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The Fixed Point Property of the Smallest Open Neighborhood of the n-dimensional Khalimsky Topological Space

Sang-Eon Han[a]

[a]*Department of Mathematics Education, Institute of Pure and Applied Mathematics
Chonbuk National University, Jeonju-City Jeonbuk, 54896, Republic of Korea*

Abstract. The paper aims to propose the fixed point property(*FPP* for short) of smallest open neighborhoods of the n-dimensional Khalimsky space and further, the *FPP* of a Khalimsky (K-, for short) retract. Let (X, κ^n_X) be an n-dimensional Khalimsky topological space induced by the n-dimensional Khalimsky space denoted by $(\mathbf{Z}^n, \kappa^n)$. Although not every connected Khalimsky topological space (X, κ^n_X) has the *FPP*, we prove that for every point $x \in \mathbf{Z}^n$ the smallest open K-topological neighborhood of x, denoted by $SN_K(x) \subset (\mathbf{Z}^n, \kappa^n)$, has the *FPP*. Besides, the present paper also studies the almost fixed point property (*AFPP*, for brevity) of a K-topological space. In this paper all spaces $(X, \kappa^n_X) := X$ are assumed to be connected and $|X| \geq 2$.

1. Introduction

Let us recall that a non-empty topological space X has the fixed point property if every continuous self-map of X has at least one fixed point $x \in X$. In particular, the Lefschetz number has strongly contributed to the study of the *fixed point property* (*FPP* for brevity) of topological spaces [20, 21]. By using this number, we can recognize the existence of a fixed point of a continuous self-map of a compact topological space X in terms of traces of the induced mappings on the homology groups of X, which implies that the *FPP* related to the Lefschetz number [21] and the Nielsen number [2] is a topological and a homotopy equivalent invariant. In many areas of applied science, for given a space X it is important to find fixed points of a certain self-map of X. In particular, given a digital topological space (X, T) the study of the *FPP* of (X, T) can be very useful to applied science [11].

Digital topology has a focus on studying various properties of n-dimensional digital spaces, which has contributed to the study of some areas of computer sciences such as computer graphics, image processing, approximation theory, mathematical morphology, optimization theory and so forth. Since digital topology is partially related to lattice theory, for complete lattices, Tarski [24] proposed an important results: “Let L be a complete lattice and consider a self order-preserving function f of L. Then the set of fixed points of f in L is also a complete lattice”. But, since almost of all digital topological spaces are not complete

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Email address: sehan@jbnu.ac.kr (Sang-Eon Han)

lattices, there is some difficulty in using the theorem. Roughly saying, in digital topology there are several approaches for studying digital spaces. First of all, we can consider the Rosenfeld model [22, 23], Alexandroff topology with T_0-separation axiom such as Khalimsky topology, Marcus-Wyse topology, axiomatic locally finite topology [19] and so forth. Meanwhile, in relation to fixed point theory for digital spaces, it turns out that the Rosenfeld model [22, 23] is not suitable for studying the *FPP* of digital spaces because a digital image $(X, k), |X| \geq 2$, does not have the *FPP* [23] associated with digitally k-continuous maps(see [23], for more details, see [5–7]). Thus we conclude that we need to study the *FPP* of digital spaces by using T_0 Alexandroff topological structure. In fixed point theory, we may assume that all topological spaces are connected and in particular, it is clear that a singleton has the *FPP*.

In $(\mathbf{Z}^n, \kappa^n)$, we say that two distinct points x and y are *(Khalimsky) adjacent* if $y \in SN_K(x)$ or $x \in SN_K(y)$ [16], where $SN_K(x)$ is the smallest open set containing the given point x. We say that a non-empty K-topological space (X, κ^n_X) has the *almost fixed point property* (*AFPP*, for brevity) if every continuous self-map f of (X, κ^n_X) has a fixed point $x \in X$ or $f(x)$ is K-adjacent to x, where (X, κ^n_X) is a subspace of $(\mathbf{Z}^n, \kappa^n)$.

