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A Synthetic Algorithm for Families of Demicontractive and Nonexpansive Mappings and Equilibrium Problems

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Abstract. We study the rate of convergence of a new synthetic algorithm for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a pair of nonexpansive mappings and two finite families of demicontractive mappings. We then provide some numerical examples to illustrate our main result and the proposed algorithm.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$, and let Υ be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem for $\Upsilon : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$\Upsilon(x, y) \geq 0, \ \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(\Upsilon)$. In 2005, Combettes and Hirstoaga [1] introduced an iterative scheme for finding the best approximation to the initial data when $EP(\Upsilon)$ is nonempty, and proved a strong convergence theorem. Let $A : C \to \mathcal{H}$ be a nonlinear mapping. The classical variational inequality which is denoted by $VI(A, C)$ is to find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C.$$

Throughout this article, for a mapping $T : C \to C$ we write

$$Fix(T) = \{x \in C : x = Tx\}$$

to denote the fixed points of T.

Definition 1.1. *A mapping $T : C \to C$ is said to be quasi-nonexpansive if*

$$\|T(x) - x^*\| \leq \|x - x^*\|, \ \forall x \in C, \ x^* \in Fix(T).$$

If the strict inequality holds, then T is called strictly quasi-nonexpansive.

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Definition 1.2. *([2, 3]) A mapping $T : C \to C$ is said to be nonspreading if*

$$2\|T(x) - T(y)\|^2 \leq \|T(x) - y\|^2 + 2\langle x - T(x), y - T(y)\rangle, \ \forall x, y \in C.$$

In [4], Lemoto and Takahashi introduced an equivalence relation in order that a mapping $T : C \to C$ to be nonspreading:

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + 2\langle x - T(x), y - T(y)\rangle, \ \forall x, y \in C.$$

Definition 1.3. *([5, 6]) A mapping $T : C \to C$ is said to be demicontractive (or k-demicontractive) if there exists $k \in [0, 1)$ such that*

$$\|T(x) - x^*\|^2 \leq \|x - x^*\|^2 + k\|T(x) - x\|^2, \ \forall x \in C, \ x^* \in Fix(T).$$

We note that the class of demicontractive mappings properly includes the class of quasi-nonexpansive mappings.

Definition 1.4. *([7]) A mapping T with domain $D(T)$ and range $R(T)$ in $\mathcal{H}$ is called a k-strictly pseudo-contractive mapping of Browder-Petryshyn type, if for all $x, y \in D(T)$ there exists $k \in [0, 1)$ such that*

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(x - T(x)) - (y - T(y))\|^2, \ \forall x, y \in D(T).$$

If this inequality holds for $k = 1$ then T is simply called pseudo-contractive. Note that the class of strictly pseudo-contractive mappings includes the class of nonexpansive mappings as a subclass; it suffices to put $k = 0$. Recently, Osilike and Isioguge in [8] introduced a new class of mappings in a Hilbert space which is called the class of k-strictly pseudo-nonspreading mappings:

Definition 1.5. *([8]) A mapping $T : C \to C$ is said to be k-strictly pseudo-nonspreading, if there exists $k \in [0, 1)$ such that for all $x, y \in C$, the following inequality holds:*

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(x - T(x)) - (y - T(y))\|^2 + 2\langle x - T(x), y - T(y)\rangle.$$

Kohsaka and Takahashi in [2] introduced a nonlinear mapping called nonspreading mapping. This class was studied in Banach spaces, as well as in Hilbert spaces: see [4, 9, 10].

As for nonexpansive mappings, weak convergence theorems for two nonexpansive mappings Q and R (with Lipschitz constants k_Q and k_R respectively equal to 1) of C to itself were discussed by Takahashi and Tamura in [11]:

$$\begin{cases} x_1 = x \in C, & \textit{chosen arbitrary}, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n R\{\beta_n Q(x_n) + (1 - \beta_n)x_n\}, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In this paper we want to modify this algorithm to incorporate the demi-contractive mappings. For this reason we begin with the following definition.

Definition 1.6. *An operator A is said to be a strongly positive bounded linear operator on a real Hilbert space $\mathcal{H}$, if there exists a constant $\varpi > 0$ such that*

$$\langle Ax, x\rangle \geq \varpi\|x\|^2, \ \forall x \in \mathcal{H}.$$

One of the most important issues of equilibrium and optimization problems is the problem of minimizing a quadratic function over the set of fixed points of a nonexpansive mapping on a real Hilbert space $\mathcal{H}$:

$$\min_{x \in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle. \tag{3}$$

Moudafi in [12] introduced a viscosity approximation method for finding a fixed point of nonexpansive mappings. Later on, inspired by [12], Xu in [13] and Marino and Xu in [14] introduced the following iterative scheme:

$$x_{n+1} = a_n \gamma f(x_n) + (I - a_n A)T(x_n), \ \forall n \geq 0, \tag{4}$$

where f is a contraction and T is a nonexpansive mapping. They proved that under some appropriate conditions on the parameters, the sequence given by (4) converges strongly to the unique solution of the following $VI(A, C)$ problem

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \ \forall x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in Fix(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (that is $h^{'}(x) = \gamma f(x)$ for all $x \in \mathcal{H}$).

Takahashi and Takahashi in [15] introduced a viscosity approximation method for finding a common element of $EP(\Upsilon)$ and $Fix(T)$. Afterward, Plibtieng and Punpaeng in [16] by combining the schemes (4) in [13] and using the algorithm in [15] introduced the following algorithm

$$\begin{cases} \Upsilon(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = a_n \gamma f(x_n) + (I - a_n A)T(x_n), \ \forall n \geq 0. \end{cases} \tag{5}$$

They proved that the sequences $\{x_n\}$ and $\{u_n\}$ in this algorithm converge strongly to the unique solution z of the $VI(A, C)$:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \ \forall x \in Fix(T) \cap EP(\Upsilon),$$

which is the optimality condition for the minimization problem

$$\min_{x \in Fix(T) \cap EP(\Upsilon)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf.

In [8] Osilike and Isiogugu proved a strong convergence theorem somewhat related to a Halpern-type iteration algorithm for a k-strictly pseudo-nonspreading mapping in Hilbert spaces.

Theorem 1.7. *[8] Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $T : C \to C$ be a k-strictly pseudo-nonspreading mapping with a nonempty fixed point set $Fix(T)$. Let $\zeta \in [k, 1)$ and $\{\alpha_n\}$ be a real sequence in $[0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $u \in C$ and $\{z_n\}$ and $\{x_n\}$ be sequences in C generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = \sum_{k=1}^{n-1} T_\zeta^k x_n, & \forall n \geq 0, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) z_n, & \forall n \geq 0, \end{cases} \tag{6}$$

where $T_\zeta = \zeta I + (1 - \zeta)T$. Then $\{z_n\}$, $\{x_n\}$ converge strongly to $P_{Fix(T)}(u)$.

In this paper, we improve this result by combining the algorithms (2) and (5) with a particular combination of two finite families of demicontractive mappings and a pair of nonexpansive mappings, and obtain a new synthetic algorithm. Here, instead of using a Halpern-type algorithm, we shall develop a viscosity algorithm; the advantage of the viscosity iterative scheme to the Halpern-type scheme is its higher rate of convergence. There are important applications of these type of algorithms in physical sciences, optimization, and economics. Moreover, it is known that fuzzy game problems are reduced to finding a solution of the equilibrium problem, see for instance [17]. For more information on the development of the theory, we refer the reader to [18–23].

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. For a sequence $\{x_n\}$ in $\mathcal{H}$, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

Definition 2.1. *Let C be a nonempty closed convex subset of $\mathcal{H}$, and $T : C \to C$ be a mapping, then $I - T$ is said to be demiclosed at zero if for any sequence $\{x_n\}$ in C, the conditions $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$, imply $x = T(x)$.*

In a Hilbert space, it is known that:

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \tag{7}$$

for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$.

