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Addendum to: On a Simultaneous Generalization of β-Normality and Almost Normality

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Abstract. In [1], the authors prove that almost β-normality is preserved by continuous, open, closed surjections. We present examples to show that neither "open" nor "closed" can be omitted.

1. Definitions and Theorems

Definition 1.1. ([2]) A topological space is said to be *almost normal* if for every pair of disjoint closed sets E and F one of which is regularly closed, there exist disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Definition 1.2. ([1]) A topological space is *almost β-normal* if for every pair of disjoint closed sets E and F, one of which is regularly closed, there are open sets U and V such that $\overline{E \cap U} = E$, $\overline{F \cap V} = F$, and $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 1.3. ([1]) *An almost normal space is almost β-normal.*

Theorem 1.4. ([1]) *Suppose that X and Y are topological spaces, X is almost β-normal, and $f : X \to Y$ is onto, continuous, open, and closed. Then Y is almost β-normal.*

2. Examples

Example 2.1. *Almost β-normality is not preserved by continuous open surjections even in the case where the domain space is a compact Hausdorff space.*

Let $A = \mathbb{N}\times\{0\}$, $B = \mathbb{N}\times\{1\}$, $C = \mathbb{N}\times\{2\}$, $r, s \notin A\cup B\cup C$ be two distinct points, and $X = A\cup B\cup C\cup\{r\}\cup\{s\}$. For $x \in A\cup B\cup C$, let $\mathcal{B}_x = \{\{x\}\}$. Let $\mathcal{B}_r = \{(\{r\}\cup A\cup B)\setminus F : F \subseteq A\cup B, |F| < \aleph_0\}$ and $\mathcal{B}_s = \{(\{s\}\cup C)\setminus F : F \subseteq C, |F| < \aleph_0\}$. Let $\mathcal{T}_X$ be the topology on X generated by taking $\bigcup\limits_{x\in X} \mathcal{B}_x$ as a base. Note that X is compact and Hausdorff. Now let $Y = A \cup B \cup \{r\} \cup \{s\} = X \setminus C$. For $y \neq s$, let $\mathcal{B}_y$ be as above. Let $\mathcal{B}_s = \{(\{s\} \cup A) \setminus F : F \subseteq A, |F| < \aleph_0\}$. Let $\mathcal{T}_Y$ be the topology on Y generated by taking $\bigcup\limits_{y\in Y} \mathcal{B}_y$ as a base. To see that Y is not almost β-normal

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consider the regularly closed set $B \cup \{r\}$ and the closed set $\{s\}$. Suppose that U and V are open sets such that $\overline{U \cap (B \cup \{r\})} = B \cup \{r\}$ and $\overline{V \cap \{s\}} = \{s\}$. Then $B \subseteq U$ which implies that $r \in \overline{U}$. Also, $V \cap A$ is infinite (cofinite relative to A) which implies that $r \in \overline{V}$. Therefore, $\overline{U} \cap \overline{V} \neq \emptyset$. Now define $f : X \to Y$ as follows. For $x \notin C$, $f(x) = x$. For $x = (n, 2) \in C$, $f(x) = (n, 0) \in A$. Note that f is the quotient map that joins together the points $(n, 0)$ and $(n, 2)$ for each $n \in \mathbb{N}$. It is readily verified that f is continuous, open, and onto. Note that f is not closed since $C \cup \{s\}$ is closed in X while $f(C \cup \{s\}) = A \cup \{s\}$ is not closed in Y.

Example 2.2. *Almost β-normality is not preserved by continuous closed surjections even in the case where the domain space is almost normal.*