The paper aims to study both the *FPP* and the *AFPP* for Khalimsky topological spaces. Up to now, since there is no homotopy or homology associated with the Lefschetz number which can be suitable for studying the *FPP* of K-topological spaces [8], we can study these properties by using general topological tools. The present paper studies the *FPP* and the *AFPP* by using general topological tools and some T_0 Alexandroff topological structures, in particular, a K-retract.

Then we may raise the following two questions:

$$\begin{Bmatrix} (\star 1)\ \text{What about the } FPP \text{ of a Khalimsky topological retract?} \\ (\star 2)\ \text{What about the } AFPP \text{ of a Khalimsky topological space?} \end{Bmatrix}$$

To address these issues ($\star$1) and ($\star$2), we study the *FPP* of a retract in the category of Khalimsky topological spaces.

The rest of the paper is organized as follows: Section 2 provides basic notions on K-topology. Section 3 studies the *FPP* of a K-topological retract. Section 4 investigates the *AFPP* of K-topological spaces. Section 5 concludes the paper with some remarks.

2. Preliminaries

As mentioned in Section 1, since almost of all studies of digital topologies are based on Alexandroff topology and a digital space [13], let us recall basic notions of these structures. We say that a topological space X is Alexandroff if for each point $x \in X$ there is the smallest open set $O(x)$ containing x [1]. We say that a *digital space* is a pair (X, R) [13], where X is a nonempty set and R is a binary symmetric relation on X such that X is R-connected. Here, we say that X is *R-connected* if for any two elements x and y of X there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbf{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in R$ for $j \in [0, l-1]_{\mathbf{Z}}$. For instance, it is clear that for two distinct points in an Alexandroff T_0-space (X, T), we have an *R-connected* relation between them induced by the connectedness of (X, T). Thus it is obvious that a K-topological space is a digital space. Besides, we also have an *R-connected* relation for a digital space in terms of the digital k-connectivity followed from the Rosenfeld model [22].
For $a, b \in \mathbf{Z}$ with $a \precsim b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \mid a \leq n \leq b\}$ with 2-adjacency is called a digital interval [18].

To address the question ($\star$1) in Section 1, we need to recall some notions on K-topology associated with the *FPP* of K-topological spaces. To study fixed point theory from the viewpoint of Khalimsky topology, we assume that every K-topological space is K-connected and in particular, it is obvious that a singleton has the *FPP*. Unlike the study of the *FPP* for digital images in the graph theoretical approach in terms of the Rosenfeld model [23], the *FPP* for K-topological spaces has its own feature which is quite different from the *FPP* for digital images.

Motivated by the Alexandroff space [1], the *Khalimsky line topology* on $\mathbf{Z}$ is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} : n \in \mathbf{Z}\}$ as a subbase [1]. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or *Khalimsky n-dimensional space*) which is denoted by $(\mathbf{Z}^n, \kappa^n)$ [15]. For convenience, we say that a point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *pure open* if all coordinates are odd; and it is *pure closed* if each of the coordinates is even [16]. The other points in $\mathbf{Z}^n$ are called *mixed* [16].

Furthermore, for a subset $X \subset \mathbf{Z}^n$ we will consider (X, κ^n_X), $n \geq 1$ [12] as a subspace of $(\mathbf{Z}^n, \kappa^n)$, and it is called a K-topological space. Besides, for a point $p \in (\mathbf{Z}^n, \kappa^n)$, we denote by $SN_K(p)$ the smallest open K-topological neighborhood of p [16]. For instance, for a point $p := (p_1, p_2)$ in $(\mathbf{Z}^2, \kappa^2)$, its smallest open neighborhood $SN_K(p) \subset \mathbf{Z}^2$ is obtained [16], as follows (see Figure 1(1)-(3)):

$$SN_K(p) := \left\{ \begin{array}{l} \{p\} \text{ if } p \text{ is pure open,} \\ \{(p_1 - 1, p_2), p, (p_1 + 1, p_2)\} \text{ if } p \text{ is closed-open,} \\ \{(p_1, p_2 - 1), p, (p_1, p_2 + 1)\} \text{ if } p \text{ is open-closed,} \\ \ [2m - 1, 2m + 1]_{\mathbf{Z}} \times [2n - 1, 2n + 1]_{\mathbf{Z}} \\ \ \text{if } p = (2m, 2n), m, n \in \mathbf{Z}, \end{array} \right\} \tag{2.1}$$

where the point $p := (p_1, p_2)$ is called *closed-open* (resp. *open-closed*) if p_1 is even (resp. odd) and p_2 is odd (resp. even).

