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Visualization of Enneper's Surface by Line Graphics

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Abstract. Here we study Enneper's minimal surface and some of its properties. We compute and visualize the lines of self–intersection, lines of intersections with planes, lines of curvature, asymptotic and geodesic lines of Enneper's surface. For the graphical representations of all the results we use our own software for line graphics.

1. Introduction and Known Results

The study of minimal surfaces plays an important role in differential geometry. Minimal surfaces are surfaces with identically vanishing mean curvature [1, 2]. The term *minimal* comes from the fact that if a surface which is bounded by a closed curve has minimal surface area then it has identically vanishing mean curvature. Due to their attractive shapes and other properties, they have applications in areas such as architecture, design and material science [3–5].

One example of a minimal surface is Enneper's surface. It has a non–trivial shape and some interesting properties such as symmetry, containing straight lines and self-intersections. Some of the recent results on Enneper's surface are given in [6–9].

The main stream to represent surfaces in modern computer graphics is their *approximation by a polygon mesh*. A problem in this approach arises when we want to represent curves on surfaces. We can draw curves rather precisely, but, if the surface is approximated, the curves look as if they are not exactly on the represented surface, but rather as if they "float" around the surface. To overcome this disadvantage we use line graphics. We do not approximate surfaces, but represent them by families of curves on them.

Unlike the wire model, line graphics take into account the visibility of the surface. For this we need to compute the intersections of straight line segments and the surface. The visibility problem is treated in Section 2.

To complete the representation of a surface in line graphics we need to draw the contour by which we mean the curve of all points of the surface at which the surface normal vector is orthogonal to the projection ray. Analytically this means we have to find the zeros of a real valued function of two variables on a rectangle. The corresponding numerical method can be found in [10]. We implemented the procedure for representing the contour line of Enneper's surface with the appropriate function in our software.

In this paper we also compute the lines of self–intersection of Ennepers surface. Furthermore, we visualize the lines of intersections with planes. In the last section we compute and visualize the lines of curvature, asymptotic and geodesic lines.

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We use our own software to visualize the results. The basic principles of the software are given in [10]. Some of its extensions are applied in the papers [11, 12].

Enneper's surface *ES* is given by a parametric representation

$$\vec{x}(u^i) = \left\{u^1 - \frac{(u^1)^3}{3} + u^1(u^2)^2, u^2 - \frac{(u^2)^3}{3} + u^2(u^1)^2, (u^1)^2 - (u^2)^2\right\} \quad ((u^1, u^2) \in \mathbb{R}^2). \tag{1}$$

It is a minimal surface.

First, we prove a few results that will be used in the sequel.

Lemma 1.1. *We introduce polar coordinates ρ and ϕ in $\mathbb{R}^2$ by*

$$u^1(\rho, \phi) = \rho\cos\phi \quad \textit{and} \quad u^2(\rho, \phi) = \rho\sin\phi \quad \textit{for } (\rho, \phi) \in (0, \infty) \times (0, 2\pi).$$

Then ES has a parametric representation

$$\vec{x}(\rho, \phi) = \left\{\rho\cos\phi - \frac{\rho^3}{3}\cos(3\phi), \rho\sin\phi + \frac{\rho^3}{3}\sin(3\phi), \rho^2\cos(2\phi)\right\} \quad ((\rho, \phi) \in (0, \infty) \times (0, 2\pi)). \tag{2}$$

Proof. We write $x^k(u^i)$ and $x^k(\rho, \phi)$ $(k = 1, 2, 3)$ for the components of the parametric representations of *ES* in (1) and (2), respectivly. Then we obtain

$$x^3(\rho, \phi) = x^3(u^i(\rho, \phi)) = (u^1(\rho, \phi))^2 - (u^2(\rho, \phi))^2 = \rho^2\left(\cos^2\phi - \sin^2\phi\right) = \rho^2\cos(2\phi),$$

and using

$$\cos(3\phi) = \mathrm{Re}(\exp(\mathrm{i}\phi)) = \mathrm{Re}\left(\cos\phi + \mathrm{i}\sin\phi\right)^3 = \cos^3\phi - 3\cos\phi\sin^2\phi$$

and

$$\cos(3\phi) = \mathrm{Im}(\exp(\mathrm{i}\phi)) = \mathrm{Im}\left(\cos\phi + \mathrm{i}\sin\phi\right)^3 = -\sin^3\phi + 3\cos^2\phi\cos\phi,$$

we conclude

$$\begin{aligned} x^1(\rho, \phi) = x^1(u^i(\rho, \phi)) &= \rho\cos\phi - \frac{\rho^3}{3}\cos^3\phi + \rho^3\cos\phi\sin^2\phi \\ &= \rho\cos\phi - \frac{\rho^3}{3}\left(\cos^3\phi - 3\cos\phi\sin^2\phi\right) = \rho\cos\phi - \frac{\rho^3}{3}\cos 3\phi \end{aligned}$$

and

$$\begin{aligned} x^2(\rho, \phi) = x^2(u^i(\rho, \phi)) &= \rho\sin\phi - \frac{\rho^3}{3}\sin^3\phi + \rho^3\sin\phi\cos^2\phi \\ &= \rho\sin\phi - \frac{\rho^3}{3}\left(\sin^3\phi - 3\sin\phi\cos^2\phi\right) = \rho\sin\phi + \frac{\rho^3}{3}\sin(3\phi). \end{aligned}$$

□

Lemma 1.2. *The components $x^k(\rho, \phi)$ of the parametric representation (2) satisfy the relation*

$$\left(x^1(\rho, \phi)\right)^2 + \left(x^2(\rho, \phi)\right)^2 + \frac{4}{3}\left(x^3(\rho, \phi)\right)^2 = \left(\rho + \frac{\rho^3}{3}\right)^2. \tag{3}$$

