Filomat 31:19 (2017), 6205–6210
https://doi.org/10.2298/FIL1719205Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Topological Semigroups and their Prequantale Models

Bin Zhao[a], Changchun Xia[a], Kaiyun Wang[a]

[a]*College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, P.R. China*

Abstract. In this paper, we introduce a condition (Δ) on topological semigroups, and prove that every T_1 topological semigroup satisfying condition (Δ) has a bounded complete algebraic prequantale model. On the basis of this result, we also show that every T_0 topological semigroup satisfying condition (Δ) can be embedded into a compact and locally compact sober topological semigroup.

1. Introduction

Domain theory was originated from the work by Scott (see [17]) in order to provide the mathematical foundation for denotational semantics of programming languages. Moreover, based on the work of Lawson, Hofmann and Stralka (see [8, 9, 12]), domain theory possessed the background of pure mathematics. After about 40 years of domain theory, one can recognize that topology and order theory have been beneficial to each other.

As one of the most central problems in domain theoretic studies of topological spaces, the maximal point space problem can be traced at least thirty years ago, to the work of Scott (see [18]), Kamimura and Tang (see [10]) and Abramsky and Jung (see [1]). This problem requires the characterization of those T_1 spaces which are homeomorphic to the spaces of maximal elements of some special posets. More precisely, a poset model of a topological space X is a poset P together with a homeomorphism $\phi\colon X \longrightarrow \max(P)$ ($\max(P)$ is the subspace of the Scott space $\Sigma(P)$ consisting of maximal points of P). In [13], Lawson proved that a space has an ω-continuous dcpo model satisfying Lawson condition iff the space is a Polish space. Liang and Keimel (see [14]) proved that a space has a continuous poset model satisfying the Lawson condition iff the space is Tychonoff. In [11], Kopperman, Künzi and Waszkiewicz proved that every complete metric space has a bounded complete continuous dcpo model. In [2], Ali-Akbari, Honari and Pourmahdian showed that any T_1 space has a continuous poset model. In [20] (also in [5]), it was proved that every T_1 space has a bounded complete algebraic poset model. On the basis of the above result, Zhao and Xi (see [21]) proved that every T_1 space has a dcpo model. Furthermore, Xi and Zhao (see [19]) showed that a T_1 space X is well-filtered iff the dcpo model of X given in [21] is well-filtered.

2010 *Mathematics Subject Classification*. Primary 22A25; Secondary 06B35, 06F05
Keywords. Topological semigroup, prequantale, Scott topology, prequantale model
Received: 08 January 2017; Revised: 11 July 2017; Accepted: 19 July 2017
Communicated by Ljubiša D.R. Kočinac
Corresponding author: Kaiyun Wang
This research is supported by the National Natural Science Foundation of China (Grant nos. 11531009, 11301316) and the Fundamental Research Funds for the Central Universities (Grant no. GK201501001)
Email addresses: zhaobin@snnu.edu.cn (Bin Zhao), xiachangchun@snnu.edu.cn (Changchun Xia), wangkaiyun@snnu.edu.cn (Kaiyun Wang)

Motivated by considerations from the maximal point space problem, one may ask the following question: whether every T_1 topological semigroup has a similar model? We propose the notions of maximal point topological semigroups and prequantale models of topological semigroups. A prequantale model of a topological semigroup S is a prequantale P together with a topological isomorphism $\phi : S \longrightarrow \max(P)$ ($\max(P)$ is the maximal point topological semigroup of P). We prove that every T_1 topological semigroup satisfying condition (Δ) has a bounded complete algebraic prequantale model. Hence the above question is partly solved. As we know, compactness is a rather weak property in the case of non-Hausdorff spaces. For example, every space having a least element in its specialization order is trivially compact. Just as compactness plays an important role in the study of Hausdorff topological semigroups, it seems appropriate to look for a stronger notion than that of compactness in the setting of the general topological semigroups. In non-Hausdorff topology, compact and local compact sober spaces have been shown to provide a proper language for the study of domain theory (see [7]). So the notion of compact and local compact sober topological semigroups is a promising candidate for such a strengthened notion. On the basis of prequantale models of topological semigroups, we show that every T_0 topological semigroup satisfying condition (Δ) can be embedded into a compact and locally compact sober topological semigroup. For notions and concepts concerned, but not explained, please refer to [3, 4, 6, 16].