For a subset $X \subset \mathbf{Z}^n$ and a point $x \in (X, \kappa^n_X)$, we will use the notation $SN_K(x) \cap X := SN_K(x)$ in (X, κ^n_X) again if there is no danger of the ambiguity.

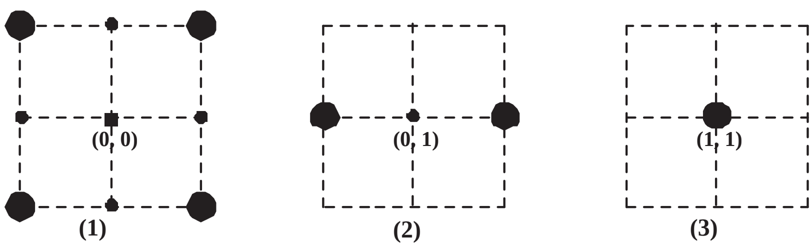

Figure 1: Configuration of $SN_K(p)$ depending on the given point $p \in \mathbf{Z}^2$. [16]

In $(\mathbf{Z}^n, \kappa^n)$, let us now recall some properties of the K-continuity of maps between two K-topological spaces [14–16] as follows: for two K-topological spaces $(X, \kappa^{n_0}_X) := X$ and $(Y, \kappa^{n_1}_Y) := Y$, a function $f : X \to Y$ is said to be *K-continuous* at a point $x \in X$ if f is continuous at the point x from the viewpoint of Khalimsky product topology as usual, *i.e.*

$$f(SN_K(x)) \subset SN_K(f(x)) \tag{2.2}$$

because the spaces $(X, \kappa^{n_0}_X)$ and $(Y, \kappa^{n_1}_Y)$ are Alexandroff spaces. Furthermore, we say that a map $f : X \to Y$ is K-continuous if it is K-continuous at every point $x \in X$. In addition, we recall the notion of a *K-homeomorphism* as follows: for two spaces $(X, \kappa^{n_0}_X)$ and $(Y, \kappa^{n_1}_Y)$, a map $h : X \to Y$ is called a K-homeomorphism if h is a K-continuous bijection and further, $h^{-1} : Y \to X$ is K-continuous [15].

By using the K-continuity of the map f, we obtain the *K-topological category* denoted by *KTC* [3], consisting of the following two sets:
• the set of objects (X, κ^n_X), denoted by $Ob(KTC)$;
• for every ordered pair of objects $(X, \kappa^{n_0}_X)$ and $(Y, \kappa^{n_1}_Y)$, the set of all K-continuous maps $f : (X, \kappa^{n_0}_X) \to (Y, \kappa^{n_1}_Y)$ as morphisms.

3. The fixed point property of a retract in the category of Khalimsky topological spaces

Let us now recall some basic notions and terminology for studying Khalimsky adjacency.
Let us recall the following terminology for studying the *FPP* of K-topological spaces.

Definition 3.1. *[12] Let $(X, \kappa^n_X) := X$ be a K-topological space. Then we define the following: A simple K-path in X is an injective and finite sequence [12] $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ such that x_i and x_j are K-adjacent if and only if $|i-j|=1$. Furthermore, we say that a simple closed K-curve with l elements $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$ is a K-homeomorphic image of a Khalimsky circle $\mathbf{Z}/l\mathbf{Z}$ as a quotient topological space, where l is an even integer $l \geq 4$ [17]. We denote it by $SC^{n,l}_K := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}, l \geq 4$ [12] (see Figure 2(a)(1) and (2)).*

In addition, for a point $x \in (\mathbf{Z}, \kappa^n)$ we use the notation $AN(x) = \{y \in \mathbf{Z}^n \mid y$ is K-adjacent to $x.\}$ (see Figure 2(b)). Besides, we call $AN(x)$ a Khalimsky adjacent neighborhood of x [12].