Proof. We have

$$\left(x^1(\rho,\phi)\right)^2+\left(x^2(\rho,\phi)\right)^2+\frac{4}{3}\left(x^3(\rho,\phi)\right)^2=$$
$$\rho^2\cos^2\phi-\frac{2\rho^4}{3}\cos\phi\cos(3\phi)+\frac{\rho^6}{9}\cos^2(3\phi)+\rho^2\sin^2\phi+\frac{2\rho^4}{3}\sin\phi\sin(3\phi)+\frac{\rho^6}{9}\sin^2(3\phi)+\frac{4}{3}\rho^4\cos^2(2\phi)=$$
$$\rho^2+\frac{\rho^6}{9}-\frac{2\rho^4}{3}\Big(\cos\phi\cos(3\phi)-\sin\phi\sin(3\phi)\Big)+\frac{4}{3}\rho^4\cos^2(2\phi)=\rho^2+\frac{\rho^6}{9}-\frac{2\rho^4}{3}\cos(4\phi)+\frac{4}{3}\rho^4\cos^2(2\phi)=$$
$$\rho^2+\frac{\rho^6}{9}-\frac{2\rho^4}{3}\Big(\cos^2(2\phi)-\sin^2(2\phi)\Big)+\frac{4}{3}\rho^4\cos^2(2\phi)=\rho^2+\frac{\rho^6}{9}-\frac{2\rho^4}{3}\Big(2\cos^2(2\phi)-1\Big)+\frac{4}{3}\rho^4\cos^2(2\phi)=$$
$$\rho^2+\frac{\rho^6}{9}+\frac{2\rho^4}{3}=\left(\rho+\frac{\rho^3}{3}\right)^2.$$

□

Next result gives us the lines of self–intersection of Enneper's surface. (Figure 1). We also visualize the lines of intersections of Enneper's surface with planes (Figure 2).

Lemma 1.3. *The lines of self–intersection of Enneper's surface with the parametric representation (2) are given by*

$$f_1(\rho,\phi)=\cos\phi-\frac{\rho^2}{3}\cos(3\phi)=0 \quad \textit{and} \quad f_2(\rho,\phi)=\sin\phi+\frac{\rho^2}{3}\sin(3\phi)=0; \tag{4}$$

consequently they are in the planes $x=0$ and $y=0$, respectively (Figure 1).

Proof. The points of self–intersection of Enneper's surface given by a the parametric representation (2) must satisfy

$$\vec{x}(\rho_1,\phi_1)=\vec{x}(\rho_2,\phi_2), \quad \text{that is} \quad x^k(\rho_1,\phi_1)=x^k(\rho_2,\phi_2) \quad \text{for } k=1,2,3, \tag{5}$$

and it follows from (3) that

$$\rho_1+\frac{\rho_1^3}{3}=\rho_2+\frac{\rho_2^3}{3}.$$

Since the function $f:\mathbb{R}\to\mathbb{R}$ with $f(t)=t+t^3/3$ obviously is one–to-one, this implies $\rho_1=\rho_2=\rho$. Thus it follows from $x^3(\rho,\phi_1)=x^3(\rho,\phi_2)$, that $\cos(2\phi_1)=\cos(2\phi_2)$, hence $\phi_2=\pi-\phi_1$ or $\phi_2=2\pi-\phi_2$.
If $\phi_2=\pi-\phi_1$, then $x^1(\rho,\phi_1)=x^1(\rho,\pi-\phi_1)$ implies

$$\cos\phi_1-\frac{\rho^2}{3}\cos(3\phi_1)=\cos(\pi-\phi_1)-\frac{\rho^2}{3}\cos(3(\pi-\phi_1))=-\left(\cos\phi_1-\frac{\rho^2}{3}\cos(3\phi_1)\right)$$

that is $x^1(\rho,\phi_1)=-x^1(\rho,\phi_1)=f_1(\rho,\phi_1)=0$.
If $\phi_2=2\pi-\phi_1$, then it can similarly be shown that $x^2(\rho,\phi_1)=-x^2(\rho,\phi_1)=f_2(\rho,\phi_1)=0$.
It is easy to see that if $\phi_2=\pi-\phi_1$ or $\phi_2=2\pi-\phi_1$, then $\vec{x}(\rho,\phi_1)=\vec{x}(\rho,\phi_2)$. □

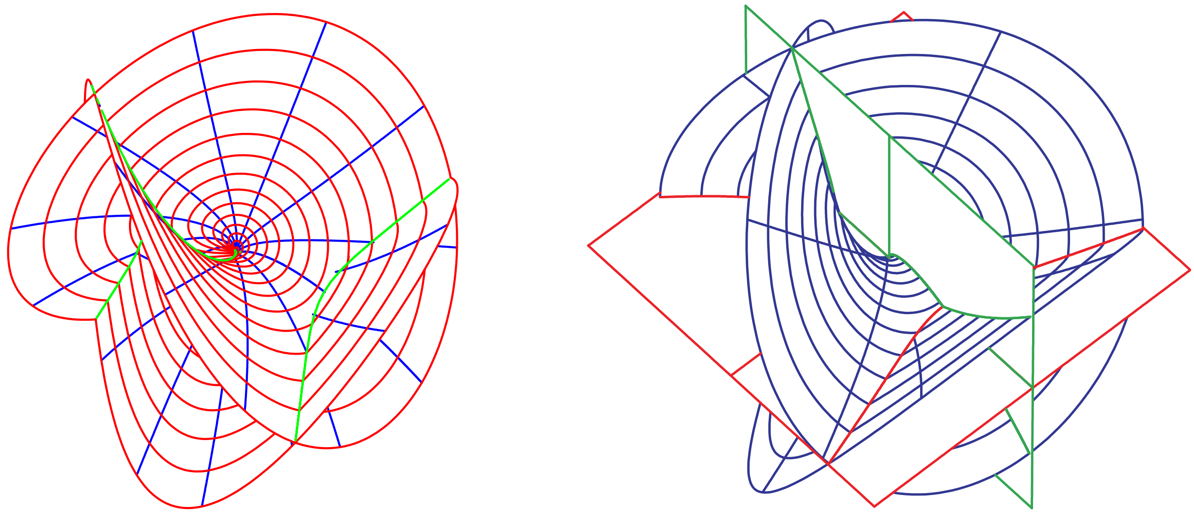

Figure 1: Left: Enneper's surface given by (2). Right: Enneper's surface and its intersections with the planes $x = 0$ and $y = 0$.

