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An Inequality for Warped Product Semi-Invariant Submanifolds of a Normal Paracontact Metric Manifold

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Abstract. The aim of this paper is to study the warped product semi-invariant submanifolds in a normal paracontact metric space form. We obtain some characterization and new geometric obstructions for the warped product type $M_\perp \times_f M_T$. We establish a general inequality among the trace of the induced tensor, laplace operator, the squared norms of the second fundamental form and warping function . These inequalities are discussed and we obtain some new results.

1. Introduction

The geometric inequalities of warped product submanifolds have been studied since B-Y. Chen introduced the notion of a CR-warped product submanifold in a Kaehler manifold and established inequalities for the fundamental form in terms of warping function[3].

In a natural way, warped products appeared in differential geometry generalizing the class of Riemannian product manifolds to much larger one, called warped product manifolds, which are applied in general relativity to model the standard space time.

Recently, Uddin, et al [6, 9] obtained some inequalities of warped product submanifolds in cosymplectic and nearly trans-Sasakian manifolds. They obtained an inequality for the length of the second fundamental form of the warped product submanifold a nearly cosymplectic manifold in terms of warping function, discussed this inequality and found some new results.

In [4], authors obtained a characterization for warped product submanifolds in terms of warping function and shape operator and gave an inequality for squared norm of the second fundamental form.

Motivated by the studies of the above authors, in this paper, we extend this idea into a normal paracontact metric manifold, which has not been attempted so far, and derive the geometric inequalities of non-trivial warped product semi-invariant submanifold and obtain an inequality involving the trace of the induced tensor and warping function.

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2. Preliminaries

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with structure tensors (ϕ, ξ, η, g), where ϕ is (1,1)-type tensor field, ξ is a vector field, η is dual of ξ and g is also Riemannian metric tensor on $\bar{M}$. If we have

$$\phi^2 X = X - \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2}$$

for any vector fields X, Y on $\bar{M}$, then $\bar{M}$ is called almost paracontact metric manifold. An almost paracontact metric manifold $\bar{M}$ is said to be normal if

$$(\bar{\nabla}_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3}$$

for any vector fields on $\bar{M}$, where $\bar{\nabla}$ denotes the Riemannian connection on $\bar{M}$[7]. (3) implies that

$$\bar{\nabla}_X \xi = \phi X \ \ and \ \ (\bar{\nabla}_X \eta)Y = g(\phi X, Y). \tag{4}$$

On the other hand, if a normal paracontact metric manifold $\bar{M}$ has a constant-c, denoted by $\bar{M}(c)$, then its the Riemannian curvature tensor $\bar{R}$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} \\ &+ \frac{1}{4}(c-1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &- g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned} \tag{5}$$

for any vector fields X, Y, Z on $\bar{M}$[7].

Now, let M be an isometrically immersed submanifold in a normal paracontact metric manifold $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ be the differentiable vector fields set tangent and normal to M, respectively. Also we denote by ∇ and $\nabla^{\perp}$ induced connections on $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{7}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are the second fundamental form and shape operatory for the immersed of M into $\bar{M}$, respectively. They are related as

$$g(h(X, Y), V) = g(A_V X, Y). \tag{8}$$

By R, we denote the Riemannian curvature tensor of ∇, then we have

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \tag{9}$$

where the covariant derivative of h is defined by

$$(\nabla_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{10}$$

for any $X, Y, Z \in \Gamma(TM)$.

Let M be an immersed submanifold of a normal paracontact metric manifold $\bar{M}$. For any $X \in \Gamma(TM)$, we can set

$$\phi X = TX + NX, \tag{11}$$

where TX and NX denote the tangential and normal components of ϕX, respectively. In the same way, for any $V \in \Gamma(T^{\perp}M)$, we can write

$$\phi V = BV + CV, \tag{12}$$

where BV(resp. CV) are the tangential(resp. normal) components of ϕV.
The squared norm and trace of T at $p \in M$ are, respectively, defined by

$$\|T\|^2 = \sum_{i,j=1}^{n} g^2(Te_i, e_j), \quad trace(T) = \sum_{i=1}^{n} g(Te_i, e_i), \tag{13}$$

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal basis of the tangent space $\Gamma(TM)$.