Remark 3.2. *(1) Consider a simple K-path $(X := (x_i)_{i\in[0,m]_{\mathbf{Z}}}, m \geq 3, \kappa^n_X)$. Then for a point $x_i \in X$ it is clear that $|SN_K(x_i)| = 3$, $|SN_K(x_i)| = 2$, or $|SN_K(x_i)| = 1$ depending on $i \in [0,m]_{\mathbf{Z}}$.*

(2) An unbounded simple K-path does not have the FPP because an unbounded simple K-path is K-homeomorphic to the Khalimsky topological line $(\mathbf{Z}, \kappa)$ and in KTC the FPP is a K-topological invariant. To be specific, it is clear that $(\mathbf{Z}, \kappa)$ does not have the FPP with the following map. Consider the self-map of $(\mathbf{Z}, \kappa)$ given by $f(t) = t+2n, n \in \mathbf{Z}\setminus\{0\}$. While the map f is certainly a K-continuous map, we see that there is no point $x \in \mathbf{Z}$ such that $f(x) = x$.

(3) The n-dimensional Khalimsky topological space $(\mathbf{Z}^n, \kappa^n)$ does not have the FPP.

(4) Any $SC^{n,l}_K$ does not have the FPP [8]. To be specific, consider a self-map of $SC^{n,l}_K := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}, l \geq 4$ given by $f(x_i) = x_{i+2(mod\, l)}$. While the map f is a K-continuous map, it has no fixed point.

To address the query ($\star$1), we need to recall the notion of a retraction in *KTC* [4], as follows: we say that a K-continuous map $r : (X', \kappa^n_{X'}) \to (X, \kappa^n_X)$ is a K-retraction [4] if
(1) (X, κ^n_X) is a subspace of $(X', \kappa^n_{X'})$, and
(2) $r(a) = a$ for all $a \in (X, \kappa^n_X)$.
Then we say that (X, κ^n_X) is a K-retract of $(X', \kappa^n_{X'})$. Furthermore, we say that the point $a \in X' \setminus X$ is K-retractable.

Hereafter, let us remind again that every K-topological space (X, κ^n_X) has the cardinality $1 \lneq |X| \lneq \infty$ and is K-connected. To address the issue ($\star$1), we need the following:

Theorem 3.3. *For every point $x \in (\mathbf{Z}^2, \kappa^2)$ $SN_K(x) (\subset (\mathbf{Z}^2, \kappa^2))$ has the FPP.*

Proof: Depending on the choice of the point $x \in (\mathbf{Z}^2, \kappa^2)$, $SN_K(x)$ is determined (see Figure 1(1)-(3) in $(\mathbf{Z}^2, \kappa^2)$), *i.e.* x can be a pure closed, a mixed or a pure open point. We prove combinatorially the *FPP* of $SN_K(x)$ according to the point x as follows:

(Case 1) Let us consider any K-continuous self-map f of $SN_K(x)$, where x is a pure closed point.

(1-1) If x is mapped into x by the map f, then the proof is obviously completed.

(1-2) We may assume that x is mapped into x_1 by the map f, where x_1 is a mixed point in $SN_K(x)$. For convenience, we may assume $x := (0,0)$ and $x_1 := (0,1)$ (see Figure 2(b)(2)). Owing to the K-topological structures of $SN_K(x)$ and $SN_K(x_1)$, it is clear that $f(SN_K(x)) \subset SN_K(x_1) = \{(-1,1), x_1, (1,1)\}$. Then we need to investigate the mapping of the element of $SN_K(x_1)$ under f, as follows:
In case $f(x_1) = x_1$, the proof is completed.
In case $f(x_1) = i \in \{(-1,1), (1,1)\}$, we should get $f(i) = i$ because $SN_K(i) = \{i\}$, which completes the proof.

(1-3) Assume that x is mapped into x_2 by the map f, where x_2 is a pure open point in $SN_K(x)$. Then, owing to the K-topological structure of $SN_K(x_2)$, it is clear $f(SN_K(x)) \subset SN_K(x_2) = \{x_2\}$. Hence we have the point x_2 as a fixed point of f.