Zegeye and Shahzad in [20] generalized the equation (7) and obtained the following result.

Lemma 2.2. *[20]. For each $x_1, x_2, \cdots, x_m \in \mathcal{H}$ and $\alpha_1, \alpha_2, \cdots, \alpha_m \in [0,1]$ with $\sum_{i=1}^m \alpha_i = 1$, we have*

$$\|\alpha_1 x_1 + \cdots + \alpha_m x_m\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{1\le i\le j\le m} \alpha_i\alpha_j \|x_i - x_j\|^2. \tag{8}$$

Lemma 2.3. *Let $\mathcal{H}$ be a real Hilbert space. Then we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x, y \in \mathcal{H}.$$

Lemma 2.4. *[14]. Assume that A is a strongly positive self-adjoint bounded linear operator on a Hilbert space $\mathcal{H}$ with coefficient $\overline{\gamma}$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 2.5. *[13]. Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \ \forall n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

1. $\sum_{n=1}^\infty \gamma_n = \infty$,
2. $\limsup_{n\to\infty} \gamma_n \le 0$ *or* $\sum_{n=1}^\infty |\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. *[24]. Let $\{u_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \le u_{n_{i+1}}$ for all $i \ge 0$. For every $n \ge n_0$, define an integer sequence $\{\tau(n)\}$ as $\tau(n) = \max\{k \le n : u_{n_i} < u_{n_{i+1}}\}$. Then $\tau(n) \to \infty$ as $n \to \infty$, moreover for all $n \ge n_0$,*

$$\max\{u_{\tau(n)}, u_n\} \le u_{\tau(n)+1}.$$

For solving the equilibrium problem, we shall make the following assumptions on the bifunction $\Upsilon : C \times C \to \mathbb{R}$:

(C1) $\Upsilon(x,x) = 0$ for all $x \in C$,

(C2) Υ is monotone, that is

$$\Upsilon(x,y) + \Upsilon(y,x) \le 0, \quad \forall x, y \in C,$$

(C3) Υ is upper-hemicontinuous, that is

$$\limsup_{h\to 0^+} \Upsilon(hz + (1-h)x, y) \le \Upsilon(x,y), \quad \forall x, y, z \in C,$$

(C4) $\Upsilon(x, 0)$ is convex and lower semicontinuous for each $x \in C$.

Lemma 2.7. *[25]. Let C be a nonempty closed convex subset of $\mathcal{H}$, and let Υ be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying $(C1)-(C4)$. Let $r > 0$ and $x \in \mathcal{H}$. Then, there exists $z \in C$ such that*

$$\Upsilon(u_n, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8. *[1]. Assum that $\Upsilon : C\times C \to \mathbb{R}$ satisfies $(C1)-(C4)$. For $r > 0$ and $x \in \mathcal{H}$ define a set-valued mapping $T_r : \mathcal{H} \rightrightarrows C$ in the following way:*

$$T_r(x) = \{z \in C : \Upsilon(z, y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then we have

1. *T_r is single valued.*
2. *T_r is firmly nonexpansive, that is for any $x, y, z \in \mathcal{H}$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle,$$

3. *$Fix(T_r) = EP(\Upsilon)$,*
4. *$EP(\Upsilon)$ is closed and convex.*

Lemma 2.9. *[8]. Let C be a nonempty closed convex subset of $\mathcal{H}$, and let $T : C \to C$ be a k-strictly pseudo-nonspreading mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.10. *[26]. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $T : C \to C$ be a k-strictly pseudo-contractive mapping. Then $I - T$ is demiclosed at zero.*

Takahashi in [27] proved that if $T : C \to C$ is a nonexpansive mapping, then $Fix(T)$ is closed and convex. In the following, we prove that this claim is true for demi-contractive mappings.

Lemma 2.11. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $T : C \to C$ be a demicontractive mapping. If $Fix(T) \neq \emptyset$, then $Fix(T)$ is closed and convex.*

Proof. Let $\{x_n\} \subset Fix(T)$ be a sequence which converges to $x \in C$. We show that $x \in Fix(T)$. Observe that

$$\begin{aligned} \|T(x) - x\| &= \|T(x) - x_n + x_n - x\| \\ &\leq \|T(x) - T(x_n)\| + \|x_n - x\|. \end{aligned} \tag{9}$$

Since T is demicontractive and $\{x_n\} \subset Fix(T)$, there exists $k \in [0, 1)$ such that

$$\begin{aligned} \|T(x) - T(x_n)\|^2 &= \|T(x) - x_n\|^2 \\ &\leq \|x_n - x\|^2 + k\|x - T(x)\|^2 \\ &\leq (\|x_n - x\| + \sqrt{k}\|x - T(x)\|)^2. \end{aligned} \tag{10}$$

By using (10) in (9), we obtain

$$0 \leq \|T(x) - x\| \leq \Big[\frac{2}{1 - \sqrt{k}}\Big]\|x_n - x\| \to 0, \quad as \ n \to \infty.$$

Hence $x \in Fix(T)$. To prove the convexity, suppose that $p_1, p_2 \in Fix(T)$ and $\lambda \in [0, 1]$; it is enough to show that $\lambda p_1 + (1 - \lambda)p_2 \in Fix(T)$. Let $z = \lambda p_1 + (1 - \lambda)p_2$, then

$$\begin{aligned} p_1 - z &= (1 - \lambda)(p_1 - p_2), \\ p_2 - z &= \lambda(p_2 - p_1). \end{aligned}$$

Now, by using (7) we conclude that

$$\begin{aligned}\|z - T(z)\|^2 &= \|\lambda(p_1 - T(z)) + (1-\lambda)(p_2 - T(z))\|^2\\ &= \lambda\|p_1 - T(z)\|^2 + (1-\lambda)\|p_2 - T(z)\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2\\ &\le \lambda\Big[\|p_1 - z\|^2 + k\|z - T(z)\|^2\Big] + (1-\lambda)\Big[\|p_2 - z\|^2 + k\|z - T(z)\|^2\Big] - \lambda(1-\lambda)\|p_1 - p_2\|^2\\ &= k\|z - T(z)\|.\end{aligned}$$

Thus $(1-k)\|z - T(z)\|^2 = 0$, or equivalently $z \in Fix(T)$. □

3. Main Results

By using the mappings Q and R, defined above, we begin this section with the following theorem on demicontractive mappings.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $\Upsilon : C \times C \to \mathbb{R}$ be a bifunction satisfying $(\boldsymbol{C1}) - (\boldsymbol{C4})$. Let for $i = 1, 2, \cdots, m$, $T_i : C \longrightarrow C$ be a finite family of κ-demicontractive mappings and $S_i : C \longrightarrow C$ be a finite family of ι-demicontractive mappings such that $I - T_i$ and $I - S_i$ are demiclosed at 0. Assume that*

$$F := \left(\bigcap_{i=1}^{m} Fix(T_i)\right) \cap \left(\bigcap_{i=1}^{m} Fix(S_i)\right) \cap Fix(Q) \cap Fix(R) \cap Ep(\Upsilon) \neq \emptyset.$$

Let f be a contraction of C into itself with constant $b \in (0, 1)$ and A be a strongly positive self-adjoint bounded linear operator on $\mathcal{H}$ with coefficient $\overline{\gamma}$ such that $0 < \gamma < \frac{(1+\overline{\gamma})-k_R}{b}$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases}\Upsilon(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall y \in C,\\ y_n = \alpha_n u_n + \beta_n x_n + \sum_{i=1}^{m}\gamma_{n,i} T_i u_n, &\\ w_n = \Lambda_{n,0} Q(y_n) + \sum_{i=1}^{m}\Lambda_{n,i} S_i(Qy_n), &\\ x_{n+1} = a_n\gamma f(R(w_n)) + (I - a_n A)R(w_n). &\end{cases} \tag{11}$$

Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}, \{\Lambda_{n,i}\}, \{r_n\}$ and $\{a_n\}$ satisfy the following conditions:

1. $\alpha_n + \beta_n + \sum_{i=1}^{m}\gamma_{n,i} = 1$ *and* $\Lambda_{n,0} + \sum_{i=1}^{m}\Lambda_{n,i} = 1$,
2. $\{a_n\} \subset (0, 1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
3. $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n > 0$,
4. $\kappa < \alpha_n < 1$, $\iota < \Lambda_{n,0} < 1$ *and* $\liminf_{n\to\infty}(\Lambda_{n,0} - L)\Lambda_{n,i} > 0$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$ which solves the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \ge 0, \quad \forall x \in F. \tag{12}$$

Proof. Step 1. It is easy to see that P_F has a fixed point:

$$\begin{aligned}\|P_F(I - A + \gamma f)(Rx) - P_F(I - A + \gamma f)(Ry)\| &\le \Big\|(I - A + \gamma f)(Rx) - (I - A + \gamma f)(Ry)\Big\|\\ &\le \|(I - A)(Rx) - (I - A)(Ry)\| + \gamma\|fx - fy\|\\ &\le (k_R - \overline{\gamma})\|x - y\| + \gamma b\|x - y\|\\ &\le (k_R - (\overline{\gamma} - \gamma b))\|x - y\|.\end{aligned}$$

This means that P_F is a contraction of C into itself. Thus there exists a unique element $q \in C$ such that $q = P_F(I - A + \gamma f)q$, or equivalently for all $p \in F$, we have $\langle (I - A + \gamma f)q - p, q - p\rangle \geqslant 0$. Since $\lim_{n\to\infty} a_n = 0$, we may assume that $0 < a_n < \|A\|^{-1}$, for all $n \geqslant 0$. By Lemma (2.4) we have $\|I - a_nA\| \leqslant 1 - a_n\overline{\gamma}$. Now take $p \in F$, since $u_n = T_{r_n}x_n$ and $p = T_{r_n}p$, from Lemma (2.8) for any $n \geqslant 0$ we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leqslant \|x_n - p\|.$$

In the following we show that $\{x_n\}$ is bounded. If we define $\mathbf{k} := \max\{ k_i ;\ 1 \leq i \leq m\}$, then

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n u_n + \beta_n x_n + \sum_{i=1}^{m} \gamma_{n,i}T_i u_n - p\|^2 \\
&\leq \alpha_n\|u_n - p\|^2 + \beta_n\|x_n - p\|^2 + \sum_{i=1}^{m} \gamma_{n,i}\|T_i u_n - p\|^2 - \alpha_n\beta_n\|x_n - u_n\|^2 - \alpha_n\gamma_{n,i}\|u_n - T_i u_n\|^2 \\
&\leq \alpha_n\|u_n - p\|^2 + \beta_n\|x_n - p\|^2 + \sum_{i=1}^{m} \gamma_{n,i}(\|u_n - p\|^2 + k\|u_n - T_i u_n\|^2) \\
&\quad - \alpha_n\beta_n\|x_n - u_n\|^2 - \alpha_n\gamma_{n,i}\|u_n - T_i u_n\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n\beta_n\|x_n - u_n\|^2 - (\alpha_n - \mathbf{k})\sum_{i=1}^{m} \gamma_{n,i}\|u_n - T_i u_n\|^2. \qquad (13)
\end{aligned}$$

By using Lemma (2.2) and a similar argument as before, and using the fact that $k_Q^2 \leq k_Q$, we can write

$$\begin{aligned}
\|w_n - p\|^2 &= \|\Lambda_{n,0}Q(y_n) + \sum_{i=1}^{m} \Lambda_{n,i}S_i(Qy_n) - p\|^2 \\
&\leq \Lambda_{n,\circ}\|Q(y_n) - p\|^2 + \sum_{i=1}^{m} \Lambda_{n,i}\|S_i(Qy_n) - p\|^2 - \Lambda_{n,0}\Lambda_{n,i}\|Q(y_n) - S_i(Q(y_n))\|^2 \\
&= \Lambda_{n,0}\|Q(y_n) - Q(p)\|^2 + \sum_{i=1}^{m} \Lambda_{n,i}\|S_i(Qy_n) - S_i p\|^2 - \Lambda_{n,0}\Lambda_{n,i}\|Qy_n - S_i(Qy_n)\|^2 \\
&\leq k_Q\Lambda_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \Lambda_{n,i}(\|Qy_n - p\|^2 + \iota\|Qy_n - S_i(Qy_n)\|^2) - \Lambda_{n,0}\Lambda_{n,i}\|Qy_n - S_i(Qy_n)\|^2 \\
&\leq k_Q\Lambda_{n,0}\|y_n - p\|^2 + \sum_{i=1}^{m} \Lambda_{n,i}(k_Q^2\|y_n - p\|^2 + \iota\|Qy_n - S_i(Qy_n)\|^2) - \Lambda_{n,0}\Lambda_{n,i}\|Qy_n - S_i(Qy_n)\|^2 \\
&= k_Q\|y_n - p\|^2 - (\Lambda_{n,0} - \iota)\sum_{i=1}^{m} \Lambda_{n,i}\|Qy_n - S_i(Qy_n)\|^2 \\
&\leq k_Q\|x_n - p\|^2 - (\Lambda_{n,0} - \iota)\sum_{i=1}^{m} \Lambda_{n,i}\|Qy_n - S_i(Qy_n)\|^2 \\
&\quad - \alpha_n\beta_n\|x_n - u_n\|^2 - (\alpha_n - \mathbf{k})\sum_{i=1}^{m} \gamma_{n,i}\|u_n - T_i u_n\|^2. \qquad (14)
\end{aligned}$$

Thus $\|w_n - p\| \leq \|x_n - p\|$. Now, using Lemma (2.4) we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n(\gamma f(R(w_n) - Ap) + (I - a_nA)(R(w_n) - p)\| \\
&= \|a_n(\gamma f(R(w_n) - Ap) + (I - a_nA)(R(w_n) - R(p))\| \\
&\leq a_n\|\gamma f(R(w_n) - Ap\| + \|I - a_nA\|\|R(w_n) - R(p)\| \\
&\leq a_n\gamma\|f(R(w_n) - fp\| + a_n\|\gamma fp - Ap\| + k_R(1 - a_n\overline{\gamma})\|w_n - p\| \\
&= a_n\gamma\|f(R(w_n) - f(R(p))\| + a_n\|\gamma fp - Ap\| + k_R(1 - a_n\overline{\gamma})\|w_n - p\| \\
&\leq a_n\gamma\, b\|Rw_n - Rp\| + a_n\|\gamma fp - Ap\| + k_R(1 - a_n\overline{\gamma})\|w_n - p\|
\end{aligned}$$

$$\leq k_R a_n \gamma\, b\|w_n - p\| + a_n\|\gamma f p - Ap\| + k_R(1 - a_n\overline{\gamma})\|w_n - p\|$$
$$\leq k_R(1 - a_n(\overline{\gamma} - \gamma\, b))\|x_n - p\| + a_n\|\gamma f p - Ap\|.$$

Finally, we use an induction argument on $n \in \mathbb{N}$ to obtain

$$\|x_n - p\| \leq \max\Big\{\|x_0 - p\|, \frac{1}{k_R(\overline{\gamma} - \gamma\, b)}\|\gamma f p - Ap\|\Big\}.$$

This means that $\{x_n\}$ is bounded. It now follows from (13) that $\{y_n\}$ is bounded too. Similarly, it can be shown that $\{u_n\}$, $\{w_n\}$ and $\{f(R(w_n)\}$ are bounded sequences.
*S*tep 2. We show that for $i = 1, 2, \cdots, m$

$$\lim_{n\to\infty} \|u_n - T_i u_n\| = 0, \qquad \lim \|y_n - S_i(Qy_n)\| = 0.$$