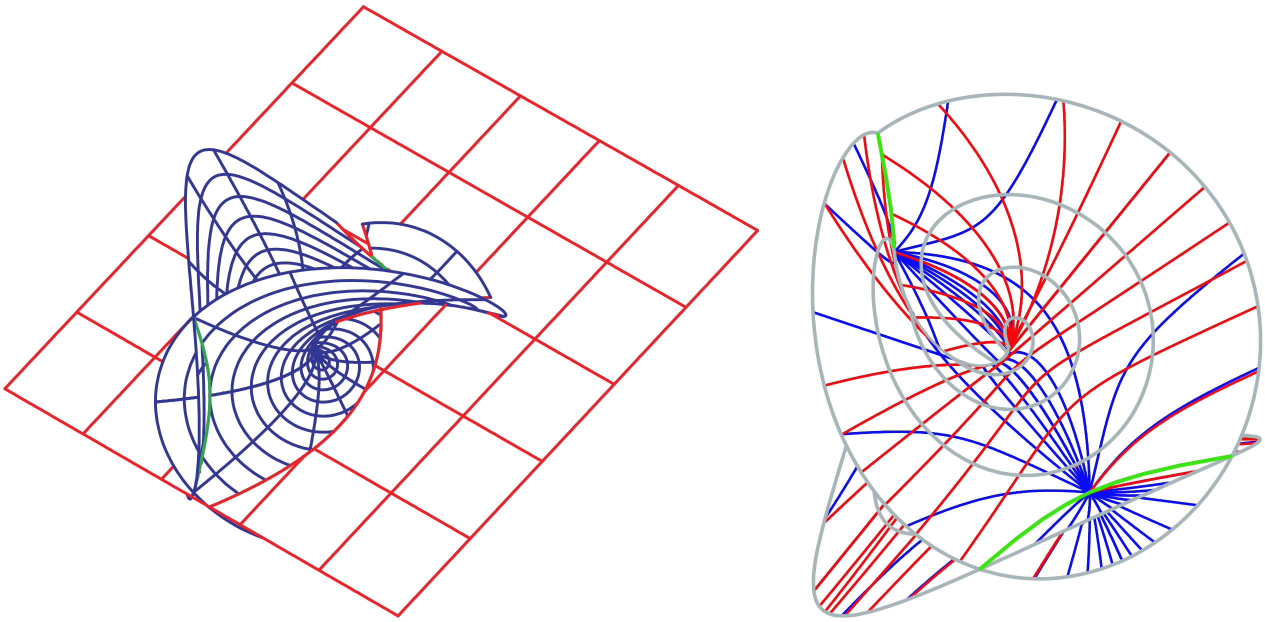

Figure 2: Left: Enneper's surface and its intersections with a tangent plane.
Right: Enneper's surface and its lines of intersections with planes.

2. The Visibility Problem for Enneper's Surface

Now we deal with the visibility problem for Enneper's surface ES. First we find the points of intersection of a straight line γ with ES.

We need the following result.

Lemma 2.1. *Let $\vec{p} = \{p^1, p^2, p^3\}$ and $\vec{v} = \{v^1, v^2, v^3\}$ be vectors with $\|\vec{v}\| = 1$. We put*

$$a = 1 + \frac{(v^3)^2}{3}, \ b = \vec{p} \bullet \vec{v} + \frac{1}{3}p^3 v^3 \quad \text{and} \quad c(\rho) = \|\vec{p}\|^2 + \frac{(p^3)^2}{3} - \left(\rho + \frac{\rho^3}{3}\right)^2. \tag{6}$$

(a) Then we have

$$b^2 - ac(\rho) \geq 0 \tag{7}$$

if and only if

$$\rho \geq 2\sinh\left(\frac{1}{3}\log\left(\frac{1}{2}\left(3d + \sqrt{9d^2 + 4}\right)\right)\right) \tag{8}$$

where

$$d = \frac{\sqrt{a\left(\|\vec{p}\|^2 + \frac{1}{3}(p^3)^2\right) - \left(\vec{p} \bullet \vec{v} + \frac{2}{3}p^3 v^3\right)^2}}{a}.$$

(b) We also have $c(p) = 0$ if and only if

$$\rho \geq \rho_1 = 2\sinh\left(\frac{1}{3}\log\left(\frac{3}{2}\sqrt{\|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3}} + \sqrt{\frac{9}{4}\left(\|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3}\right)^2 + 1}\right)\right); \tag{9}$$

furthermore, $\rho_1 \geq \rho_0$.

Proof. Obviously the condition in (7) means

$$a\left(\rho + \frac{\rho^3}{3}\right)^2 \geq a\left(\|\vec{p}\|^2 + \frac{1}{3}(p^3)^2\right) - \left(\vec{p} \bullet \vec{v} + \frac{1}{3}p^3 v^3\right)^2.$$