Based on these cases (1-1)~(1-3) and the property (2.2), we have the following:

(1-4) Assume a mixed point in $SN_K(x)$ is mapped into a pure closed point x by the map f. Then, according to the cases (1-1)~(1-3), the proof is completed.

(1-5) In case a mixed point in $SN_K(x)$ is mapped into a mixed point in terms of f. Then, according to the cases (1-1)~(1-3), the proof is completed.

(1-6) In case a mixed point in $SN_K(x)$ is mapped into a pure open point under f. Then, according to the cases (1-1)∼(1-3), the proof is completed.

(1-7) Assume a pure open point in $SN_K(x)$ is mapped into a pure closed point x by the map f. Then, according to the cases (1-1)∼(1-3), the proof is completed.

(1-8) Assume a pure open point in $SN_K(x)$ is mapped into a mixed point under f. Then, according to the cases (1-1)∼(1-3), the proof is completed.

(1-9) Assume a pure open point in $SN_K(x)$ is mapped into a pure open point in $SN_K(x)$ in terms of f. Then, according to the cases (1-1)∼(1-3), the proof is completed.

(Case 2) Let us consider any K-continuous self-map f of $SN_K(x)$, where x is a mixed point (see Figure 1(2)).

(2-1) In case x is mapped into x by the map f, the proof is obviously completed.

(2-2) We may assume that x is mapped into a pure open point in $SN_K(x)$ by the map f.
For convenience, put $x := (0, 1)$ and $x_2 := (1, 1)$ (see Figure 2(b)(2)). Owing to the K-topological structures of $SN_K(x)$ and $SN_K(x_2)$, it is clear that $f(SN_K(x)) \subset SN_K(x_2) = \{x_2\}$. Since we have $f(x_2) = x_2$, the proof is completed.

Based on these cases (2-1)∼(2-2) and the property (2.2), we have the following:

(2-3) Assume a pure open point in $SN_K(x)$ is mapped into a mixed point x by the map f. Then the given mixed point x should be a fixed point of f owing to the property (2.2).

(2-4) In case a pure open point in $SN_K(x)$ is mapped into a pure open in $SN_K(x)$ by the map f, according to (2-1)∼(2-2), the proof is completed.

(Case 3) Let us consider any K-continuous self-map f of $SN_K(x)$, where x is a pure open point. Then the proof is trivial owing to the K-continuity of the given map. □

By the method similar to the proof of Theorem 3.3, it is clear that in $(\mathbf{Z}, \kappa)$, $SN_K(x)(\subset (\mathbf{Z}, \kappa))$ has the *FPP*. As a generalization of Theorem 3.3, we obtain the following:

Corollary 3.4. *For every point $x \in (\mathbf{Z}^n, \kappa^n)$, $SN_K(x)(\subset (\mathbf{Z}^n, \kappa^n))$ has the FPP, $n \in \mathbf{N}$.*

In *KTC* we have the following property of which the study of the *FPP* of a K-retract [4] contributes to fixed point theory in *KTC*.

Theorem 3.5. *In KTC, let (A, κ^n_A) be a K-retract of (X, κ^n_X). If (X, κ^n_X) has the FPP, then (A, κ^n_A) also has the FPP, where A need not be a singleton.*

Proof: Consider the inclusion map $i : (A, \kappa^n_A) \to (X, \kappa^n_X)$ and the retraction $r : (X, \kappa^n_X) \to (A, \kappa^n_A)$. Under the hypothesis, to prove the *FPP* of (A, κ^n_A), take any K-continuous self-map f of (A, κ^n_A). Then consider the composition $i \circ f \circ r := h$ which is a K-continuous self-map of (X, κ^n_X) because K-continuity supports the composite. By the hypothesis, we should have a point $x \in X$ such that $h(x) = x$. Then it is clear that $h(x) \in A$. Owing to the K-retraction from (X, κ^n_X) to (A, κ^n_A), the point x should be an element of A. Based on this approach, we can take a point x in (A, κ^n_A) such that $h(x) = i \circ f \circ r(x) = i \circ f(x) = f(x)$ because of the property of the inclusion map i. □