By Lemma (2.4) and the inequality (14) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|a_n\gamma f(R(w_n)) + (1 - a_n A)R(w_n) - p\|^2\\
&= \|a_n(\gamma f(R(w_n)) - Ap) + (I - a_n A)(R(w_n) - p)\|^2\\
&= \|a_n(\gamma f(R(w_n)) - Ap) + (I - a_n A)(R(w_n) - R(p))\|^2\\
&\leq a_n^2\|\gamma f(R(w_n)) - Ap\|^2 + (1 - a_n\overline{\gamma})^2\|(R(w_n) - R(p))\|^2\\
&\quad + 2a_n(1 - a_n\overline{\gamma})\|\gamma f(R(w_n)) - Ap\|\|(R(w_n) - R(p))\|\\
&\leq a_n^2\|\gamma f(R(w_n)) - Ap\|^2 + k_R^2(1 - a_n\overline{\gamma})^2\|w_n - p\|^2\\
&\quad + 2k_R a_n(1 - a_n\overline{\gamma})\|\gamma f(R(w_n)) - Ap\|\|w_n - p\|\\
&\leq a_n^2\|\gamma f(R(w_n)) - Ap\|^2 + k_R^2(1 - a_n\overline{\gamma})^2\|x_n - p\|^2\\
&\quad + 2k_R a_n(1 - a_n\overline{\gamma})\|\gamma f(R(w_n)) - Ap\|\|x_n - p\| - (1 - a_n\overline{\gamma})^2\alpha_n\beta_n\|x_n - u_n\|^2\\
&\quad - (1 - a_n\overline{\gamma})^2(\Lambda_{n,0} - \iota)\Lambda_{n,i}\|Qy_n - S_iQy_n\|^2 - (1 - a_n\overline{\gamma})^2(\alpha_n - \mathbf{k})\gamma_{n,i}\|u_n - T_iu_n\|^2.
\end{aligned}$$

The last inequality simplifies to

$$\begin{aligned}
(1 - a_n\overline{\gamma})^2(\Lambda_{n,0} - \iota)\Lambda_{n,i}\|Qy_n - S_iQy_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\
&\quad + 2k_R a_n(1 - a_n\overline{\gamma})\|x_n - p\|\|\gamma f(R(w_n) - Ap\|\\
&\quad + a_n^2\|\gamma f(R(w_n)) - Ap\|^2. \qquad (15)
\end{aligned}$$

*S*tep 3. We prove that $x_n \to q$ as $n \to \infty$. To prove this, we consider two possible cases.
Case 1. Assume that $\{\|x_n - q\|\}_{n\geq 1}$ is a monotone sequence. In other words, for N_0 large enough, $\{\|x_n - q\|\}_{n\geq N_0}$ is either nondecreasing or non-increasing. Since $\|x_n - q\|$ is bounded, we conclude that $\|x_n - q\|$ is convergent. Since $\lim_{n\to\infty} a_n = 0$ and $\{f(Rw_n)\}$ and $\{x_n\}$ are bounded, we have

$$\lim_{n\to\infty}(1 - a_n\overline{\gamma})^2(\Lambda_{n,0} - \iota)\Lambda_{n,i}\|Qy_n - S_iQy_n\|^2 = 0.$$

From $\lim_{n\to\infty} a_n = 0$, we may assume that for some $\varsigma \in (0, 1)$, $0 < \varsigma < (1 - a_n\overline{\gamma})^2$. By assumption that $\liminf_{n\to\infty}(\Lambda_{n,0} - \iota)\Lambda_{n,i} > 0$, we obtain

$$\lim_{n\to\infty} \|Qy_n - S_iQy_n\| = 0, \ \ i = 1, 2, \cdots, m. \qquad (16)$$

With a similar reasoning as in the inequality (15), we conclude that

$$\begin{aligned}
(1 - a_n\overline{\gamma})^2(\alpha_n - \mathbf{k})\gamma_{n,i}\|u_n - T_iu_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\
&\quad + 2k_R a_n(1 - a_n\overline{\gamma})\|x_n - p\|\|\gamma f(R(w_n) - Ap\|\\
&\quad + a_n^2\|\gamma f(R(w_n)) - Ap\|^2, \qquad (17)
\end{aligned}$$

and that

$$\lim_{n\to\infty} \|u_n - T_i u_n\| = 0, \quad i = 1, 2, \cdots, m. \tag{18}$$

Also, we obtain

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{19}$$

By using the second equation in (11), we obtain

$$\|y_n - u_n\| \leqslant \beta_n \|x_n - u_n\| + \sum_{i=1}^{m} \gamma_{n,i} \|T_i u_n - u_n\|,$$

this, together with (18) and (19) yields

$$\lim_{n\to\infty} \|y_n - u_n\| = 0.$$

In the following, we show that

$$\limsup_{n\to\infty} \langle (A - \gamma f)q, q - x_n \rangle \leq 0,$$

where $q = P_F(I - A + \gamma f)q$ is the unique solution of the variational inequality (12). We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle (A - \gamma f)q, q - x_{n_i} \rangle = \limsup \langle (A - \gamma f)q, q - x_n \rangle.$$

In the previous step, we observed that the sequence x_{n_i} is bounded, therefore there exists a subsequence $x_{n_{i_j}}$ of x_{n_i} which converges weakly to ν. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup \nu$. We have already proved that $\lim_{n\to\infty} \|u_n - x_n\| = 0$, therefore $u_{n_i} \rightharpoonup \nu$. To complete the proof, we need to show that $\nu \in F$. But, as a byproduct of Takahashi and Tamura's argument in [11], we know that $\nu \in Fix(Q) \cap Fix(R)$. Now, we show that $\nu \in Ep(\Upsilon)$. Since $u_n = T_{r_n} x_n$, we have

$$\Upsilon(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

Since Υ is monotone, we can write

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \Upsilon(y, u_n),$$

therefore

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq \Upsilon(y, u_{n_i}).$$

But $u_{n_i} - x_{n_i} \to 0$ and $u_{n_i} \rightharpoonup \nu$, however, by using (**C4**) for all $y \in C$, we conclude that $\Upsilon(y, \nu) \leq 0$. Now, for $t \in (0, 1]$ and $y \in C$, set $z_t = ty + (1 - t)\nu$, since $y, \nu \in C$ and C is convex, we have $z_t \in C$ and hence $\Upsilon(z_t, \nu) \leq 0$. So from (**C1**) and (**C4**) we have

$$0 = \Upsilon(z_t, z_t) \leq t\Upsilon(z_t, y) + (1 - t)\Upsilon(z_t, \nu) \leq t\Upsilon(z_t, y),$$

since $t \in (0, 1]$. Thus $\Upsilon(z_t, y) \geq 0$, and from (**C3**) we conclude that

$$0 \leq \Upsilon(y, \nu), \qquad \forall y \in C.$$

This means that $\nu \in Ep(\Upsilon)$. So far, we have proved that

$$\nu \in Fix(Q) \cap Fix(R) \cap Ep(\Upsilon).$$

It is easy to show that $\nu \in (\bigcap_{i=1}^m Fix(T_i))$ and $\nu \in (\bigcap_{i=1}^m Fix(S_i))$. Since Q is nonexpansive and $\lim_{n\to\infty} \|y_n - u_n\| = 0$, we have

$$\lim_{n\to\infty} \|Q(y_n) - Q(u_n)\| = 0.$$

Due to (16) and our assumption that $I - S_i$ is demiclosed at 0, it follows that $\nu \in (\bigcap_{i=1}^m Fix(S_i))$. In the same way, we can prove that $\nu \in (\bigcap_{i=1}^m Fix(T_i))$, hence $\nu \in F$. Since $q = P_F(I - A + \gamma f)q$ and $\nu \in F$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (A-\gamma f)q, q - x_n\rangle &= \lim_{i\to\infty}\langle (A-\gamma f)q, q - x_{n_i}\rangle \\ &= \langle (A-\gamma f)q, q-\nu\rangle \le 0.\end{aligned}$$