2. Preliminaries

Let $(L, \leq)$ be a poset. We use $\max(L)$ to denote the set of all maximal elements of L. A subset D of L is *directed* provided that it is nonempty and every finite subset of D has an upper bound in D. The poset L is a *directed complete partially ordered set* (abbreviated: *dcpo*) if every directed subset of L has a supremum. It is *bounded complete* if every upper bounded subset has a supremum.

Let $(L, \leq)$ be a poset. We say that x is *way below* y, in symbol $x \ll y$, if and only if for all directed subsets D of L for which $\sup D$ exists, the relation $y \leq \bigvee D$ always implies the existence of a $d \in D$ with $x \leq d$. An element satisfying $x \ll x$ is said to be *compact*. $K(L)$ denotes the set of all compact elements of L. The poset L is called *continuous* if for every $x \in L$, the subset $\Downarrow x = \{u \in L : u \ll x\}$ is directed and $x = \bigvee\{u \in L : u \ll x\}$. It is *algebraic* if for every $x \in L$, the subset $\downarrow x \cap K(L)$ is directed and $x = \bigvee(\downarrow x \cap K(L))$.

Definition 2.1. ([6]) Let L be a poset and $U \subseteq L$. Then U is called *Scott open* if and only if it satisfies:
(1) U is an upper set;
(2) For all directed subsets D of L with $\bigvee D$ existing, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$.

The collection of all Scott open subsets of L forms a topology σ which is called the *Scott topology*.

Definition 2.2. A triple $(X, \leq, \cdot)$ is called a *prequantale* if it satisfies:
(1) $(X, \leq)$ is a poset;
(2) $(X, \cdot)$ is a semigroup;
(3) For all directed subsets D of X with $\bigvee D$ existing, $a \cdot \bigvee D = \bigvee(a \cdot D)$ and $(\bigvee D) \cdot a = \bigvee(D \cdot a)$, where $a \cdot D = \{a \cdot d : d \in D\}$ and $D \cdot a = \{d \cdot a : d \in D\}$.

Remark 2.3. Note that a prequantale defined in [15] must be a dcpo, but in Definition 2.2, we do not ask that a prequantale is a dcpo.

A prequantale $(X, \leq, \cdot)$ is called *continuous (algebraic)*, if $(X, \leq)$ is continuous (algebraic).

Remark 2.4. Obviously, an algebraic prequantale must be continuous. Conversely, it is not true. For example, $((0, 1], \leq, \times)$ is a continuous prequantale but not algebraic, where $\times$ is the usual multiplication.

3. Prequantale Models of Topological Semigroups

A *topological semigroup* consists of a semigroup S and a topology τ on the set S such that the multiplication $\cdot$ in S, as a mapping of $S \times S$ to S, is continuous when $S \times S$ is endowed with the product topology, or equivalently, for each x and y in S and each open neighborhood W of $x \cdot y$, there exist open neighborhoods U of x and V of y such that $U \cdot V \subseteq W$, where $U \cdot V = \{u \cdot v : u \in U, v \in V\}$. Let $(S, \tau_1, \cdot)$, $(T, \tau_2, \bullet)$ be topological semigroups. A mapping $f : S \longrightarrow T$ is called a *topological (embedding) isomorphism*, if it is both a topological (embedding) homeomorphism and a semigroup homomorphism.

Definition 3.1. Let $(P, \leq, *)$ be an ordered semigroup. A triple $(\max(P), \sigma|_{\max(P)}, *)$ is called a *maximal point topological semigroup* of P, if it satisfies:
(1) $(\max(P), *)$ is a subsemigroup of P;
(2) $(\max(P), \sigma|_{\max(P)}, *)$ is a topological semigroup.