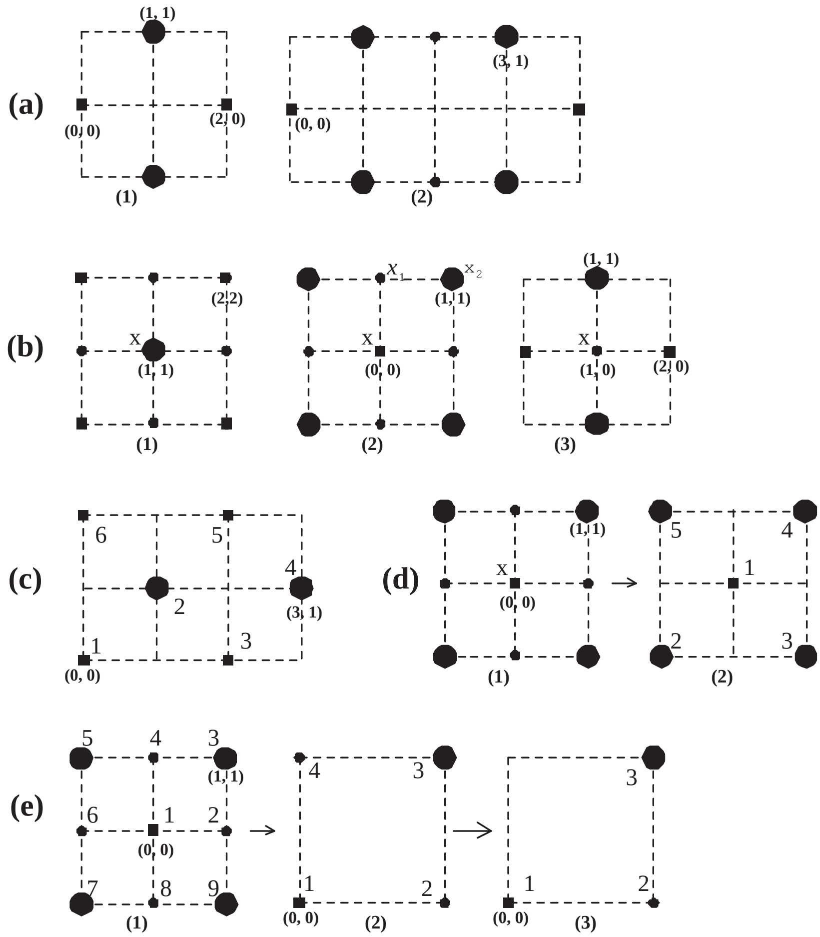

Figure 2: (a) $SC_K^{2,4}$ and $SC_K^{2,8}$; (b) $AN_K(x)$ in $(\mathbf{Z}^2, \kappa^2)$, where x can be a pure open, a pure closed and a mixed point; (c) some examples for the *FPP* in *KTC*; (d-e) explanation of the *FPP* of *K*-retracts.

Example 3.6. *(1) Consider the space $(T_1, \kappa^2_{T_1})$ in Figure 2(d)(2), $T_1 := \{1,2,3,4,5\}$. Then we prove that $(T_1, \kappa^2_{T_1})$ has the FPP. Owing to the K-topological structure of $(T_1, \kappa^2_{T_1})$, we have roughly several kinds of K-continuous self-maps f of T_1 as follows:*
(Case 1) In case $f(1) = 1$, the proof is completed.
(Case 2) In case $f(1) = j \in \{2,3,4,5\}$, since $SN_K(1) = T_1$ in $(T_1, \kappa^2_{T_1})$ and $SN_K(j) = \{j\}, j \in \{2,3,4,5\}$, it is clear that the point j should be a fixed point of f.
(Case 3) In case $f(j) = 1, j \in \{2,3,4,5\}$, depending on the map f, according to Case (1)∼(2), the proof is completed.
(Case 4) The other cases such as the case that a pure open point is mapped into a pure open point by f are also proved by using the methods similar to the Cases (1)∼(3).