Definition 2.1. *A submanifold M of a normal paracontact metric manifold $\bar{M}$ is said to be semi-invariant submanifold if there exist two orthogonal distributions $D^{\perp}$ and D^{T} such that*
i.) $TM = D^{\perp} \oplus D^{T}$,
ii.) $D^{\perp}$ is anti-invariant distribution under ϕ, i.e., $\phi(D^{\perp}) \subseteq T^{\perp}M$,
iii.)D^{T} is an invariant distribution ϕ, i.e., $\phi(D^{T}) \subseteq TM$.

Next, let us suppose that M be a semi-invariant submanifold of a normal paracontact metric manifold $\bar{M}$, then the normal bundle $T^{\perp}M$ can be decomposed as follow as;

$$T^{\perp}M = \phi(D^{\perp}) \oplus \mu, \tag{14}$$

where μ is an invariant subbundle of $T^{\perp}M$.

For a differentiable function f on M, the gradient and Hessian form are, respectively, defined by

$$Xf = g(\nabla f, X), \quad \nabla f = grad f, \tag{15}$$

and

$$H^f(X, Y) = X(Yf) - (\nabla_X Y)f = g(\nabla_X grad f, Y), \tag{16}$$

for any $X, Y \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{n}(e_i(f))^2. \tag{17}$$

The laplacian of f is defined by

$$\begin{aligned} \Delta f &= \sum_{i=1}^{n}\{(\nabla_{e_i}e_i)f - e_i(e_i f)\} = -\sum_{i=1}^{n} g(\nabla_{e_i} grad f, e_i) \\ &= -\sum_{i=1}^{n} H^{\ln f}(e_i, e_i). \end{aligned} \tag{18}$$

From the integration on the manifolds theory, for M is a compact, orientable Riemannian manifold without boundary, we have

$$\int_M \Delta f dV = 0, \tag{19}$$

where dV denote the volume element of M[8].

3. Warped Product Manifolds

Bishop and O'Neill defined the notion of warped product manifolds to construct examples of Riemannian manifolds with a negative curvature. These manifolds are naturel generalizations of Riemannian product manifolds. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f be a positive defined differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its canonical projections

$$\pi_1 : M_1 \times M_2 \to M_1, \ \ \pi_2 : M_1 \times M_2 \to M_2.$$

The warped product $M = M_1 \times M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_{1*}(X)\|^2 + f^2(\pi_1(p))\|\pi_{2*}(X)\|^2, \tag{20}$$

for any $X \in \Gamma(TM)$, where $*$ is the stand for the tangent map. So we have $g = \pi_1^* g_1 + (f \circ \pi_1)^2 \pi_2^* g_2$. The function f is called the warping function on M[2].

Next we will give the following Lemma for later use.

Lemma 3.1. *Let $M = M_1 \times_f M_2$ be a warped product manifold. We have*
i.) $\nabla_X Y \in \Gamma(TM_1)$
ii.) $\nabla_Z X = \nabla_X Z = X(\ln f)Z$
iii.) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,
for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, where ∇ and ∇' denote the Riemannian connections on M and M_2, respectively.

We note that a warped product manifold $M = M_1 \times_f M_2$ is characterized by the fact that M_1 and M_2 are totally geodesic and totally umbilical submanifolds of M, respectively. If warped function f is constant, then warped product manifold is said to be Riemannian product.

4. Warped Product Semi-Invariant Submanifolds of A Normal Paracontact Metric Manifold

In this section, we establish warped product semi-invariant submanifolds which are form $M = M_\perp \times_f M_T$, where $M_\perp$ and M_T are anti-invariant and invariant submanifolds of $\bar{M}$, respectively. Furthermore, the co-vector field ξ is tangent to $M_\perp$. Otherwise, the warping function f is constant.

Next, we will give an example for the method presented in this paper is effective.