We put

$$\tilde{d} = a\left(\|\vec{p}\|^2 + \frac{1}{3}(p^3)^2\right) - \left(\vec{p} \bullet \vec{v} + \frac{1}{3}p^3 v^3\right)^2,$$

and observe that by the Cauchy–Schwarz inequality and the fact that $\|\vec{v}\| = 1$

$$\begin{aligned}
\tilde{d} &= \left(1 + \frac{(v^3)^2}{3}\right)\left(\|\vec{p}\|^2 + \frac{1}{3}(p^3)^2\right) - (\vec{p} \bullet \vec{v})^2 - \frac{2(\vec{p} \bullet \vec{v})p^3 v^3}{3} - \frac{1}{9}(p^3)^2(v^3)^2 \\
&= \left(1 + \frac{(v^3)^2}{3}\right)\|\vec{p}\| + \frac{(p^3)^2}{3} + \frac{(v^3)^2(p^3)^2}{9} - (\vec{p} \bullet \vec{v})^2 - \frac{2(\vec{p} \bullet \vec{v})p^3 v^3}{3} - \frac{(p^3)^2(v^3)^2}{9} \\
&\geq \left(1 + \frac{(v^3)^2}{3}\right)(\vec{p} \bullet \vec{v})^2 + \frac{(p^3)^2}{3} - (\vec{p} \bullet \vec{v})^2 - \frac{2(\vec{p} \bullet \vec{v})p^3 v^3}{3} \\
&\geq \frac{(v^3)^2\,(\vec{p} \bullet \vec{v})^2}{3} - \frac{2(\vec{p} \bullet \vec{v})p^3 v^3}{3} + \frac{(p^3)^2}{3} = \frac{1}{3}\left(v^3(\vec{p} \bullet \vec{v}) - p^3\right)^2 \geq 0.
\end{aligned}$$

So we may write $d = \sqrt{\tilde{d}}/a$, and since $a > 0$ and $\rho > 0$, the condition in (7) is equivalent to

$$g(\rho) = \rho^3 + 3\rho - 3d \geq 0.$$

We put $\tilde{p} = 1$, $\tilde{q} = -(3/2)d$, $r = \mathrm{sgn}(q) = -1$ and $\sinh(\varphi) = \tilde{q}/r^3 = -\tilde{q}$. Then the cubic equation $g(\rho) = 0$ has one real solution, given by

$$\rho_0 = -2r\sinh\left(\frac{\varphi}{3}\right) = 2\sinh\left(\frac{1}{3}\mathrm{Arsinh}(-\tilde{q})\right).$$

Therefore, the condition in (7) is equivalent to $\rho \geq \rho_0$, but

$$\rho_0 = 2\sinh\left(\frac{1}{3}\log\left(-\tilde{q} + \sqrt{\tilde{q}^2 + 1}\right)\right) = 2\sinh\left(\frac{1}{3}\log\left(\frac{1}{2}\left(3d + \sqrt{9d^2 + 4}\right)\right)\right).$$

Thus we have shown that the conditions in (7) and (8) are equivalent.
We observe that

$$\rho_0 = \sqrt[3]{\frac{1}{2}\left(3d + \sqrt{9d^2 + 4}\right)} - \frac{1}{\sqrt[3]{\frac{1}{2}\left(3d + \sqrt{9d^2 + 4}\right)}}.$$

(b) We put

$$d_1 = \sqrt{\|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3}}.$$

Since $\rho > 0$, the condition $c(\rho) > 0$ is equivalent to

$$\rho^3 + 3\tilde{p}\rho + 2\tilde{q} = 0 \quad \text{where} \quad \tilde{p} = 1 \quad \text{and} \quad \tilde{q} = -\frac{3}{2}d_1.$$

Putting $r = -1$ and $\sinh\psi = \tilde{q}/r^3$, we obtain, as in the proof of Part (a), that $c(\rho) = 0$ if and only if

$$\begin{aligned}\rho = \rho_1 &= -2\sinh\left(\frac{\psi}{3}\right) = 2\sinh\left(\frac{1}{3}\log\left(\frac{\tilde{q}}{r^3} + \sqrt{\frac{\tilde{q}^2}{r^6} + 1}\right)\right)\\ &= 2\sinh\left(\frac{1}{3}\log\left(\frac{3}{2}\sqrt{\|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3}} + \sqrt{\frac{9}{4}\left(\|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3}\right)^2 + 1}\right)\right),\end{aligned}$$

which is (9).
Furthermore, since $\rho^3 + 3\rho = 3d$ if and only if $\rho = \rho_0$ by Part (a), $\rho^3 + 3\rho = 3d_1$ if and only if $\rho = \rho_1$ by what we have just shown, and $d_1 \geq d$, it follows that $\rho_1 \geq \rho_0$. □

2.1. *The intersection of Enneper's surface with a straight line*

We need to determine the points of intersection of Enneper's surface *ES* with an arbitrary straight line (Figure 3) to be able to solve the visibility problem for Enneper's surface.

Let γ be a straight line given by a parametric representation

$$\vec{y}(t) = \vec{p} + t\vec{v} \quad (t \in \mathbb{R}),$$

where $\vec{p} = \{p^1, p^2, p^3\}$ is the position vector of a point P of γ, and $\vec{v} = \{v^1, v^2, v^3\}$ is a unit vector along γ. To find the points of intersection of γ and ES when ES is given by a parametric representation (2), we have to find $(\rho, \phi) \in I_\rho \times I_\phi$ and $t \in \mathbb{R}$ such that

$$\vec{x}(\rho, \phi) = \vec{y}(t). \tag{10}$$

It follows from (2) in Lemma 1.1 and (10) that

$$\begin{aligned}\|\vec{y}(t)\|^2 + \frac{\left(y^3(t)\right)^2}{3} &= \|\vec{p}\|^2 + 2(\vec{p} \bullet \vec{v})t + \|\vec{v}\|^2 + \frac{1}{3}\left(\left(p^3\right)^2 + 2p^3v^3t + v^3t^2\right) \\ &= \left(1 + \frac{\left(v^3\right)^2}{3}\right)t^2 + 2\left(\vec{p} \bullet \vec{v} + \frac{p^3v^3}{3}\right)t + \|\vec{p}\|^2 + \frac{\left(p^3\right)^2}{3} = \left(\rho + \frac{\rho^3}{3}\right)^2.\end{aligned}$$