As another proof by using Theorems 3.3 and 3.5, for $x = (0,0)$ in Figure 2(d)(1) we can consider a map $r : SN_K(x) \to T_1$ given by $r(\{(-1,0),(0,-1),(1,0),(0,1)\}) = \{x\}$ and $r(i) = i, i \in \{1,2,3,4,5\}$ (see Figure 2(d)(1) and (2)), where $x = (0,0) = 1$ in Figure 2(d)(2). Then it is clear that r is a K-retraction. By Theorems 3.3 and 3.5, the proof is completed.

(2) Consider the space $(T_2, \kappa^2_{T_2})$ in Figure 2(c), $T_2 := \{1,2,3,4,5,6\}$. Then we prove that $(T_2, \kappa^2_{T_2})$ does not have the FPP as follows: Owing to K-topological structure of $(T_2, \kappa^2_{T_2})$, consider the self-maps g of T_2 as follows:

$$g(\{1,5,6\}) = \{3\}, g(3) = 5, g(2) = 4, g(4) = 2.$$

While g is a K-continuous map, it cannot have any fixed point.
Indeed, we can observe that the space in Figure 2(c) is not a K-retract of $([0,3]_{\mathbb{Z}} \times [0,2]_{\mathbb{Z}}, \kappa^2_{[0,3]_{\mathbb{Z}}\times[0,2]_{\mathbb{Z}}})$.

(3) Consider the space $(T_3, \kappa^2_{T_3})$ in Figure 2(d)(2), $T_3 := \{1,2,3,4\}$. Then we prove that $(T_3, \kappa^2_{T_3})$ has the FPP as follows: Owing to K-topological structure of $(T_3, \kappa^2_{T_3})$, we prove that $(T_3, \kappa^2_{T_3})$ is a K-retract of $(SN_K(0_2), \kappa^2_{SN_K(0_2)})$ by mapping both all mixed points and the point $(-1,1)$ in $SN_K(0_2)$ into to the point 0_2 and the other points remain, where $0_2 := (0,0)$. Then it is clear that the map r is a K-retraction. By Theorem 3.3, since $(SN_K(0_2), \kappa^2_{SN_K(0_2)})$ has the FPP, by Theorem 3.5, the proof is completed.

Remark 3.7. *While the K-topological space $(T_4 := \{1,2,3\}, \kappa^2_{T_4})$ in Figure 2(e) is not a simple K-path, it has the FPP.*

Proof: We see that $(T_4 := \{1,2,3\}, \kappa^2_{T_4})$ is a K-retract of $([0,1]_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}} := Y, \kappa^2_Y)$ in terms of the map $r : (Y, \kappa^2_Y) \to (T_4, \kappa^2_{T_4})$ given by

$$r(i) = i, i \in \{1,2,3\} \text{ and } r(4) \in \{1,2,3\}.$$

Then it is clear that r is a K-retraction. By using the method similar to the assertion of Example 3.6(3), since (Y, κ^2_Y) has the *FPP*, by Theorem 3.5, the proof is completed.
Meanwhile, we can also prove the *FPP* of $(T_4 := \{1,2,3\}, \kappa^2_{T_4})$ without using Theorem 3.5, as follows: let us consider any K-continuous self-maps h of $(T_4, \kappa^2_{T_4})$ in such a way:
In case $h(1) = 1$, the proof is completed.
In case $h(1) = 2$, we should take $h(\{2,3\}) \subset \{2,3\}$.
Then if $h(2) = 2$, the proof is completed and if $h(2) = 3$, then we should take $h(3) = 3$ because of the K-continuity of h at the point 3. □

4. The *AFPP* of *K*-topological spaces

This section studies the *AFPP* of a K-topological space and addresses the issue (⋆2) posed in Section 1.

Definition 4.1. *We say that a space $(X, \kappa^n_X) := X$ in KTC has the AFPP if every K-continuous self-map f of X has a point $x \in X$ such that $f(x) = x$ or $f(x)$ is K-adjacent to x.*

Let us prove the K-homeomorphic invariant of the *AFPP*.