Now by using the algorithm used in the definition of x_{n+1} and Lemma (2.3), we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \|(I-a_nA)(R(w_n)-q)\|^2 + 2a_n\langle \gamma f(R(w_n)) - Aq, x_{n+1}-q\rangle \\ &\le (1-a_n\overline{\gamma})^2\|R(w_n)-q\|^2 + 2a_n\gamma\langle f(R(w_n)) - fq, x_{n+1}-q\rangle + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &= (1-a_n\overline{\gamma})^2\|R(w_n)-R(q)\|^2 + 2a_n\gamma\langle f(R(w_n)) - f(Rq), x_{n+1}-q\rangle + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &\le k_R^2(1-a_n\overline{\gamma})^2\|w_n-q\|^2 + 2k_Ra_nb\gamma\|w_n-q\|\|x_{n+1}-q\| + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &\le k_R^2(1-a_n\overline{\gamma})\|x_n-q\|^2 + 2k_Ra_nb\gamma\|x_n-q\|\|x_{n+1}-q\| + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &\le k_R^2(1-a_n\overline{\gamma})^2\|x_n-q\|^2 + k_Ra_nb\gamma(\|x_n-q\|^2 + \|x_{n+1}-q\|^2) + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &= \left(k_R^2(1-a_n\overline{\gamma})^2 + a_nb\gamma k_R\right)\|x_n-q\|^2 + k_Ra_nb\gamma\|x_{n+1}-q\|^2 + 2a_n\langle \gamma fq - Aq, x_{n+1}-q\rangle.\end{aligned}$$

Arranging the above inequality, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 &\le \frac{k_R^2(1-a_n\overline{\gamma})^2 + a_nb\gamma k_R}{1-k_Ra_nb\gamma}\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &= \left(\frac{k_R^2 - 2k_R^2a_n\overline{\gamma} + k_R^2a_n^2\overline{\gamma}^2 + a_nb\gamma k_R}{1-k_Ra_nb\gamma}\right)\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &= \left(\frac{k_R^2 - 2k_R^2a_n\overline{\gamma} + a_nb\gamma k_R}{1-k_Ra_nb\gamma}\right)\|x_n-q\|^2 + \frac{k_R^2a_n^2\overline{\gamma}^2}{1-k_Ra_nb\gamma}\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &\leqslant \left(\frac{k_R - 2k_R^2a_n\overline{\gamma} + a_nb\gamma k_R + a_nb\gamma k_R^2 - a_nb\gamma k_R^2}{1-k_Ra_nb\gamma}\right)\|x_n-q\|^2 \\ &\quad + \frac{k_R^2a_n^2\overline{\gamma}^2}{1-k_Ra_nb\gamma}\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &= \left(\frac{k_R(1-k_Ra_nb\gamma) - 2k_R^2a_n\overline{\gamma} + a_nb\gamma k_R + a_nb\gamma k_R^2}{(1-k_Ra_nb\gamma)}\right)\|x_n-q\|^2 \\ &\quad + \frac{k_R^2a_n^2\overline{\gamma}^2}{1-k_Ra_nb\gamma}\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle \\ &\leqslant \left(k_R + \frac{-2k_R^2a_n\overline{\gamma} + a_nb\gamma k_R + a_nb\gamma k_R^2}{(1-k_Ra_nb\gamma)}\right)\|x_n-q\|^2 \\ &\quad + \frac{k_R^2a_n^2\overline{\gamma}^2}{1-k_Ra_nb\gamma}\|x_n-q\|^2 + \frac{2a_n}{1-k_Ra_nb\gamma}\langle \gamma fq - Aq, x_{n+1}-q\rangle\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(k_R + \frac{-2k_R^2 a_n\overline{\gamma} + a_n b\gamma k_R + a_n b\gamma k_R}{(1-k_R a_n b\gamma)}\right)\|x_n - q\|^2 \\
&\quad + \frac{k_R^2 a_n^2 \overline{\gamma}^2}{1-k_R a_n b\gamma}\|x_n - q\|^2 + \frac{2a_n}{1-k_R a_n b\gamma}\langle \gamma f q - Aq, x_{n+1} - q\rangle \\
&= \left(k_R - \frac{2k_R^2 a_n\overline{\gamma} - 2k_R a_n b\gamma}{(1-k_R a_n b\gamma)}\right)\|x_n - q\|^2 + \frac{k_R^2 a_n^2 \overline{\gamma}^2}{1-k_R a_n b\gamma}\|x_n - q\|^2 + \frac{2a_n}{1-k_R a_n b\gamma}\langle \gamma f q - Aq, x_{n+1} - q\rangle \\
&\leqslant \left(1 - \frac{2k_R a_n(k_R\overline{\gamma} - b\gamma)}{1-k_R a_n b\gamma}\right)\|x_n - q\|^2 + \frac{k_R^2 a_n^2 \overline{\gamma}^2}{1-k_R a_n b\gamma}\|x_n - q\|^2 + \frac{2a_n}{1-k_R a_n b\gamma}\langle \gamma f q - Aq, x_{n+1} - q\rangle \\
&\leqslant \frac{2k_R a_n(k_R\overline{\gamma} - b\gamma)}{1-k_R a_n b\gamma}\left[\frac{k_R a_n \overline{\gamma}^2 M}{2(k_R\overline{\gamma} - b\gamma)} + \frac{1}{k_R(k_R\overline{\gamma} - b\gamma)}\langle \gamma f q - Aq, x_{n+1} - q\rangle\right] + \left(1 - \frac{2k_R a_n(k_R\overline{\gamma} - b\gamma)}{1-k_R a_n b\gamma}\right)\|x_n - q\|^2 \\
&= \sigma_n\eta_n + (1-\sigma_n)\|x_n - q\|^2,
\end{aligned}
$$

where $M = \sup\left\{\|x_n - q\|^2 : \quad n \geq 0\right\}$, $\sigma_n = \dfrac{2k_R a_n(k_R\overline{\gamma} - b\gamma)}{1-k_R a_n b\gamma}$ and

$$
\eta_n = \frac{k_R a_n \overline{\gamma}^2 M}{2(k_R\overline{\gamma} - b\gamma)} + \frac{1}{k_R(k_R\overline{\gamma} - b\gamma)}\langle \gamma f q - Aq, x_{n+1} - q\rangle.
$$

It is easy to sea that $\sigma_n \to 0$, $\sum_{n\geq 1}\sigma_n = \infty$ and $\limsup_{n\to\infty}\eta_n \leq 0$. Hence, by Lemma (2.5) the sequence $\{x_n\}$ converges strongly to q.