Remark 3.2. Let $(P, \leq, *)$ be an ordered semigroup. If $(P, \sigma, *)$ is a topological semigroup and $(\max(P), *)$ is a subsemigroup of P, then $(\max(P), \sigma|_{\max(P)}, *)$ is a topological semigroup.

Definition 3.3. Let $(S, \tau, \cdot)$ be a topological semigroup. A *prequantale model* is a prequantale P together with a topological isomorphism ϕ: $(S, \tau, \cdot) \longrightarrow (\max(P), \sigma|_{\max(P)}, *)$, where $(\max(P), \sigma|_{\max(P)}, *)$ is the maximal point topological semigroup of P. We shall use (P, ϕ) to denote a prequantale model of S.

Similarly, we can define *bounded complete (continuous, algebraic) prequantale models* of topological semigroups.

Lemma 3.4. ([20]) *Every T_1 space has a bounded complete algebraic poset model.*

Condition (Δ) Let $(S, \tau, \cdot)$ be a topological semigroup and $U, V \in \tau$. Then $U \cdot V \in \tau$.

Example 3.5. (1) Clearly, every topological group satisfies condition (Δ).
(2) $([0, 1], \tau|_{[0,1]}, \times)$ is a topological semigroup satisfying condition (Δ), but not a topological group, where τ and $\times$ are the usual topology and multiplication on the set of real numbers $\mathbb{R}$, respectively.
(3) Let $L = \{\perp, a, \top\}$ be a chain with $\perp < a < \top$. Define a binary operation $*$ on L as follows:

$*$	$\perp$	a	$\top$
$\perp$	$\perp$	$\perp$	$\perp$
a	$\perp$	a	a
$\top$	$\perp$	$\top$	$\top$

Then $(L, *)$ is a semigroup. One can see that $(L, \tau, *)$ is a topological semigroup, where τ is the Alexandroff topology on L. Clearly, $\{\top\}$ and $\{\perp, a, \top\}$ are open sets, but $\{\top\} * \{\perp, a, \top\} = \{\perp, \top\}$ is not an open set. Thus $(L, \tau, *)$ does not satisfy condition (Δ).

Theorem 3.6. *Every T_1 topological semigroup satisfying condition (Δ) has a bounded complete algebraic prequantale model.*

Proof. Let $(S, \tau, \cdot)$ be a T_1 topological semigroup satisfying condition (Δ) and $Filt^l(\tau^*)$ be the set of all filters of nonempty open sets of S that has a nonempty intersection, that is, $Filt^l(\tau^*) = \{\mathcal{A} \in Filt(\tau^*) : \bigcap \mathcal{A} \neq \emptyset\}$, where $\tau^* = \tau \backslash \{\emptyset\}$. Then $K(Filt^l(\tau^*)) = \{L(U) : U \in \tau^*\}$ and $\max(Filt^l(\tau^*)) = \{N(x) : x \in S\}$, where $L(U) = \{V \in \tau^* : U \subseteq V\}$ and $N(x) = \{U \in \tau^* : x \in U\}$ is the open neighborhood filter of $x \in S$. The binary operation $\otimes$ on $Filt^l(\tau^*)$ is defined by $\mathcal{A} \otimes \mathcal{B} = \uparrow \{U \cdot V : U \in \mathcal{A}, V \in \mathcal{B}\}$, where $U \cdot V = \{u \cdot v : u \in U, v \in V\}$. Since $(S, \tau, \cdot)$ satisfies condition (Δ), $\mathcal{A} \otimes \mathcal{B}$ is a filter of τ^*. As $\bigcap \mathcal{A} \neq \emptyset$ and $\bigcap \mathcal{B} \neq \emptyset$, we have that $\bigcap(\mathcal{A} \otimes \mathcal{B}) \neq \emptyset$. Then the binary operation $\otimes$ is well defined.

Define a mapping $\phi : (S, \tau, \cdot) \longrightarrow (\max(Filt^l(\tau^*)), \sigma|_{\max(Filt^l(\tau^*))}, \otimes)$ as follows:

$$\forall\, x \in S,\ \ \phi(x) = N(x).$$

It follows from Lemma 3.4 that $(Filt^l(\tau^*), \phi)$ is a bounded complete algebraic poset model of (S, τ). Next, we shall prove that $(Filt^l(\tau^*), \phi)$ is a bounded complete algebraic prequantale model of $(S, \tau, \cdot)$.