Proposition 4.2. *The AFPP in KTC is a K-homeomorphic invariant.*

Proof: Suppose that (X, κ^n_X) has the *AFPP* and there exists a K-homeomorphism $h : (X, \kappa^n_X) \to (Y, \kappa^n_Y)$. Then we prove that (Y, κ^n_Y) has the *AFPP*. Assume that g is any K-continuous self-map of (Y, κ^n_Y). Then consider the composition $h \circ f \circ h^{-1} := g : (Y, \kappa^n_Y) \to (Y, \kappa^n_Y)$, where f is a K-continuous self-map of (X, κ^n_X). Owing to the hypothesis, assume that $x \in X$ is an almost fixed point for a K-continuous self-map f of (X, κ^n_X). Since h is a K-homeomorphism, there is a point $y \in Y$ such that $h(x) = y$. Let us consider the mapping

$$f(x) = h^{-1} \circ g \circ h(x) = h^{-1}(g(h(x))) = h^{-1}(g(y)). \tag{4.1}$$

Thus, from (4.1) we see $h(f(x)) = g(y)$ and further, owing to the hypothesis of the *AFPP* of (X, κ^n_X) and the K-homeomorphism between (X, κ^n_X) and (Y, κ^n_Y), (*i.e.* the preservation of the K-connectedness by h)

$$g(y) = h(f(x)) \in h(AN(x)) = AN(h(x)) = AN(y), \tag{4.2}$$

which implies that the point $h(x)$ is an almost fixed point of the map g, which implies that (Y, κ^n_Y) has the *AFPP*. □

Theorem 4.3. *In KTC, let (A, κ_A^n) be a K-retract of (X, κ_X^n). If (X, κ_X^n) has the AFPP, then (A, κ_A^n) has also the AFPP, where A need not be a singleton.*

Proof: Consider the inclusion map $i : (A, \kappa_A^n) \to (X, \kappa_X^n)$ and the retraction $r : (X, \kappa_X^n) \to (A, \kappa_A^n)$. Under the hypothesis, to prove the *AFPP* of (A, κ_A^n), take any K-continuous self-map f of (A, κ_A^n). Then consider the composition $i \circ f \circ r := h$ which is a K-continuous self-map of (X, κ_X^n) because the K-continuity supports the composite. Under the hypothesis, we have a point $x \in X$ such that $h(x) \in AN(x)$. Then it is clear that $h(x) \in A$. Owing to the K-retraction from (X, κ_X^n) to (A, κ_A^n), the point x should be an element of A. Based on this approach, take a point x in (A, κ_A^n) such that

$$h(x) = i \circ f \circ r(x) = i \circ f(x) = f(x)$$

because of the property of the inclusion map i. Thus we have $f(x) \in AN(x)$. □

Remark 4.4. *Every $SC_K^{n,l}$ does not have the AFPP.*

In view of Theorem 4.3 and Remark 4.4, we obtain the following:

Corollary 4.5. *Not every compact and connected K-topological space has the AFPP.*

5. Summary and further work

We have studied the *FPP* of a K-retract in *KTC*, which can be helpful to study *FPP* for digital spaces. Furthermore, in n-dimensional Khalimsky space we proved that $SN_K(x) \subset (X, \kappa_X^n)$ has the *FPP*, where $SN_K(x)$ is the smallest open K-topological neighborhood of x. Besides, we have proved that not every compact K-topological space has the *FPP* and the *AFPP*.
The recent paper [9] developed a new type of locally finite space motivated by the *ALF*-space in [19]. However, the notion of a boundary of a given element was missing in Definition 2.5 of [9]. Thus we need to add it as follows:
[Definition of a boundary of a given element of an *SST*(a Space Set Topological space)]: Let $C := (X, N, dim)$ be an *AC* complex, where $X := \{c_j^i | i \in M, j \in M_i'\}$. For each m-cell c^m in X its boundary, denoted by $\partial(\{c^m\})$ (or ∂c^m), is defined as follows:
$\partial c^m := \{c_j^i \mid c_j^i$ is adjacent to (or joins) $c^m, i \lneq m\}$.

As a further work, we can study the *FPP* for category of topological graphs based on *KTC* and the product property of the *FPP* of Khalimsky topological spaces. Besides, by using the several types of continuities in [4], we can investigate the *FPP* of the given categories in [4, 10].

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