Using the notations of (6) in Lemma 2.1, we have to solve the quadratic equation

$$at^2 + 2bt^2 + c(\rho) = 0, \tag{11}$$

which, by Lemma 2.1 (a), has real solutions given by

$$t_\pm(\rho) = \frac{-b \pm \sqrt{b^2 - ac(\rho)}}{a} \text{ if and only if } \rho \geq \rho_0, \tag{12}$$

where ρ_0 is given by (8); we also note that by Lemma 2.1 (b),

$$t_\pm(\rho) = \frac{-b \pm |b|}{a} \text{ if and only if } \rho = \rho_1, \tag{13}$$

where ρ_1 is given by (9). We define the interval $I_{\rho_0} = [\rho_0, \infty) \cap I_\rho$. Now it follows from the equation of the third component in (10) that

$$\cos(2\phi) = \frac{p^3 + t_\pm(\rho)v^3}{\rho^2}, \tag{14}$$

and so

$$\cos^2\phi = \frac{\cos(2\phi) + 1}{2} = \frac{p^3 + t_\pm(\rho)v^3 + \rho^2}{2\rho^2}. \tag{15}$$

Furthermore, since

$$\begin{aligned}\cos(3\phi) &= \cos(2\phi)\cos\phi - \sin(2\phi)\sin\phi = \cos\phi\left(\cos(2\phi) - 2\sin^2\phi\right) \\ &= \cos\phi\left(\cos(2\phi) - 2\left(1 - \cos^2\phi\right)\right) = \cos\phi\left(\cos(2\phi) - 2 + \cos(2\phi) + 1\right) \\ &= \cos\phi\left(2\cos(2\phi) - 1\right),\end{aligned}$$

the equation for the first component in (10) yields

$$\rho\cos\phi - \frac{\rho^3}{3}\cos(3\phi) = \rho\cos\phi\left(1 - \frac{\rho^2\left(2\cos(2\phi) - 1\right)}{3}\right) = p^1 + t_\pm(\rho)v^1.$$

Squaring the last identity and substituting (14) and (15), we obtain

$$\begin{aligned}\rho^2\cos^2\phi\left(1 - \frac{\rho^2}{3}\cos^2(3\phi)\right)^2 &= \frac{p^3 + t_\pm(\rho)v^3 + \rho^2}{2}\left(1 - \frac{\rho^2}{3}\left(2\left(p^3 + t_\pm(\rho)v^3\right) - 1\right)\right)^2 \\ &= \frac{p^3 + t_\pm(\rho)v^3 + \rho^2}{2}\left(1 + \frac{\rho^2}{2} - \frac{2\left(p^3 + t_\pm(\rho)v^3\right)}{2}\right)^2 = \left(p^1 + t_\pm v^1\right)^2.\end{aligned}$$

We define the functions f_+ and f_- on I_{ρ_0} by

$$f_\pm(\rho) = \frac{p^3 + t_\pm(\rho)v^3 + \rho^2}{2}\left(1 + \frac{\rho^2}{2} - \frac{2\left(p^3 + t_\pm(\rho)v^3\right)}{2}\right)^2 - \left(p^1 + t_\pm v^1\right)^2,$$

where $t_\pm(\rho)$ is given by (12) for $\rho \in I_{\rho_0} \setminus \{\rho_1\}$ and by (13) for $\rho = \rho_1$, and determine the zeros of $f_\pm(\rho)$. Let $\rho_\pm^{(k)}$ denote the zeros of $f_\pm$. By (14), we have to check

$$\frac{|p^3 + t_\pm(\rho_\pm^{(k)})v^3|}{\left(\rho_\pm^{(k)}\right)^2} \leq 1. \tag{16}$$

Let $\rho_\pm^{(j)}$ denote the zeros of $f_\pm$ that satisfy (16). Since

$$\cos\phi = \pm\sqrt{\frac{\cos(2\phi) + 1}{2}},$$

we obtain the values

$$\begin{aligned}
\phi_{\pm;1}^{(j);+} &= \arccos\left(+\sqrt{\frac{p^3 + t_\pm(\rho_\pm^{(j)}v^3) + \left(\rho_\pm^{(j)}\right)^2}{2\left(\rho_\pm^{(j)}\right)^2}}\right),\\
\phi_{\pm;2}^{(j);+} &= 2\pi - \phi_{\pm;1}^{(j);+}\\
\phi_{\pm;1}^{(j);-} &= \arccos\left(-\sqrt{\frac{p^3 + t_\pm(\rho_\pm^{(j)}v^3) + \left(\rho_\pm^{(j)}\right)^2}{2\left(\rho_\pm^{(j)}\right)^2}}\right),\\
\phi_{\pm;2}^{(j);-} &= 2\pi - \phi_{\pm;1}^{(j);-}.
\end{aligned}$$

Finally, the points of intersection of Enneper's surface with the straight line γ are given by the pairs $(\rho_\pm^{(j)}, \phi_{\pm;1,2}^{(j);\pm}) \in I_{\rho_0} \times I_\phi$ that satisfy

$$\left\|\vec{x}\left(\rho_\pm^{(j)}, \phi_{\pm;1,2}^{(j);\pm}\right) - \vec{y}\left(t_\pm\left(\rho_\pm^{(j)}\right)\right)\right\| = 0.$$

Now let COP denote the center of projection and P be a point in $\mathbb{R}^3$. We write $\vec{p}$ for the position vector of P and $\vec{v}$ for the unit vector from P to COP. Then P is hidden by Enneper's surface, if we find $(\rho, \phi) \in I_\rho \times I_\phi$ and $t > 0$ by the methods described above, such that (10) is satisfied; in the same way, a point P on Enneper's surface is invisible with respect to Enneper's surface ES; now $t_\pm(\rho)$ is given by (13) on I_ρ, since $P \in ES$ implies $c(\rho) = 0$ (Figure 3).

Next we represent the intersections of Enneper's surface with spheres. Similarly as in a case of the contour, analytically this means to find the zeros of real valued functions of two variables on a rectangle. We implemented the procedure for drawing the lines of intersection of spheres with Enneper's surface in our software (Figure 4).