Case 2. If $\{\|x_n - q\|\}_{n\geq 1}$ is not a monotone sequence, then we can define an integer sequence $\{\varrho(n)\}$ for all $n \geq n_0$ (for some n_0 large enough):

$$
\varrho(n) := \max\{k \in \mathbb{N};\ k \leq n :\ \|x_k - q\| < \|x_{k+1} - q\|\}.
$$

Clearly, ϱ is a nondecreasing sequence such that $\varrho(n) \to 0$, as $n \to \infty$ and for all $n \geq n_0$, we can write

$$
\|x_{\varrho(n)} - q\| < \|x_{\varrho(n)+1} - q\|.
$$

From (17) we conclude that

$$
\lim_{n\to\infty}\|u_{\varrho(n)} - T_i u_{\varrho(n)}\| = 0, \quad \lim_{n\to\infty}\|u_{\varrho(n)} - x_{\varrho(n)}\| = 0.
$$

Using a similar argument as in Case 1, we have

$$
\|x_{\varrho(n)+1} - q\|^2 \leq (1-\sigma_{\varrho(n)})\|x_{\varrho(n)} - q\|^2 + \sigma_{\varrho(n)}\eta_{\varrho(n)},
$$

where $\sigma_{\varrho(n)} \to 0$, $\sum_{n\geq 1}\sigma_{\varrho(n)} = \infty$ and $\limsup_{n\to\infty}\eta_{\varrho(n)} \leq 0$. Hence, by Lemma (2.5), we obtain

$$
\lim_{n\to\infty}\|x_{\varrho(n)+1} - q\| = 0, \quad \lim_{n\to\infty}\|x_{\varrho(n)} - q\| = 0.
$$

Therefore, by Lemma (2.6) we conclude that

$$
0 \leq \|x_n - q\| \leq \max\left\{\|x_{\varrho(n)} - q\|,\ \|x_n - q\|; \quad n \geq 0\right\} \leq \|x_{\varrho(n)+1} - q\|.
$$

So, the sequence $\{x_n\}$ converges strongly to q. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $\Upsilon : C \times C \to \mathbb{R}$ be a bifunction satisfying (**C1**) – (**C4**). Let for $i = 1, 2, \cdots, m$, $T_i : C \longrightarrow C$ be a finite family of κ-stricly pseudo-nonspreading mappings and $S_i : C \longrightarrow C$ be a finite family of ι-stricly pseudo-nonspreading mappings such that $I - T_i$ and $I - S_i$ are demiclosed at 0. Assume that*

$$
F := \left(\bigcap_{i=1}^{m} Fix(T_i)\right) \cap \left(\bigcap_{i=1}^{m} Fix(S_i)\right) \cap Fix(Q) \cap Fix(R) \cap Ep(\Upsilon) \neq \emptyset.
$$

Let f be a contraction of C into itself with constant $b \in (0,1)$ and A be a strongly positive self-adjoint bounded linear operator on $\mathcal{H}$ with coefficient $\overline{\gamma}$ such that $0 < \gamma < \frac{(1+\overline{\gamma})-k_R}{b}$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} \Upsilon(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ y_n = \alpha_n u_n + \beta_n x_n + \sum_{i=1}^m \gamma_{n,i} T_i u_n, \\ w_n = \Lambda_{n,0} Q(y_n) + \sum_{i=1}^m \Lambda_{n,i} S_i(Q y_n), \\ x_{n+1} = a_n \gamma f(R(w_n)) + (I - a_n A) R(w_n). \end{cases}$$

Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}, \{\Lambda_{n,i}\}, \{r_n\}$ and $\{a_n\}$ satisfy the following conditions:

1. $\alpha_n + \beta_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\Lambda_{n,0} + \sum_{i=1}^m \Lambda_{n,i} = 1$,
2. $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$,
3. $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$,
4. $\kappa < \alpha_n < 1$, $\iota < \Lambda_{n,0} < 1$ *and* $\liminf_{n\to\infty}(\Lambda_{n,0} - L)\Lambda_{n,i} > 0$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$ which solves the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F.$$

Proof. First, we claim that every κ-strictly pseudo-nonspreading mapping T_i is demicontractive. To prove this, let $x^* \in Fix(T_i)$ and $x \in C$. Then we have

$$\begin{aligned} \|T_i(x) - x^*\|^2 &= \|T_i(x) - T_i(x^*)\|^2 \\ &\leq \|x - x^*\|^2 + \kappa\|(x - T_i(x)) - (x^* - T_i(x^*))\|^2 + 2\langle x - T_i(x), x^* - T_i(x^*)\rangle \\ &= \|x - x^*\|^2 + \kappa\|x - T_i(x)\|^2. \end{aligned}$$

According to Lemma (2.9), for every κ-strictly pseudo-nonspreading mapping T_i, $I - T_i$ is demiclosed at 0. By the way, the same conclusion holds for each S_i. Therefore, the result follows from Theorem (3.1). □

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $\Upsilon : C \times C \to \mathbb{R}$ be a bifunction satisfying (**C1**) – (**C4**). Let for $i = 1, 2, \cdots, m$, $T_i : C \longrightarrow C$ be a finite family of κ-strictly pseudo-contractive mappings and $S_i : C \longrightarrow C$ be a finite family of ι-strictly pseudo-contractive mappings such that $I - T_i$ and $I - S_i$ are demiclosed at 0. Assume that*

$$F := \left(\bigcap_{i=1}^m Fix(T_i)\right) \cap \left(\bigcap_{i=1}^m Fix(S_i)\right) \cap Fix(Q) \cap Fix(R) \cap Ep(\Upsilon) \neq \emptyset.$$

Let f be a contraction of C into itself with constant $b \in (0,1)$ and A be a strongly positive self-adjoint bounded linear operator on $\mathcal{H}$ with coefficient $\overline{\gamma}$ such that $0 < \gamma < \frac{(1+\overline{\gamma})-k_R}{b}$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} \Upsilon(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ y_n = \alpha_n u_n + \beta_n x_n + \sum_{i=1}^m \gamma_{n,i} T_i u_n, \\ w_n = \Lambda_{n,0} Q(y_n) + \sum_{i=1}^m \Lambda_{n,i} S_i(Q y_n), \\ x_{n+1} = a_n \gamma f(R(w_n)) + (I - a_n A) R(w_n). \end{cases}$$

Suppose that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{n,i}\}, \{\Lambda_{n,i}\}, \{r_n\}$ and $\{a_n\}$ satisfy the following conditions:

1. $\alpha_n + \beta_n + \sum_{i=1}^m \gamma_{n,i} = 1$ *and* $\Lambda_{n,0} + \sum_{i=1}^m \Lambda_{n,i} = 1$,
2. $\{a_n\} \subset (0,1)$, $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$,
3. $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$,
4. $\kappa < \alpha_n < 1$, $\iota < \Lambda_{n,0} < 1$ *and* $\liminf_{n\to\infty}(\Lambda_{n,0} - L)\Lambda_{n,i} > 0$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F$ which solves the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F.$$

Proof. First, we claim that every κ-strictly pseudo-contractive mapping T_i is demicontractive. To prove this, let $x^* \in Fix(T_i)$ and $x \in C$. Then we have

$$\begin{aligned}\|T_i(x) - x^*\|^2 &= \|T_i(x) - T_i(x^*)\|^2 \\ &\leq \|x - x^*\|^2 + \kappa\|(x - T_i(x)) - (x^* - T_i(x^*))\|^2 \\ &= \|x - x^*\|^2 + \kappa\|x - T_i(x)\|^2.\end{aligned}$$

According to Lemma (2.10), for every κ-strictly pseudo-contractive mapping T_i, $I - T_i$ is demiclosed at 0. By the way, the same conclusion holds for each S_i. Therefore, the result follows from Theorem (3.1). □

4. Application

In the following, we provide some numerical examples to illustrate the rate of convergence of our algorithm (11). Let $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ which is a nonempty closed convex subset of the real Hilbert space $\mathbb{R}$. For $i = 1, 2$ define the mappings $T_i : C \longrightarrow C$ by

$$T_i(x) = \begin{cases} x & x \in [\frac{-1}{\pi}, 0], \\ \frac{x}{i+1}|\sin\frac{1}{x}|, & x \in (0, \frac{1}{\pi}], \end{cases}$$

and $S_i : C \longrightarrow C$ by $S_i(x) = \frac{i}{i+1}x$. Clearly, zero is the only fixed point of the mappings T_i (see Figure 1).