Step 1. One can easily see that$(Filt^l(\tau^*), \otimes)$ is a semigroup. For each directed family $\{\mathcal{D}_i\}_{i\in I} \subseteq Filt^l(\tau^*)$ with $\bigvee_{i\in I} \mathcal{D}_i$ existing and $\mathcal{A} \in Filt^l(\tau^*)$, since $\bigvee_{i\in I} \mathcal{D}_i = \bigcup_{i\in I} \mathcal{D}_i$, one can verify that $\mathcal{A} \otimes (\bigvee_{i\in I} \mathcal{D}_i) = \bigvee_{i\in I}(\mathcal{A} \otimes \mathcal{D}_i)$ and $(\bigvee_{i\in I} \mathcal{D}_i) \otimes \mathcal{A} = \bigvee_{i\in I}(\mathcal{D}_i \otimes \mathcal{A})$.

Step 2. It follows from Lemma 3.4 that ϕ is a topological homeomorphism. We shall prove that ϕ is a semigroup homomorphism. Clearly, $N(x)\otimes N(y) \subseteq N(x\cdot y)$. Let $x, y \in S$. For all $U \in N(x \cdot y)$, there exist open neighborhoods A of x and B of y such that $A \cdot B \subseteq U$. Then $U \in N(x)\otimes N(y)$ and hence $N(x\cdot y) = N(x)\otimes N(y)$, that is, $\phi(x \cdot y) = \phi(x) \otimes \phi(y)$. Moreover, $(\max(Filt^l(\tau^*)), \otimes)$ is a semigroup. Therefore, ϕ is a semigroup homomorphism. □

Proposition 3.7. *Let $(S, \tau, \cdot)$ be a topological semigroup with condition (Δ). Then $(Filt^l(\tau^*), \sigma, \otimes)$ is a topological semigroup.*

Proof. By the proof of Theorem 3.6, $(Filt^l(\tau^*), \otimes)$ is a semigroup and the subsets of $Filt^l(\tau^*)$ of the form $\{\uparrow L(U) : U \in \tau^*\}$ form a basis of the Scott topology on $Filt^l(\tau^*)$. For all $\mathcal{A}, \mathcal{B} \in Filt^l(\tau^*)$ and $\mathcal{A} \otimes \mathcal{B} \in \uparrow L(U)$, we have that

$$L(U) \subseteq \mathcal{A} \otimes \mathcal{B} = \bigcup_{W\in\mathcal{A}} L(W) \otimes \bigcup_{V\in\mathcal{B}} L(V) = \bigcup_{W\in\mathcal{A}, V\in\mathcal{B}} L(W) \otimes L(V).$$

Since $L(U)$ is a compact element of $Filt^l(\tau^*)$, there exist $L(W_1) \subseteq \mathcal{A}$ and $L(V_1) \subseteq \mathcal{B}$ such that $L(U) \subseteq L(W_1) \otimes L(V_1)$. We claim that $\uparrow L(W_1)\otimes \uparrow L(V_1) \subseteq\uparrow L(U)$, where $\uparrow L(W_1)\otimes \uparrow L(V_1) = \{\mathcal{C}\otimes\mathcal{D} : \mathcal{C} \in\uparrow L(W_1), \mathcal{D} \in\uparrow L(V_1)\}$. For all $\mathcal{F} \in Filt^l(\tau^*)$, $\mathcal{F} \in\uparrow L(W_1)\otimes \uparrow L(V_1)$, there exist $\mathcal{C}_1$, $\mathcal{D}_1 \in Filt^l(\tau^*))$ such that $L(W_1) \subseteq \mathcal{C}_1$, $L(V_1) \subseteq \mathcal{D}_1$ and $\mathcal{F} = \mathcal{C}_1 \otimes \mathcal{D}_1$. Since $L(U) \subseteq L(W_1) \otimes L(V_1) \subseteq \mathcal{C}_1 \otimes \mathcal{D}_1 = \mathcal{F}$, we have that $\mathcal{F} \in\uparrow L(U)$. One can conclude that $\uparrow L(W_1)\otimes \uparrow L(V_1) \subseteq\uparrow L(U)$. Since $\uparrow L(W_1)$ and $\uparrow L(V_1)$ are Scott open sets of $Filt^l(\tau^*)$, $(Filt^l(\tau^*), \sigma, \otimes)$ is a topological semigroup. □