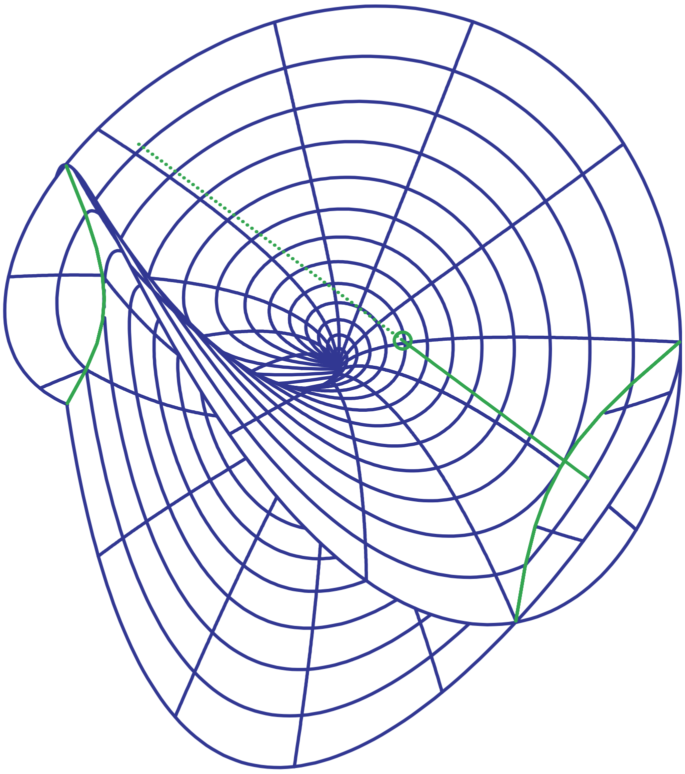

Figure 3: Enneper's surface and its intersection with a straight line.

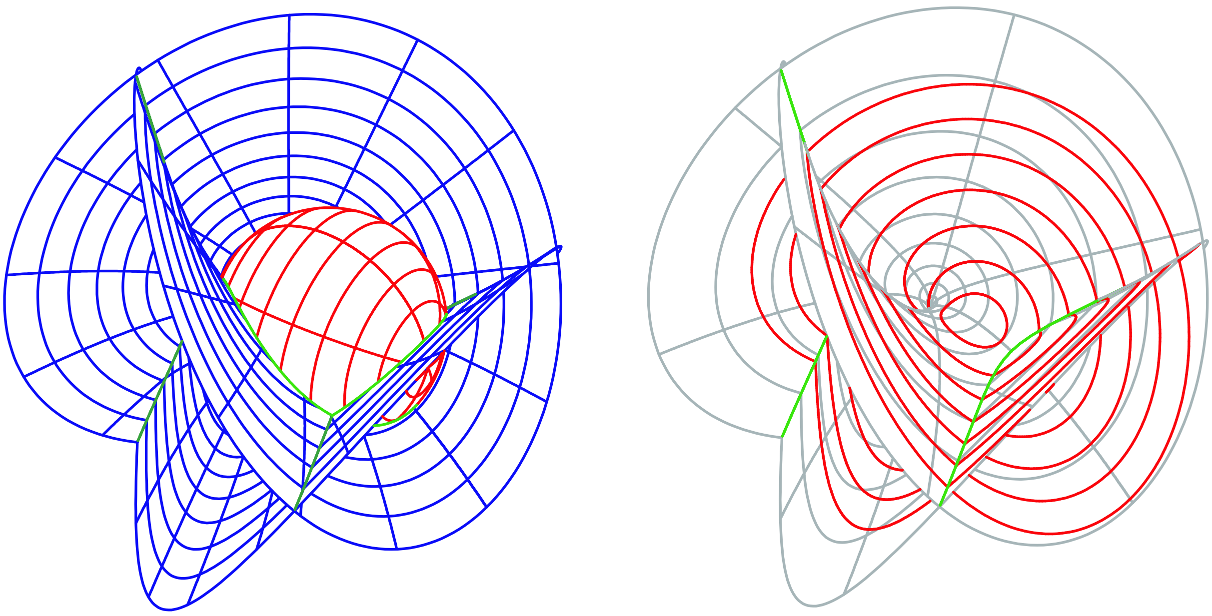

Figure 4: Enneper's surface and its intersections with spheres.

3. Lines of Curvature, Asymptotic and Geodesic Lines on Enneper's surface

Here we determine the lines of curvature, and the asymptotic and geodesic lines on Enneper's surface. We need the following results.

Lemma 3.1. *Let Enneper's surface be given by a parametric representation (1).*
(a) Then the first fundamental coefficients are given by

$$g_{11}(u^1,u^2) = g_{22}(u^1,u^2) = \left(1+(u^1)^2+(u^2)^2\right)^2, \ g_{12}(u^1,u^2) = 0 \ \text{ and } \ g(u^1,u^2) = \left(1+(u^1)^2+(u^2)^2\right)^4. \quad (17)$$

(b) The second fundamental coefficients are given by

$$L_{11}(u^1,u^2) = -L_{22}(u^1,u^2) = 2, \ L_{12}(u^1,u^2) = 0 \ \text{ and } \ L(u^1,u^2) = -4. \quad (18)$$

