

$$g(x) = -9x^2 + n(4n-9)x - n^2 - 36n + 180 \qquad \text{for } 1 \leq x \leq n-4.$$

Then $g'(x) = n\,(4n-9) - 18x \geq 4n^2 - 27n + 72 > 0$ for $n \geq 7$. Therefore $g(x)$ is an increasing function on $1 \leq x \leq n-4$ and hence $g(x) \geq g(1) = 3(n^2 - 15n + 57) > 0$ for $n \geq 7$. Therefore,

$$n(4n+9)(r+2) - 9(n+r)^2 - 18(3n-10) = -9r^2 + n(4n-9)r - n^2 - 36n + 180 > 0\,,$$

which implies that (5) holds. This completes the proof of Claim 1. $\square$

One can easily see that $a\,(G_0)$ satisfies the following system of equations:

$$\begin{cases} (x-p-r)\,y_1 = -(p-1)\,y_2 - y_3 - ry_5, \\ (x-1)\,y_2 = -y_1, \\ (x-q-r)\,y_3 = -y_1 - (q-1)y_4 - ry_5, \\ (x-1)\,y_4 = -y_3, \\ (x-2)\,y_5 = -y_1 - y_3. \end{cases}$$

Thus, $a\,(G_0)$ satisfies $h(x;\, p,\, q,\, r) = 0$, where

$$\begin{aligned} h(x;\, p,\, q,\, r) = x^4 &- (p+q+2r+4)\,x^3 + ((q+r)(p+r) + 3(p+q) + 6(r+1))\,x^2 \\ &- (2(q+r)(p+r) + 3(p+q) + 2(3r+2))\,x + (r+2)(p+q+r). \end{aligned}$$

Let $s = \dfrac{18}{4n+9}$. Since $n = p + q + r$ and $q \geq p \geq 2$, we have

$$\begin{aligned} &s^2\left(s^2 - (n+r+4)\,s + (p+r)(q+r) + 6(r+1) + 3(p+q)\right) - s(3r+14) \\ \geq &s^2\left(s^2 - (n+r+4)\,s + 4 + (n-r)r + r^2 + 6(r+1) + 3(n-r)\right) - s(3r+14) \\ = &\frac{18}{(4n+9)^4}\left((96n^3 + 864n^2 + 1134n - 729)r - 32n^3 - 576n^2 - 4374n - 1458\right) \\ \geq &\frac{18(64n^3 + 288n^2 - 3240n - 2187)}{(4n+9)^4} > 0. \end{aligned}$$

Combining this with $n = p + q + r$, we have

$$\begin{aligned} h(s; p, q, r) &= s^2 \left(s^2 - (n + r + 4)s + (p + r)(q + r) + 6(r + 1) + 3(p + q)\right) \\ &\quad - (2(q + r)(p + r) + 3(p + q) + 2(3r + 2))s + n(r + 2) \\ &\geq s(3r + 14) + n(r + 2) - s(2(n - p)(n - q) + 3n + 3r + 4) \\ &= n(r + 2) - 2s(n - p)(n - q) - s(3n - 10). \end{aligned}$$

Thus, by **Claim 1**, it follows that $h(s; p, q, r) > 0$.

Moreover, since $h(1; p, q, r) = -(p-1)(q-1) < 0$, $h(2; p, q, r) = (n-2)r > 0$, $h(q+r+1; p, q, r) = -(q-1)(p-1) < 0$ and $h(n; p, q, r) = (q - 1)(p - 1)n(n - 2) > 0$, we can conclude that inequality (4) holds. □

Theorem 2.5. *If $G \ncong H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is a connected graph with $n \geq 6$ vertices and independence number $\alpha(G) = 2$, then*

$$a(G) \geq a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right) > a\left(H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) \quad (6)$$

with equality holding if and only if $G \cong H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$.

Proof. By Lemma 2.2, we have $a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right) > a\left(H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right)$. Thus, it suffices to show that $a(G) \geq a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right)$ with equality holding if and only if $G \cong H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$.