Lemma 3.8. *Let $(S, \tau, \cdot)$ be a T_0 topological semigroup satisfying condition (Δ). Then we can obtain a topological embedding $\phi\colon (S, \tau, \cdot) \longrightarrow (Filt^l(\tau^*), \sigma, \otimes)$ defined by $\phi(x) = N(x)$.*

4. Embeddings of T_0 Topological Semigroups into Compact and Locally Compact Sober Topological Semigroups

Following the results of section 3, in this section we shall investigate embeddings of T_0 topological semigroups into compact and locally compact sober topological semigroups.

Definition 4.1. ([6]) A binary relation $\prec$ on a poset $(P, \leq)$ is called an *auxiliary relation* if it satisfies the following conditions for all $p, q, x, y \in P$:

(1) $x \prec y$ implies $x \leq y$;

(2) $p \leq x \prec y \leq q$ implies $p \prec q$;

(3) If F is a finite subset of P, $F \prec y$ implies that there exists $r \in P$ such that $F \prec r \prec y$, where we write $F \prec r$ if $a \prec r$ for all $a \in F$.

For convenience, we denote by $\Uparrow p$ the set $\{x \in P : p \prec x\}$ and denote by $\Downarrow p$ the set $\{x \in P : x \prec p\}$.

Let $(P, \leq)$ be a poset with an auxiliary relation $\prec$. A directed lower subset I of P is called a *round ideal* provided that for each $p \in I$, there exists $q \in I$ such that $p \prec q$. We use $\mathcal{I}(P)$ to denote the family of all round ideals of P. Then $\mathcal{I}(P)$ is a dcpo under the order of inclusion.

We call an auxiliary relation $\prec$ on a poset P *approximating* if for all $p, q \in P$, $\Downarrow p \subseteq\Downarrow q$ implies $p \leq q$. One can easily see that the way-below relation in every continuous poset is an approximating auxiliary relation. We call the topology generated by the set $\{\Uparrow p : p \in P\}$ *pseudo-Scott topology* and denote it by $\mathbf{P}\sigma$.

Definition 4.2. Let $(S, \leq, \cdot)$ be an ordered semigroup. The auxiliary relation $\prec$ on S is called *stable*, if it satisfies the following conditions for all $x_1, x_2, y_1, y_2 \in S$:

(1) $x_1 \prec y_1$ and $x_2 \prec y_2$ imply $x_1 \cdot x_2 \prec y_1 \cdot y_2$;

(2) $x \prec y_1 \cdot y_2$ implies that there exist $x_1 \prec y_1, x_2 \prec y_2$ such that $x \leq x_1 \cdot x_2$.

If the auxiliary relation $\prec$ on S is stable, we call the quadruple $(S, \leq, \cdot, \prec)$ a *stable ordered semigroup*.

Example 4.3. If $(S, \leq, \cdot)$ is an ordered semigroup, then $D(S)$ is an algebraic quantale and the way-below relation on $D(S)$ is stable, where $D(S)$ is the family of all lower sets of S.