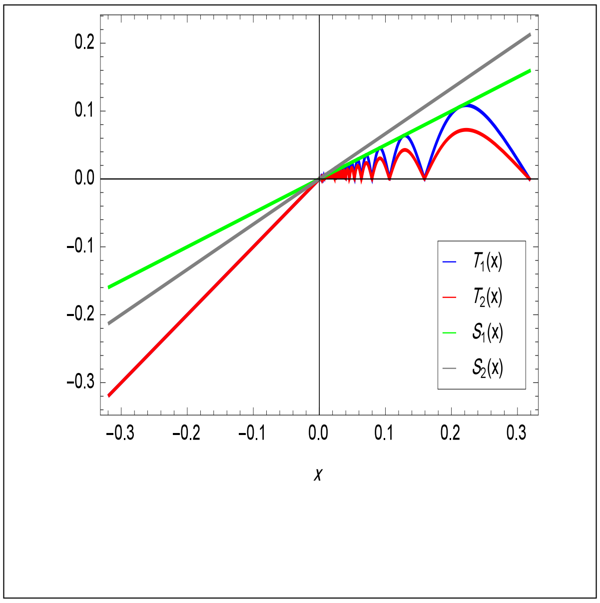

Figure 1: For $i = 1, 2$.

It is easy to see that each T_i is demicontractive for $x \in [\frac{-1}{\pi}, 0]$. For $x \in (0, \frac{1}{\pi}]$ we have

$$\|T_i x - 0\|^2 = \|T_i x\|^2 = \|\frac{x}{i+1}|\sin\frac{1}{x}|\|^2 \leq \|\frac{x}{i+1}\|^2 \leq \|x - 0\|^2 + \kappa\|T_i x - x\|^2,$$

for some $\kappa < 1$. Thus, T_i is demicontractive. Similarly, we can show that each S_i is demicontractive. Now, we define the bifunction Υ by

$$\begin{cases} \Upsilon : C \times C \longrightarrow \mathbb{R} \\ \Upsilon(x, y) = y^2 + xy - 2x^2. \end{cases}$$

It is easy to see that Υ satisfies the conditions (**C1**) – (**C4**). To have a better understanding of this issue, we sketch the graph of Υ in three-dimensional space (see Figure 2).

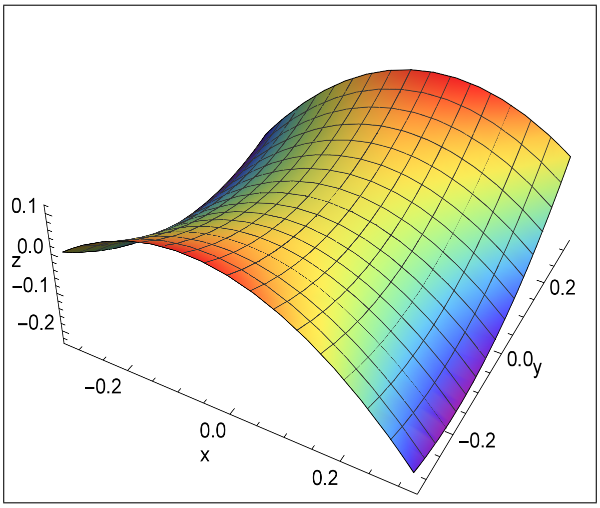

Figure 2: $0 \in Ep(\Upsilon)$, since $\Upsilon(0, y) = y^2 \geq 0$, for all $y \in C$.

In [28], Singthong and Suantai obtained the following sequence for $r_n = 1$, $u_n = T_{r_n}(x_n) = \frac{x_n}{3r_n + 1} = \frac{x_n}{4}$. For $i = 1, \cdots, m$ we define $\theta_0 := LCM(1, 2, \cdots, m)$, where $LCM(1, 2, \cdots, m)$ is the lowest common multiple of the integers $1, 2, \cdots, m$. Now, we set $\gamma = 1, a_n = \frac{1}{n+1}, \alpha_n = \beta_n = \gamma_{n,1} = \gamma_{n,2} = \frac{1}{4}$, and $\Lambda_{n,0} = \Lambda_{n,1} = \Lambda_{n,2} = \frac{1}{3}$. Note that when we run this algorithm, the parameter θ_0 helps us in running faster and leads to better answers.

We set $R(x) = Q(x) = f(x) = \frac{x}{\theta_0}$ and $A = I$. Now for any $x_0 \in [\frac{-1}{\pi}, \frac{1}{\pi}]$, our algorithm is the following:

$$\begin{cases} y_n = \frac{x_n}{16} + \frac{x_n}{4} + \frac{1}{4}T_1(\frac{x_n}{4}) + \frac{1}{4}T_2(\frac{x_n}{4}), \\ w_n = \frac{y_n}{4\theta_0} + \frac{1}{4}S_1(\frac{y_n}{\theta_0}) + \frac{1}{4}S_2(\frac{y_n}{\theta_0}) = \frac{y_n}{4\theta_0} + \frac{1}{4}(\frac{1}{2}\frac{y_n}{\theta_0}) + \frac{1}{4}(\frac{2y_n}{3\theta_0}) \\ \quad = \frac{y_n}{4\theta_0} + \frac{y_n}{8\theta_0} + \frac{y_n}{6\theta_0} = \frac{y_n}{4\theta_0} + \frac{y_n}{6\theta_0} + \frac{y_n}{8\theta_0} = \frac{13}{24\theta_0}y_n, \\ x_{n+1} = \frac{w_n}{(n+1)\theta_0^2} + \frac{n}{n+1}\frac{w_n}{\theta_0} = \frac{1+n\theta_0}{(n+1)\theta_0^2}w_n. \end{cases} \tag{20}$$

Let $x_0 = \frac{1}{2\pi}$, then for all $n \geq 0$, the algorithm (20) becomes

$$\begin{cases} y_n = \frac{x_n}{16} + \frac{x_n}{4} + \frac{1}{4}\frac{x_n}{8}|\sin\frac{4}{x_n}| + \frac{1}{4} + \frac{x_n}{12}|\sin\frac{4}{x_n}| = \frac{5x_n}{16}(1 + \frac{1}{6}|\sin\frac{4}{x_n}|), \\ w_n = \frac{65x_n}{384\theta_0}(1 + \frac{1}{6}|\sin\frac{4}{x_n}|), \\ x_{n+1} = \frac{1+n\theta_0}{(n+1)\theta_0^2}\frac{65x_n}{384\theta_0}(1 + \frac{1}{6}|\sin\frac{4}{x_n}|) = \frac{65(1+n\theta_0)}{384(n+1)\theta_0^3}(1 + \frac{1}{6}|\sin\frac{4}{x_n}|)x_n. \end{cases}$$

Due to the fact that $(1 + \frac{1}{6}|\sin\frac{4}{x_n}|) \geq 0$, we conclude that x_n converges to zero. On the other hand

$$F = \left(\bigcap_{i=1}^{2} Fix(T_i)\right) \cap \left(\bigcap_{i=1}^{2} Fix(S_i)\right) \cap Fix(Q) \cap Fix(R) \cap Ep(\Upsilon) = \{0\}.$$

Here comes the table of numerical results for the first step $x_0 = \frac{1}{2\pi}$ (see Table 1):