Let $d(G)$ be the diameter of graph G. For $d(G) \geq 4$, then one can easily see that $\alpha(G) \geq 3$, a contradiction. For $d(G) = 1$ and hence $G \cong K_n$. By Corollary 2.3 and inequality (3), $a(G) = n > a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right)$ and inequality (6) holds. For $d(G) = 2$, by Lemma 2.1, Corollary 2.3 and inequality (3), $a(G) \geq 1 > a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right)$ and inequality (6) holds. Otherwise, $d(G) = 3$.

Since $d(G) = 3$, we must have $\{u, v, w, z\} \subseteq V(G)$ such that $\{uv, vw, wz\} \subseteq E(G)$ and the distance between u and z, denoted by $d(u, z)$, is equal to three. Combining this with $\alpha(G) = 2$, we must have $H_{p,q} \subseteq G$ ($q \geq p \geq 2$, $p + q \leq n$), $d_{H_{p,q}}(v) = p$ and $d_{H_{p,q}}(w) = q$. Denote by $W = N_G(v) \cap N_G(w)$ and $U = N_G(v) \cup N_G(w)$. We now divided the proof into the following two cases:

Case (i) : $|U| < n$.

Then, $|V(G) \backslash U| \geq 1$. Since $d(u, z) = 3$, we have $\{u, z\} \cap W = \emptyset$. If there exists some vertex $u_1 \in V(G) \backslash U$ such that $uu_1 \notin E(G)$, then $\{u, w, u_1\}$ is an independent set of G, a contradiction. Thus, u is adjacent to every vertex of $V(G) \backslash U$. Similarly, z is adjacent to every vertex of $V(G) \backslash U$. In this case, $d(u, z) = 2$, a contradiction.

Case (ii) : $|U| = n$.

We first suppose that $|W| = 0$. Then, $H_{p,q} \subseteq G$, where $q \geq p \geq 2$, and $p + q = n$. If $G \cong H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$, then the equality holds in (6). Otherwise, $G \ncong H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$. Then by Lemma 2.2 and interlacing theorem,

$$a(G) \geq a(H_{p,q}) > a(H_{p+1,q-1}) > \cdots > a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right) > a\left(H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right).$$

Now, we consider the case of $|W| \neq 0$. In this case, we have $G_0 \subseteq G$, where $2 \leq p \leq q$, and $p + q < n$. By interlacing theorem, we have $a(G) \geq a(G_0)$. By Lemma 2.4, we can conclude that

$$a(G) \geq a(G_0) > a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right).$$

This completes the proof of this result. □

We now give the main result of this paper.

Theorem 2.6. *If $G \ncong H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is a connected graph with $n \geq 6$ vertices, independence number $\alpha(G)$ and algebraic connectivity $a(G)$, then*

$$a(G) + \alpha(G) \geq a\left(H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}\right) + 2 > a\left(H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) + 2 \tag{7}$$

with equality holding if and only if $G \cong H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$.

Proof. For $\alpha(G) = 1$, we have $G \cong K_n$ and hence the inequality (7) is strict by Corollary 2.3, Lemma 2.2 and inequality (3). For $\alpha(G) \geq 3$, one can easily see that the inequality (7) is also strict by Corollary 2.3, Lemma 2.2 and inequality (3). Otherwise, $\alpha(G) = 2$. Now, the result follows from Theorem 2.5. □