Proof. (a) it follows from (1) that

$$\begin{aligned}
\vec{x}_1(u^1,u^2) &= \left\{1-(u^1)^2+(u^2)^2, 2u^1u^2, 2u^1\right\},\\
\vec{x}_2(u^1,u^2) &= \left\{2u^1u^2, 1-(u^2)^2+(u^1)^2, -2u^2\right\},\\
g_{11}(u^1,u^2) &= \vec{x}_1(u^1,u^2)\bullet\vec{x}_2(u^1,u^2)\\
&= \left(1-\left((u^1)^2-(u^2)^2\right)\right)^2 + 4(u^1)^2(u^2)^2 + 4(u^1)^2\\
&= 1-2(u^1)^2+2(u^2)^2+(u^1)^4-2(u^1)^2(u^2)^2+(u^2)^4+4(u^1)^2(u^2)^2+4(u^1)^2\\
&= 1+2(u^1)^2+2(u^2)^2+(u^1)^4+2(u^1)^2(u^2)^2+(u^1)^4\\
&= 1+2(u^1)^2+2(u^2)^2+\left((u^1)^2+(u^2)^2\right)^2 = \left(1+(u^1)^2+(u^2)^2\right)^2,\\
g_{12}(u^1,u^2) &= \vec{x}_1(u^1,u^2)\bullet\vec{x}_2(u^1,u^2)\\
&= 2u^1u^2\left(1-\left((u^1)^2-(u^2)^2\right)-\left(1+\left((u^1)^2-(u^2)^2\right)\right)\right)-4u^1u^2\\
&= 4u^1u^2-4u^1u^2 = 0,\\
g_{22}(u^1,u^2) &= \vec{x}_2(u^1,u^2)\bullet\vec{x}_2(u^1,u^2)\\
&= 4(u^1)^2(u^2)^2+\left(1-\left((u^2)^2-(u^1)^2\right)\right)^2+4(u^2)^2\\
&= 4(u^1)^2(u^2)^2+1-2(u^2)^2+2(u^1)^2+(u^2)^4-2(u^1)^2(u^2)^2+(u^1)^4+4(u^2)^2\\
&= 1+(u^1)^2+(u^2)^2+\left((u^1)^2+(u^2)^2\right)^2 = \left(1+(u^1)^2+(u^2)^2\right)^2
\end{aligned}$$

and

$$g(u^1,u^2) = g_{11}(u^1,u^2)g_{22}(u^1,u^2) - g_{12}^2(u^1,u^2) = \left(1+(u^1)^2+(u^2)^2\right)^4.$$

(b) Furthermore, we obtain

$$\vec{x}_{11}(u^1,u^2) = 2\{-u^1,u^2,1\}, \quad \vec{x}_{12}(u^1,u^2) = 2\{u^2,u^1,0\}, \quad \vec{x}_{22}(u^1,u^2) = 2\{u^1,-u^2,-1\} = -\vec{x}_{11}(u^1,u^2),$$

$$\begin{aligned}
&\vec{x}_1(u^1,u^2)\times\vec{x}_2(u^1,u^2) =\\
&\left\{-4u^1(u^2)^2-2u^1+2u^1(u^2)^2-2(u^1)^3, 2u^2-2(u^1)^2+(u^2)^3+4(u^1)^2u^2,\right.\\
&\left.\left(1-\left((u^1)^2-(u^2)^2\right)\right)\left(1+\left((u^1)^2-(u^2)^2\right)\right)-4(u^1)^2(u^2)^2\right\} =\\
&\left\{-2u^1\left(1+(u^1)^2(u^2)^2\right), 2u^2\left(1+(u^1)^2(u^2)^2\right), 1-\left((u^1)^2-(u^2)^2\right)^2-4(u^1)^2(u^2)^2\right\} =
\end{aligned}$$

$$\left\{-2u^1\left(1+(u^1)^2(u^2)^2\right), 2u^2\left(1+(u^1)^2(u^2)^2\right), 1-(u^1)^4+2(u^1)^2(u^2)^2-(u^2)^4-4(u^1)^2(u^2)^2\right\} =$$
$$\left\{-2u^1\left(1+(u^1)^2(u^2)^2\right), 2u^2\left(1+(u^1)^2(u^2)^2\right), 1-\left((u^1)^2+(u^2)^2\right)^2\right\} =$$
$$\left(1+(u^1)^2+(u^2)^2\right)\left\{-2u^1, 2u^2, 1-\left((u^1)^2+(u^2)^2\right)\right\},$$

$$\begin{aligned}
L_{11}(u^1,u^2) &= \frac{\vec{x}_{11}(u^1,u^2)\bullet\left(\vec{x}_1(u^1,u^2)\times\vec{x}_2(u^1,u^2)\right)}{\sqrt{g(u^1,u^2)}}\\
&= \frac{2}{1+(u^1)^2+(u^2)^2}\{-u^1,u^2,1\}\bullet\left\{-2u^1,2u^2,1-\left((u^1)^2+(u^2)^2\right)\right\}\\
&= \frac{2}{1+(u^1)^2+(u^2)^2}\left(2(u^1)^2+2(u^2)^2+1-(u^1)^2-(u^2)^2\right)\\
&= \frac{2\left(1+(u^1)^2+(u^2)^2\right)}{1+(u^1)^2+(u^2)^2} = 2,\\
L_{12}(u^1,u^2) &= \frac{\vec{x}_{12}(u^1,u^2)\bullet\left(\vec{x}_1(u^1,u^2)\times\vec{x}_2(u^1,u^2)\right)}{\sqrt{g(u^1,u^2)}}\\
&= \frac{2}{\sqrt{g(u^1,u^2)}}\{u^2,u^1,0\}\bullet\left\{-2u^1,2u^2,1-\left((u^1)^2+(u^2)^2\right)\right\}\\
&= \frac{4(-u^1u^2+u^1u^2)}{\sqrt{g(u^1,u^2)}} = 0,\\
L_{22}(u^1,u^2) &= \frac{\vec{x}_{22}(u^1,u^2)\bullet\left(\vec{x}_1(u^1,u^2)\times\vec{x}_2(u^1,u^2)\right)}{\sqrt{g(u^1,u^2)}}\\
&= -\frac{\vec{x}_{11}(u^1,u^2)\bullet\left(\vec{x}_1(u^1,u^2)\times\vec{x}_2(u^1,u^2)\right)}{\sqrt{g(u^1,u^2)}} = -L_{11}(u^1,u^2)
\end{aligned}$$

and

$$L(u^1,u^2) = L_{11}(u^1,u^2)L_{22}(u^1,u^2) - L_{12}^2(u^1,u^2) = -4.$$

□

Theorem 3.2. *Let Enneper's surface be given by the parametric representation*

$$\vec{x}(u^i) = \left\{u^1 - \frac{(u^1)^3}{3} + u^1(u^2)^2, u^2 - \frac{(u^2)^3}{3} + u^2(u^1)^2, (u^1)^2-(u^2)^2\right\} \quad ((u^1,u^2)\in\mathbb{R}^2).$$

(a) The Gaussian curvature is given by

$$K(u^1,u^2) = -\frac{4}{(1+(u^1)^2+(u^2)^2)^4} \quad \textit{(Figure 5)}.$$

(b) The mean curvature is

$$H(u^1,u^2) = 0,$$

hence Enneper's surface is a minimal surface.
(c) The principal curvatures are

$$\kappa_{1,2} = \pm\frac{2}{(1+(u^1)^2+(u^2)^2)^2}$$

(d) The asymptotic lines are given by

$$u^1 = \pm u^2 + c, \quad \text{where } c \in \mathbb{R} \text{ is a constant.}$$

(e) The lines of curvature are the parameter lines.