Proposition 4.4. *The stable ordered semigroup $(S, \leq, \cdot, \prec)$ endowed with pseudo-Scott topology $\mathbf{P}\sigma$ is a topological semigroup.*

Proof. For all $x, y_1, y_2 \in S$ and $y_1 \cdot y_2 \in \Uparrow x$, we shall show that there exist $y_1 \in \Uparrow x_1$ and $y_2 \in \Uparrow x_2$ such that $\Uparrow x_1 \cdot \Uparrow x_2 \subseteq \Uparrow x$. Since S is a stable ordered semigroup, there exist $x_1 \prec y_1, x_2 \prec y_2$ such that $x \leq x_1 \cdot x_2$. For all $t \in \Uparrow x_1 \cdot \Uparrow x_2$, there exist $x_1 \prec a, x_2 \prec b$ such that $t = a \cdot b$. As S is a stable ordered semigroup, we have that $x_1 \cdot x_2 \prec a \cdot b = t$. Then $t \in \Uparrow x$ and $\Uparrow x_1 \cdot \Uparrow x_2 \subseteq \Uparrow x$. Therefore, $(S, \mathbf{P}\sigma, \cdot)$ is a topological semigroup. □

The round completion $\mathcal{I}(P)$ of a poset P with an auxiliary relation $\prec$ is a continuous dcpo and for all $I, J \in \mathcal{I}(P)$, $I \ll J$ if and only if there exists $j \in J$ such that $I \subseteq \Downarrow j$ (see [11]).

Proposition 4.5. *Let $(P, \leq, \cdot, \prec)$ be a stable ordered semigroup. Then the following statements hold.*

(1) If the binary operation $\odot$ on $\mathcal{I}(P)$ is defined by $A \odot B = \downarrow (A \cdot B)$ for all $A, B \in \mathcal{I}(P)$, then $\mathcal{I}(P)$ is a continuous prequantale, where $A \cdot B = \{a \cdot b : a \in A, b \in B\}$.

(2) $(\mathcal{I}(P), \sigma, \odot)$ is a topological semigroup.

Proof. (1) Obviously, the binary operation $\odot$ on $\mathcal{I}(P)$ is well defined and $(\mathcal{I}(P), \odot)$ is a semigroup. For every directed subset $\mathcal{D} \subseteq \mathcal{I}(P)$, we have that $\bigvee \mathcal{D} = \bigcup \mathcal{D}$. One can easily verify that $\mathcal{I}(P)$ is a continuous prequantale.

(2) Since $\mathcal{I}(P)$ is a continuous dcpo, one can see that the way-below relation $\ll$ on $\mathcal{I}(P)$ is an approximating auxiliary relation. For $I_1, I_2, J_1, J_2 \in \mathcal{I}(P)$, if $I_1 \ll J_1$, $I_2 \ll J_2$, there exist $j_1 \in J_1$, $j_2 \in J_2$ such that $I_1 \subseteq \Downarrow j_1$ and $I_2 \subseteq \Downarrow j_2$, which imply $I_1 \odot I_2 \subseteq \Downarrow j_1 \odot \Downarrow j_2 = \Downarrow (j_1 \cdot j_2)$. Since $j_1 \in J_1$ and $j_2 \in J_2$, we have that $j_1 \cdot j_2 \in J_1 \odot J_2$ and $I_1 \odot I_2 \ll J_1 \odot J_2$. Let $J \ll I_1 \odot I_2$. Since $\mathcal{I}(P)$ is a continuous dcpo, $I_1 \odot I_2 = \bigcup\{C \in \mathcal{I}(P) : C \ll I_1\} \odot \bigcup\{D \in \mathcal{I}(P) : D \ll I_2\} = \bigcup\{C \odot D : C \ll I_1, D \ll I_2\}$. Then there exist $C_1 \ll I_1, D_1 \ll I_2$ such that $J \subseteq C_1 \odot D_1$. So, the way-below relation $\ll$ on $\mathcal{I}(P)$ is a stable approximating auxiliary relation. It follows from Proposition 4.4 that $(\mathcal{I}(P), \sigma, \odot)$ is a topological semigroup. □

Proposition 4.6. *Let $(P, \leq, \cdot \prec)$ be a stable ordered semigroup. Then the following statements hold.*

(1) The mapping $j : P \to \mathcal{I}(P)$ defined by $j(p) = \Downarrow p$ satisfies $p \leq q \Rightarrow j(p) \subseteq j(q)$, $p \prec q \Rightarrow j(p) \ll j(q)$ and $j(p \cdot q) = j(p) \odot j(q)$ for all $p, q \in P$.