Iteration steps	Values of x_n	Iteration steps	Values of x_n
0	0.159155	26	2.08640×10^{-37}
1	0.00336754	27	9.62894×10^{-39}
2	0.000111973	28	4.03356×10^{-40}
3	4.14052×10^{-6}	29	1.72747×10^{-41}
4	1.65488×10^{-7}	30	7.94206×10^{-43}
5	7.24066×10^{-9}	31	3.65726×10^{-44}
6	3.11706×10^{-10}	32	1.71492×10^{-45}
7	1.36314×10^{-11}	33	8.21766×10^{-47}
8	5.40835×10^{-13}	34	3.67328×10^{-48}
9	2.20260×10^{-14}	35	1.53926×10^{-49}
10	9.37587×10^{-16}	36	6.49615×10^{-51}
11	3.97624×10^{-17}	37	2.87259×10^{-52}
12	1.62365×10^{-18}	38	1.39661×10^{-53}
13	6.72121×10^{-20}	39	5.88287×10^{-55}
14	3.15619×10^{-21}	40	2.85415×10^{-56}
15	1.38941×10^{-22}	41	1.33051×10^{-57}
16	5.76668×10^{-24}	42	6.40133×10^{-59}
17	2.44475×10^{-25}	43	3.00585×10^{-60}
18	1.09982×10^{-26}	44	1.30438×10^{-61}
19	5.21019×10^{-28}	45	6.27974×10^{-63}
20	2.37627×10^{-29}	46	2.81458×10^{-64}
21	1.01053×10^{-30}	47	1.36055×10^{-65}
22	4.84174×10^{-32}	48	5.79889×10^{-67}
23	2.23985×10^{-33}	49	2.74276×10^{-68}
24	9.45789×10^{-35}	50	1.24763×10^{-69}
25	4.51880×10^{-36}		

Table 1: Numerical results correspondent to $x_0 = \frac{1}{2\pi}$ for 50 steps.

Let $x_0 = -\frac{1}{2\pi}$, then for all $n \geq 0$, the algorithm (20) becomes

$$\begin{cases} y_n = \frac{x_n}{16} + \frac{x_n}{4} + \frac{1}{4}T_1(\frac{x_n}{4}) + \frac{1}{4}T_2(\frac{x_n}{4}) = \frac{x_n}{16} + \frac{x_n}{4} + \frac{x_n}{16} + \frac{x_n}{16} = \frac{7x_n}{16}, \\ w_n = \frac{13}{24\theta_0}y_n = \frac{91}{384\theta_0}x_n, \\ x_{n+1} = \frac{1+n\theta_0}{(n+1)\theta_0^3}\frac{91}{384}x_n. \end{cases}$$

Again we provide the table of numerical results for the first step $x_0 = -\frac{1}{2\pi}$ (see Table 2).

Iteration steps	Values of x_n	Iteration steps	Values of x_n
0	-0.159155	26	-2.15068×10^{-34}
1	-0.00471455	27	-1.25057×10^{-35}
2	-0.000209484	28	-7.27667×10^{-37}
3	-0.0000103424	29	-4.23672×10^{-38}
4	-5.36141×10^{-7}	30	-2.46820×10^{-39}
5	-2.85872×10^{-8}	31	-1.43869×10^{-40}
6	-1.55251×10^{-9}	32	-8.39034×10^{-42}
7	-8.54080×10^{-11}	33	-4.89552×10^{-43}
8	-4.74373×10^{-12}	34	-2.85769×10^{-44}
9	-2.65428×10^{-13}	35	-1.66885×10^{-45}
10	-1.49390×10^{-14}	36	-9.74973×10^{-47}
11	-8.44826×10^{-16}	37	-5.69815×10^{-48}
12	-4.79661×10^{-17}	38	-3.33144×10^{-49}
13	-2.73244×10^{-18}	39	-1.94840×10^{-50}
14	-1.56101×10^{-19}	40	-1.13990×10^{-51}
15	-8.93992×10^{-21}	41	-6.67093×10^{-53}
16	-5.13092×10^{-22}	42	-3.90513×10^{-54}
17	-2.95040×10^{-23}	43	-2.28668×10^{-55}
18	-1.69940×10^{-24}	44	-1.33935×10^{-56}
19	-9.80313×10^{-26}	45	-7.84676×10^{-58}
20	-5.66265×10^{-27}	46	-4.59827×10^{-59}
21	-3.27495×10^{-28}	47	-2.69525×10^{-60}
22	-1.89614×10^{-29}	48	-1.58016×10^{-61}
23	-1.09894×10^{-30}	49	-9.26612×10^{-63}
24	-6.37503×10^{-32}	50	-5.43480×10^{-64}
25	-3.70133×10^{-33}		

Table 2: Numerical results corresponding to $x_0 = -\frac{1}{2\pi}$ for 50 steps.

In this case, x_n converges to zero too. Thus, in general, the sequence $\{x_n\}_{n\geq 1}$ is convergent to zero.

In the following we show the list plot of our algorithm (see Figure 3, which shows the list plot of Table 1 and Table 2).

List plot of our algorithm for Table 1 and Table 2

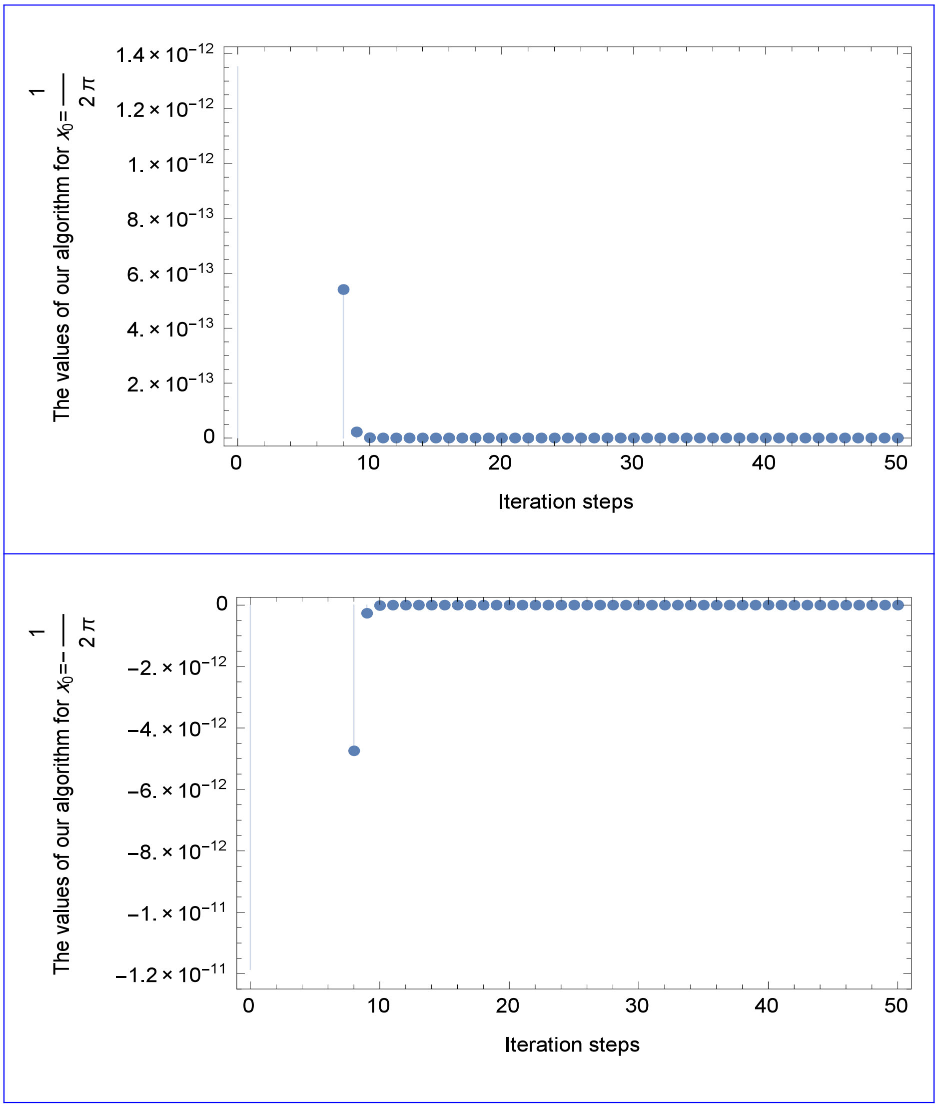

Figure 3: List plot of our algorithm for Table 1 and Table 2

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