Example 4.1. *Let M be a submanifold of $\mathbb{R}^7$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, t)$ given by*

$$x_1 = u, x_2 = u\cos\theta, x_3 = u\sin\theta, y_1 = u\cos\alpha, y_2 = usin\alpha, y_3 = -u, t = 2s,$$

where u, θ, α and s denote the arbitrary parameters. It is easy to check that the tangent bundle of M is spanned by the vectors

$$\begin{aligned} e_1 &= \frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial y_1} + \sin\alpha\frac{\partial}{\partial y_2} - \frac{\partial}{\partial y_3} \\ e_2 &= -u\sin\theta\frac{\partial}{\partial x_2} + u\cos\theta\frac{\partial}{\partial x_3}, \ \ e_3 = -u\sin\alpha\frac{\partial}{\partial y_1} + u\cos\alpha\frac{\partial}{\partial y_2} \\ e_4 &= \xi = 2\frac{\partial}{\partial t}. \end{aligned}$$

Now, we define the almost paracontact metric structure ϕ of $\mathbb{R}^7$ by

$$\phi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}, \ \ \phi(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial y_i} \ \ \phi(\frac{\partial}{\partial t}) = 0, \eta = \frac{1}{2}dt, \ \ 1 \leq i \leq n,$$

then we have $\phi^2 X = X - \eta(X)\xi$, $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, $\phi\xi = 0$ *and* $\eta(\xi) = 1$. *On the other hand, with respect to the almost paracontact metric structure* ϕ *of* $\mathbb{R}^7$, *the* $\phi\Gamma(TM)$ *becomes*

$$\begin{aligned}\phi e_1 &= \frac{\partial}{\partial x_1} + \cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial x_3} - \cos\alpha\frac{\partial}{\partial y_1} - \sin\alpha\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}\\ \phi e_2 &= e_2,\ \ \phi e_3 = -e_3\ \ \phi\xi = 0.\end{aligned}$$

Since ϕe_1 *is orthogonal to* M, ϕe_2 *and* ϕe_3 *are tangent to* M, $\Gamma(TM_\perp)$ *and* $\Gamma(TM_T)$ *can be choosen subspace* $\Gamma(TM_\perp) = sp\{e_1, e_4\}$ *and* $\Gamma(TM_T) = sp\{e_1, e_2\}$. *Furthermore, the metric tensor of* M *is given by*

$$g = 4(du^2 + dt^2) + u^2(d\theta^2 + d\alpha^2) = 4g_{M_\perp} \oplus_{u^2} g_{M_T}.$$

Thus M *is a 4-dimensional warped product semi-invariant submanifold of* $\mathbb{R}^7$ *with warping function* $f = u^2$.

Lemma 4.2. *Let* $M = M_\perp \times_f M_T$ *be a semi-invariant submanifold of a normal paracontact metric manifold* $\bar{M}$ *such that* $\xi \in \Gamma(TM_\perp)$. *Then we have*

$$g(h(X, Y), \phi U) = g(X, Y)\eta(U) - g(TX, Y)U\ln f, \tag{21}$$
$$g(h(U, V), \phi W) = -g(h(U, W), \phi V),\quad U, V, W\perp\xi \tag{22}$$

and

$$g(h(U, X), \phi V) = 0, \tag{23}$$

for any $X, Y \in \Gamma(TM_T)$ *and* $U, V, W \in \Gamma(TM_\perp)$.

Proof. By using (3), (6) and Lemma 3.1, we have

$$\begin{aligned} g(h(X, Y), \phi U) &= g(\bar\nabla_X Y, \phi U) = g(\phi\bar\nabla_X Y, U) = g(\bar\nabla_X \phi Y - (\bar\nabla_X\phi)Y, U)\\ &= g(\bar\nabla_X TY, U) - g((\bar\nabla_X\phi)Y, U)\\ &= -g(X, TY)U\ln f + g(X, Y)\eta(U),\end{aligned}$$

which gives us (21). In the same way, we have

$$\begin{aligned} g(h(U, V), \phi W) &= g(\bar\nabla_U V, \phi W) = -g(\bar\nabla_U\phi W, V)\\ &= -g((\bar\nabla_U\phi)W + \phi\bar\nabla_U W, V)\\ &= -g(-g(U, W)\xi - \eta(W)U + 2\eta(W)\eta(U)\xi, V)\\ &- g(\bar\nabla_U W, \phi V)\\ &= -g(h(U, W), \phi V)\end{aligned}$$

and

$$\begin{aligned} g(h(U, X), \phi V) &= g(\bar\nabla_U X, \phi V) = g(\phi\bar\nabla_U X, V)\\ &= g(\bar\nabla_U\phi X - (\bar\nabla_U\phi)X, V)\\ &= g(\nabla_U\phi X, V) - g(-g(U, X)\xi - \eta(X)U\\ &+ 2\eta(X)\eta(U)\xi, V)\\ &= U\ln f g(\phi X, V) = 0.\end{aligned}$$

Thus the proof is complete. □

Lemma 4.3. *Let* $M = M_\perp \times_f M_T$ *be a warped product semi-invariant submanifold of a normal paracontact metric manifold* $\bar{M}$. *Then we have*

$$g(h(\phi X, U), \phi h(X, U)) = \|h(X, U)\|^2, \tag{24}$$

for any $X \in \Gamma(TM_T)$ *and* $U \in \Gamma(TM_\perp)$.