Remark 2.7. *Let $s = \frac{18}{4n+9}$. From the proof of Corollary 2.3 and Theorem 2.6, if $f(s; p, n-p) = s^3 - (n+2)s^2 + (p(n-p) + n + 2)s - n > 0$, then $H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}, H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}, \ldots, H_{\lfloor \frac{n}{2} \rfloor - p, \lceil \frac{n}{2} \rceil + p}$ have the first to $(p+1)$-th minimum values for addition of algebraic connectivity and independence number among connected graphs with n vertices, where $1 \leq p \leq \lfloor \frac{n}{2} \rfloor - 2$. For instance, when $n \geq 11$, since*

$$f\left(s; \left\lfloor \frac{n}{2} \right\rfloor - 2, \left\lceil \frac{n}{2} \right\rceil + 2\right) \geq \frac{16n^4 + 360n^3 - 3663n^2 - 20574n - 12393}{2(4n+9)^3} > 0,$$

we can conclude that $H_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, $H_{\lfloor \frac{n}{2} \rfloor - 1, \lceil \frac{n}{2} \rceil + 1}$, $H_{\lfloor \frac{n}{2} \rfloor - 2, \lceil \frac{n}{2} \rceil + 2}$ have the first to third minimum values for addition of algebraic connectivity and independence number among connected graphs with $n \geq 11$ vertices.

3. Lower Bound for Algebraic Connectivity and Radius of Connected Graphs

In this section, we shall give a lower bound to the addition of algebraic connectivity and radius of connected graphs. The following result will play an important role in our proof.

Lemma 3.1. [2] *Let $G = (V, E)$ be a graph with a vertex subset $V' = \{v_1, v_2, \ldots, v_k\}$ having the same set of neighbors $\{v_{k+1}, v_{k+2}, \ldots, v_{k+N}\}$, where $V(G) = \{v_1, \ldots, v_k, \ldots, v_{k+N}, \ldots, v_n\}$. Also let $E^+ = E \cup E'$, where $E' \subseteq V' \times V'$. If $G' = (V', E')$ has eigenvalues $a_1 \geq a_2 \geq \cdots \geq a_k = 0$, then the eigenvalues of $L(G^+)$, where $G^+ = (V, E^+)$ are as follows: those eigenvalues of the graph $G = (V, E)$ which are equal to N ($k-1$ in number) are incremented by a_i, $i = 1, 2, \ldots, k-1$ and the remaining eigenvalues are the same.*

The join of two graphs H and G, denoted by $H \vee G$, is a graph obtained from H and G by joining each vertex of H to all vertices of G. We are now ready to give a lower bound to the addition of algebraic connectivity and radius of connected graphs and characterize the corresponding extremal graphs.

Theorem 3.2. *If G is a connected graph with $n \geq 3$ vertices, radius $r(G)$ and algebraic connectivity $a(G)$, then*

$$a(G) + r(G) \geq 2 \tag{8}$$

with equality if and only if G is isomorphic to $K_1 \vee (H_1 \cup H_2)$, where H_1 and H_2 are graphs with n_1 and n_2 vertices, respectively, such that $n_1 + n_2 = n - 1$.

Proof. Since G is connected and since $r(G)$ is the radius of G, we have $a(G) > 0$ and $r(G) \geq 1$. If $r(G) \geq 2$, then the inequality in (8) is strict. Otherwise, $r(G) = 1$. Then there exists a vertex v in G being adjacent to all other vertices and hence $d_G(v) = n - 1$. Thus we have $G \cong K_1 \vee G_1$, where G_1 is a graph of order $n - 1$. By Lemma 3.1, we have

$$\mu_1(G) = n,\ \mu_{i+1}(G) = \mu_i(G_1) + 1,\ i = 1, 2, \ldots, n-2,\ \mu_n(G) = \mu_{n-1}(G_1) = 0.$$

First we assume that G_1 is connected. Then $\mu_{n-2}(G_1) > 0$, that is, $a(G) = \mu_{n-1}(G) > 1$ and hence the inequality in (8) is strict. Next we assume that G_1 is disconnected. Thus we have $G_1 \cong H_1 \cup H_2$, where H_1 and H_2 are graphs of order n_1 and n_2, respectively, such that $n_1 + n_2 = n - 1$. In this case $\mu_{n-2}(G_1) = 0$, that is, $\mu_{n-1}(G) = 1$ and hence the equality holds in (8). This completes the proof of this theorem. □

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