Let A and B be disjoint countable infinite sets, $r, s, t \notin (A \cup B)$ be three distinct points, and $X = A \cup B \cup \{r\} \cup \{s\} \cup \{t\}$. For $x \in A \cup B$, let $\mathcal{B}_x = \{\{x\}\}$. Also, let $\mathcal{B}_r = \{(\{r\} \cup A) \setminus F : F \subseteq A, |F| < \aleph_0\}$, $\mathcal{B}_s = \{(\{s\} \cup A) \setminus F : F \subseteq A, |F| < \aleph_0\}$, and $\mathcal{B}_t = \{(\{t\} \cup B) \setminus F : F \subseteq B, |F| < \aleph_0\}$. Let $\mathcal{T}_X$ be the topology on X generated by taking $\bigcup_{x \in X} \mathcal{B}_x$ as a base. Note that the space X is the sum of the modified Fort space $(A \cup \{r\} \cup \{s\})$ and a convergent sequence of discrete points $(B \cup \{t\})$. Note that regularly closed subsets of the modified Fort space are either finite subsets of A or sets of the form $S \cup \{r\} \cup \{s\}$ where $S \subseteq A$ is infinite. So the modified Fort space is almost normal. Therefore, X is almost normal. Now let $Y = X \setminus \{t\}$. For $y \neq r$, let $\mathcal{B}_y$ be as above. Let $\mathcal{B}_r = \{(\{r\} \cup A \cup B) \setminus F : F \subseteq (A \cup B), |F| < \aleph_0\}$. Let $\mathcal{T}_Y$ be the topology on Y generated by taking $\bigcup_{y \in Y} \mathcal{B}_y$ as a base. Note that the space Y is the homeomorphic to the space Y in the previous example. Therefore, Y is not almost β-normal. Now define $f : X \to Y$ as follows. For $x \neq t$, $f(x) = x$ and $f(t) = r$. It is readily verified that f is continuous, closed, and onto. Note that f is not open since $B \cup \{t\}$ is open in X while $f(B \cup \{t\}) = B \cup \{r\}$ is not open in Y.

Remark 2.3. It is clear that Y in Example 2.1 and X, Y in Example 2.2 are not Hausdorff, so it is obvious to ask the question: Is almost β-normality preserved under continuous closed (or open) surjections if both range and domain spaces are Hausdorff ? Example 2.4 below provides an answer to this question in negative in the case of open maps.

Example 2.4. *Almost β-normality is not preserved by continuous open surjections even in the case where the domain space is a normal Hausdorff space and the range space is also Hausdorff.*

Let $X = (\mathbb{R} \times [0, \infty)) \setminus ((-\infty, 0] \times \{0\})$. For $(x, y) \in X$ with $x \neq 0$ and $y \neq 0$, let $\mathcal{B}_{(x,y)} = \{\{(x, y)\}\}$. For $x > 0$, let $\mathcal{B}_{(x,0)} = \{(\{x\} \times [0, \infty)) \setminus F : F \subseteq X, |F| < \aleph_0\}$. Finally, for $y > 0$, let $\mathcal{B}_{(0,y)}$ $\{((-\infty, 0] \times \{y\}) \setminus F : F \subseteq X, |F| < \aleph_0\}$. Let $\mathcal{T}_X$ be the topology on X generated by taking $\bigcup_{z \in X} \mathcal{B}_z$ as a base. Since X is the sum of compact Hausdorff spaces, X is a normal Hausdorff space. Now let $Y = ([0, \infty) \times [0, \infty)) \setminus \{(0, 0)\}$. For $(x, y) \in X$ with $x > 0$ and $y > 0$, let $\mathcal{B}_{(x,y)} = \{\{(x, y)\}\}$. For $x > 0$, let $\mathcal{B}_{(x,0)} = \{(\{x\} \times [0, \infty)) \setminus F : F \subseteq Y, |F| < \aleph_0\}$. Finally, for $y > 0$, let $\mathcal{B}_{(0,y)}$ $\{([0, \infty) \times \{y\}) \setminus F : F \subseteq Y, |F| < \aleph_0\}$. Let $\mathcal{T}_Y$ be the topology on Y generated by taking $\bigcup_{z \in Y} \mathcal{B}_z$ as a base. Note that Y is Hausdorff. To see that Y is not almost β-normal, let $E = \overline{\{(p, q) : p, q \in \mathbb{Q}, p > 0, q > 0\}}$ and $F = \{(x, 0) : x \in \mathbb{R} \setminus \mathbb{Q}, x > 0\}$. Note that E is regularly closed, F is closed, and the two cannot be separated by open sets since F is uncountable. Now define $f : X \to Y$ by $f(x, y) = (|x|, y)$. Note that f simply reflects rays of the form $(-\infty, 0] \times \{b\}$ about the y-axis. So it is readily verified that f is continuous, open, and onto.

Question 2.5. *Suppose that X is a Hausdorff almost β-normal spaces, Y is Hausdorff, and f : X → Y is a continuous closed map. Is Y almost β-normal?*

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References

[1] A.K. Das, P. Bhat, J.K. Tartir, On a simultaneous generalization of β-normality and almost normality, Filomat 31:2 (2017) 425430.
[2] M.K. Singal, S.P. Arya, Almost normal and almost completely regular spaces, Glasnik Mat. 25:5 (1970) 141–152.