Proof. Making use of (3),(6) and from Lemma 3.1, we have

$$\begin{aligned} g(h(\phi X,U),\phi h(U,X)) &= g(\bar{\nabla}_U\phi X-\nabla_U\phi X,\phi h(X,U)) \\ &= g((\bar{\nabla}_U\phi)X+\phi\bar{\nabla}_UX,\phi h(U,X)) \\ &- g(U\ln f\phi X,\phi h(X,U)) \\ &= -g(g(X,U)\xi+\eta(X)U-2\eta(X)\eta(U)\xi,\phi h(U,X)) \\ &+ g(\bar{\nabla}_UX,h(X,U))=g(h(X,U),h(X,U)), \end{aligned}$$

for any $X\in\Gamma(TM_T)$ and $U\in\Gamma(TM_\perp)$. □

Theorem 4.4. *Let $M=M_\perp\times_f M_T$ be a warped product semi-invariant submanifold of a normal paracontact metric manifold $\bar{M}$ such that $c\neq 1$. Then we have*

$$\begin{aligned} \|h(X,U)\|^2 &= g(X,X)\left\{\frac{1}{4}(c-1)g(\phi U,\phi U)+H^{\ln f}(U,U)-(U\ln f)^2\right\} \\ &+ g(X,TX)\eta(U)U\ln f, \end{aligned} \tag{25}$$

for any $X\in\Gamma(TM_T)$ and $U\in\Gamma(TM_\perp)$.

Proof. By using (9), (10) and taking into account of Lemma 3.1, we have

$$\begin{aligned} g(\bar{R}(U,X)\phi X,\phi U) &= g((\bar{\nabla}_Uh)(X,\phi X)-(\bar{\nabla}_Xh)(U,\phi X),\phi U) \\ &= g(\bar{\nabla}_Uh(X,\phi X)-h(\nabla_UX,\phi X) \\ &- h(X,\nabla_U\phi X),\phi U)-g(\bar{\nabla}_Xh(U,\phi X) \\ &- h(\nabla_XU,\phi X)-h(U,\nabla_X\phi X),\phi U). \end{aligned}$$

By virtue of (21) and (23), we obtain

$$\begin{aligned} g(\bar{R}(U,X)\phi X,\phi U) &= Ug(h(X,\phi X),\phi U)-g(\bar{\nabla}_U\phi U,h(X,\phi X)) \\ &- g(h(\nabla_UX,\phi X),\phi U)-g(h(\nabla_U\phi X,X),\phi U) \\ &- Xg(h(U,\phi X),\phi U)+g(h(U,\phi X),\nabla_X\phi U) \\ &+ U\ln f g(h(X,\phi X),\phi U)+g(h(\nabla_X\phi X,U),\phi U) \\ &= U[g(X,TX)\eta(U)-g(X,X)U\ln f] \\ &- g(\phi\bar{\nabla}_UU,h(\phi X,X))-U\ln f g(h(X,\phi X),\phi U) \\ &+ g(h(U,\phi X),\phi\bar{\nabla}_XU) \\ &+ g(h(\nabla_X\phi X,U),\phi U)-U\ln f g(h(\phi X,X),\phi U). \end{aligned}$$