(2) If $\prec$ is approximating, then j is a topological semigroup embedding of $(P, \mathbf{P}\sigma, \cdot)$ into $(\mathcal{I}(P), \sigma, \odot)$ such that for all $p, q \in P$, $p \leq q \Leftrightarrow j(p) \subseteq j(q)$, $p \prec q \Leftrightarrow j(p) \ll j(q)$.

Proof. (1) Obviously, $p \leq q \Rightarrow j(p) \subseteq j(q)$ and $p \prec q \Rightarrow j(p) \ll j(q)$. We shall prove that $j(p \cdot q) = j(p) \odot j(q)$ for all $p, q \in P$. For all $t \in j(p) \odot j(q)$, there exist $x \prec p$, $y \prec q$ such that $t \leq x \cdot y \prec p \cdot q$. Then $t \in j(p \cdot q)$ and $j(p) \odot j(q) \subseteq j(p \cdot q)$. For all $t \in j(p \cdot q)$, there exist $x \prec p$ and $y \prec q$ such that $t \leq x \cdot y$. Then $t \in j(p) \odot j(q)$ and $j(p \cdot q) \subseteq j(p) \odot j(q)$. Hence $j(p \cdot q) = j(p) \odot j(q)$.

(2) Immediate by Theorem 2.3(b) in [11] and (1). □

Lemma 4.7. ([6]) *If P is a continuous dcpo with a bottom element, then (P, σ) is a compact and locally compact sober space.*

Theorem 4.8. *Every T_0 topological semigroup satisfying condition (Δ) can be embedded into a compact and locally compact sober topological semigroup.*

Proof. Let $(S, \tau, \cdot)$ be a T_0 topological semigroup satisfying condition (Δ), and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2 \in Filt^l(\tau^*)$ with $\mathcal{A}_1 \ll \mathcal{B}_1$ and $\mathcal{A}_2 \ll \mathcal{B}_2$. Since $Filt^l(\tau^*)$ is algebraic, there exist compact elements $L(U)$, $L(V)$ of $Filt^l(\tau^*)$ such that $\mathcal{A}_1 \subseteq L(U) \subseteq \mathcal{B}_1$ and $\mathcal{A}_2 \subseteq L(V) \subseteq \mathcal{B}_2$. Then $\mathcal{A}_1 \otimes \mathcal{A}_2 \subseteq L(U) \otimes L(V) \subseteq \mathcal{B}_1 \otimes \mathcal{B}_2$. Since $L(U) \otimes L(V) = L(U \cdot V)$, we have that $L(U) \otimes L(V)$ is a compact element of $Filt^l(\tau^*)$ and $\mathcal{A}_1 \otimes \mathcal{A}_2 \ll \mathcal{B}_1 \otimes \mathcal{B}_2$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in Filt^l(\tau^*)$ with $\mathcal{A} \ll \mathcal{B} \otimes \mathcal{C}$. Since

$$\mathcal{B} \otimes \mathcal{C} = \bigcup_{W \in \mathcal{B}} L(W) \otimes \bigcup_{V \in \mathcal{C}} L(V) = \bigcup_{W \in \mathcal{B}, V \in \mathcal{C}} L(W) \otimes L(V),$$

there exist $W_1 \in \mathcal{B}$, $V_1 \in \mathcal{C}$ such that $L(W_1) \subseteq \mathcal{B}, L(V_1) \subseteq \mathcal{C}$ and $\mathcal{A} \subseteq L(W_1) \otimes L(V_1)$. Therefore, $(Filt^l(\tau^*), \subseteq, \otimes, \ll)$ is a stable algebraic prequantale. By Proposition 4.5, we have that $(\mathcal{I}(Filt^l(\tau^*)), \sigma, \odot)$ is a topological semigroup. Since $Filt^l(\tau^*)$ has the bottom element $\{S\}$, we conclude that $\{\{S\}\}$ is the bottom element of $\mathcal{I}(Filt^l(\tau^*))$. So $\mathcal{I}(Filt^l(\tau^*))$ is a continuous dcpo with the bottom element $\{\{S\}\}$. By Lemma 4.7, $(\mathcal{I}(Filt^l(\tau^*)), \sigma)$ is a compact and locally compact sober space. Thus $(\mathcal{I}(Filt^l(\tau^*)), \sigma, \odot)$ is a compact and locally compact sober topological semigroup. By Propositions 3.7 and 4.6, the topological semigroup $(Filt^l(\tau^*), \sigma, \otimes)$ can be embedded into a topological semigroup $(\mathcal{I}(Filt^l(\tau^*)), \sigma, \odot)$. It follows from Remark 3.8 that every T_0 topological semigroup satisfying condition (Δ) can be embedded into a compact and locally compact sober topological semigroup. $\square$

5. Acknowledgement

The authors would like to thank the Section Editor, Professor Ljubiša Kočinac and the anonymous reviewer for their valuable comments and suggestions.

References

[1] S. Abramsky, A. Jung. Domain Theory, Handbook of Logic in Computer Science. Oxford University Press, 1994.
[2] M. Ali-Akbari, B. Honari, M. Pourmahdian, Any T_1 space has a continuous poset model, Topology Appl. 156 (2009) 2240–2245.
[3] A. Arhangel'skii, M. Tkachenko, Topological Groups and Related Structures, Atlantis Press, Amsterdam-Paris, 2008.
[4] R. Engelking, General Topology, Polish Scientific Publishers, Warsaw, 1977.
[5] M. Erné, Algebraic models for T_1 spaces. Topology Appl. 158 (2011) 945–962.
[6] G. Gierz et al., Continuous Lattices and Domain, Encyclopedia of Mathematics and its Applications, vol. 93, Cambridge University Press, 2003.
[7] J. Goubault-Larrecq, Non-Hausdorff Topology and Domain Theory, New Mathematical Monograph, vol. 22, Cambridge University Press, New York, 2013.
[8] K.H. Hofmann, A. Stralka, Mapping clylinders and compact monoids, Math. Ann. 205 (1973) 219–239.
[9] K.H. Hofmann, A. Stralka, Push-outs and strict projective limits of semilattices, Semigroup Forum 5 (1973) 243–261.
[10] T. Kamimura, A. Tang, Total objects of domains, Theor. Comput. Sci. 34 (1984) 275–288.
[11] R. Kopperman, H.-P.A. Künzi, P. Waszkiewicz, Bounded complete models of topological spaces, Topology Appl. 139 (2004) 285–297.
[12] J.D. Lawson, Topological semilattices with small semilattices, J. London Math. Soc. 1 (1969) 719–724.
[13] J.D. Lawson, Spaces of maximal points, Math. Structures Comput. Sci. 7 (1997) 543–555.
[14] J. Liang, K. Keimel, Order environments of topological spaces, Acta Math. Sinica, English Ser. 20 (2004) 943–948.
[15] J. Paseka, Projective quantales: a general view, Internat. J. Theoret. Phys. 47 (2007) 291–296.
[16] K.I. Rosenthal, Quantales and their Applications, Longman Scientific & Technical, New York, 1990.
[17] D.S. Scott, Continuous lattices, Toposes, Algebraic Geometry and Logic, Lecture Notes in Mathematics, vol. 274, Springer-Verlag, Berlin, 1972, pp. 97–136.
[18] D. S. Scott, Domains for denotational semantics, in: M. Nielsen, E.M. Schmidt (Eds.), Automata, Languages and Programming, Lecture Notes in Computer Science, Springer, Berlin, 140 (1982) 577–613.
[19] X. Xi, D. Zhao, Well-filtered spaces and their dcpo models, Math. Structures Comput. Sci. 27 (2017) 507–515.
[20] D. Zhao, Poset models of topological spaces, in: Proceeding of International Conference on Quantitative Logic and Quantification of Software, Global-Link Publisher (2009) 229–238.
[21] D. Zhao, X. Xi, Directed complete poset models of T_1 spaces, Math. Proc. Cambridge Philos. Soc., in press (doi:10.1017/S0305004116000888).