Also considering Lemma 4.2 and $M_\perp$ is totally geodesic in M, we reach

$$
\begin{aligned}
g(\bar{R}(U,X)\phi X,\phi U) &= g(X,TX)\eta(\nabla_U U) - g(X,X)U^2(\ln f)\\
&- g(\phi\nabla_U U, h(X,\phi X)) - U\ln f\{g(X,TX)\eta(U)\\
&- g(X,X)U\ln f\} + \|h(X,U)\|^2\\
&+ g(h(\nabla'_X\phi X - g(X,\phi X)\nabla\ln f, U),\phi U)\\
&= g(X,TX)\eta(\nabla_U U) - g(X,X)U^2(\ln f)\\
&- g(X,TX)\eta(\nabla_U U) + g(X,X)(\nabla_U U)\ln f\\
&- g(X,TX)U\ln f\eta(U) + g(X,X)(U\ln f)^2\\
&+ g(h(\nabla'_X\phi X,U),\phi U) - g(X,TX)g(h(\nabla\ln f,U),\phi U)\\
&+ \|h(X,U)\|^2\\
&= g(X,X)\{(\nabla_U U - U^2(\ln f))\ln f\}\\
&- g(TX,X)U\ln f\eta(U) + g(X,X)(U\ln f)^2\\
&+ \|h(X,U)\|^2\\
= &- g(X,X)H^{\ln f}(U,U) - g(TX,X)U\ln f\eta(U)\\
&+ g(X,X)(U\ln f)^2 + \|h(X,U)\|^2.
\end{aligned}
$$

On the other hand, from (5), we conclude

$$g(\bar{R}(U,X)\phi X,\phi U) = \frac{1}{4}(c-1)g(X,X)g(\phi U,\phi U), \tag{26}$$

which proves our assertion. □

Now, let $\{e_1, e_2, ...e_p, e_{p+1} = \xi, e^1, e^2, ..., e^q\}$ be an orthonormal basis of $\Gamma(TM)$ such that e_i, $1 \le i \le q+1$, are tangent to $M_\perp$ and e^j, $1 \le j \le q$, are tangent to M_T. Substituting (25) into $X = e^j$ and $U = e_i$, for $1 \le i \le p+1$ and $1 \le j \le q$, we obtain

$$\sum_{i=1}^{p+1}\sum_{j=1}^{q}\|h(e_i,e^j)\|^2 = q\left\{\frac{1}{4}(c-1)p + \sum_{i=1}^{p+1}H^{\ln f}(e_i,e_i) + \sum_{i=1}^{p+1}(e_i\ln f)^2\right\} + \sum_{j=1}^{q}g(Te^j,e^j)\xi\ln f. \tag{27}$$

By means of (6) and taking account of $M = M_\perp \times_f M_T$ being warped product semi-invariant submanifold, we have

$$\xi\ln f g(X,X) = g(TX,X),$$

which implies that

$$\xi\ln f = \frac{1}{q}tr(T).$$

Thus by using (18), (27) becomes

$$\sum_{i=1}^{p+1}\sum_{j=1}^{q}\|h(e_i,e^j)\|^2 = q\left\{\frac{1}{4}(c-1)p + \|grad\ln f\|^2 - \Delta\ln f\right\} + \frac{1}{q}tr^2(T). \tag{28}$$

From the (28), we have the following Theorems.

Theorem 4.5. *Let M be a warped product semi-invariant submanifold of a normal paracontact metric manifold $\bar{M}(c)$ such that $c \neq 1$. The squared of norm of the second fundamental form h satisfies the condition*

$$\|h\|^2 \ge q\left\{\frac{1}{4}(c-1)p + \|grad\ln f\|^2 - \Delta\ln f\right\} + \frac{1}{q}tr^2(T). \tag{29}$$

Theorem 4.6. *Let M be a compact orientable warped product semi-invariant submanifold of a normal paracontact metric manifold $\bar{M}(c)$ such that $c \neq 1$. M is a semi-invariant Riemannian product if and only if the second fundamental form h of M satisfies*

$$\sum_{i=1}^{p+1}\sum_{j=1}^{q} \|h(e_i, e^j)\|^2 \geq \frac{1}{4}(c-1)pq + \frac{1}{q}(tr^2(T)). \tag{30}$$

Proof. From (19) and (28), we conclude

$$\int_M \{\|grad \ln f\|^2 \quad - \quad \frac{1}{q}\sum_{i=1}^{p+1}\sum_{j=1}^{q} \|h(e_i, e^j)\|^2 + (\frac{1}{q}tr(T))^2\} dV = Vol(M)\frac{1}{4}(c-1)pq. \tag{31}$$

Here if (30) is satisfied, the we can derive $grad \ln f$ is constant. The converse is obvious. This proves our assertion. □

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