Proof. (a) It follows from (17) and (18) that

$$K(u^1,u^2) = \frac{L(u^1,u^2)}{g(u^1,u^2)} = -\frac{4}{(1+(u^1)^2+(u^2)^2)^4}$$

(b) It follows from (17) and (18) that

$$\begin{aligned} H(u^1,u^2) &= \frac{1}{2g(u^1,u^2)}\Big(L_{11}(u^1,u^2)g_{22}(u^1,u^2) - 2L_{12}(u^1,u^2)g_{12}(u^1,u^2) + L_{22}(u^1,u^2)g_{11}(u^1,u^2)\Big) \\ &= \frac{1}{2g(u^1,u^2)}g_{11}(u^1,u^2)(2-2) = 0 \end{aligned}$$

(c) Since

$$K(u^1,u^2) = \kappa_1(u^1,u^2)\kappa_2(u^1,u^2) \quad \text{and} \quad H(u^1,u^2) = \frac{\kappa_1(u^1,u^2)+\kappa_2(u^1,u^2)}{2},$$

it follows from Parts (a) and (b) that

$$\kappa_1 = \frac{2}{(1+(u^1)^2+(u^2)^2)^2} \quad \text{and} \quad \kappa_2 = -\frac{2}{(1+(u^1)^2+(u^2)^2)^2}.$$

(d) The differential equation for the asymptotic lines

$$L_{ik}(u^1,u^2)du^i du^k = 0$$

reduces to

$$2(du^1)^2 - 2(du^2)^2 = 0,$$

and has the solutions

$$u^1(u^2) = \pm u^2 + c \ \ (u^2 \in \mathbb{R}), \text{ where } c \in \mathbb{R} \text{ is a constant.}$$

(e) Since $g_{12}(u^1,u^2) = L_{12}(u^1,u^2) = 0$ for all $(u^1,u^2) \in \mathbb{R}^2$, the parameter lines are the lines of curvature. □

Lemma 3.3. *Let Enneper's surface be given by a parametric representation (2).*
(a) Then the first fundamental coefficients are given by

$$g_{11}(\rho,\phi) = \left(1+\rho^2\right)^2, \ g_{12}(\rho,\phi) = 0, \ g_{22}(\rho,\phi) = \rho^2\left(1+\rho^2\right)^2 \ \text{ and } \ g(\rho,\phi) = \rho^2\left(1+\rho^2\right)^4. \tag{19}$$

(b) The second fundamental coefficients are given by

$$L_{11}(\rho,\phi) = 2\cos(2\phi), \ L_{12}(\rho,\phi) = -2\rho\sin(2\phi), \ L_{22}(\rho,\phi) = -2\rho^2\cos(2\phi) \ \text{ and } \ L(\rho,\phi) = -4\rho^2. \tag{20}$$

Proof. We put $u^{*1} = \rho$ and $u^{*2} = \phi$ and write

$$\vec{x}^*(u^{*i}) = \vec{x}(u^1(u^{*i}), u^2(u^{*i})) \text{ where } u^1(u^{*i}) = u^{*1}\cos u^{*2} \text{ and } u^2(u^{*i}) = u^{*1}\sin u^{*2}.$$

Then we have

$$\frac{\partial u^1}{\partial u^{*1}}(u^{*1}, u^{*2}) = \cos u^{*2}, \qquad \frac{\partial u^1}{\partial u^{*2}}(u^{*1}, u^{*2}) = -u^{*1}\sin u^{*2},$$
$$\frac{\partial u^2}{\partial u^{*1}}(u^{*1}, u^{*2}) = \sin u^{*2} \quad \text{and} \quad \frac{\partial u^2}{\partial u^{*2}}(u^{*1}, u^{*2}) = u^{*1}\cos u^{*2}$$

and

$$D = \det\left(\frac{\partial u^k}{\partial u^{*i}}\right) = \det\begin{pmatrix} \frac{\partial u^1}{\partial u^{*1}}(u^{*1}, u^{*2}) & \frac{\partial u^1}{\partial u^{*2}}(u^{*1}, u^{*2}) \\ \frac{\partial u^2}{\partial u^{*1}}(u^{*1}, u^{*2}) & \frac{\partial u^2}{\partial u^{*2}}(u^{*1}, u^{*2}) \end{pmatrix}$$
$$= \det\begin{pmatrix} \cos u^{*2} & -u^{*1}\sin u^{*2} \\ \sin u^{*2} & u^{*1}\cos u^{*2} \end{pmatrix} = u^{*1}.$$

(a) The formulae of transformation for the first fundamental coefficients

$$g^*_{jk}(u^{*i}) = g_{\ell m}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^\ell}{\partial u^{*j}}(u^{*i})\frac{\partial u^m}{\partial u^{*j}}(u^{*i}) \text{ for } j,k = 1,2$$

yield by Lemma 3.1 (a)

$$g^*_{11}(u^{*i}) = g_{11}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^1}{\partial u^{*1}}(u^{*i})\right)^2 + g_{22}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^2}{\partial u^{*1}}(u^{*i})\right)^2$$
$$= \left(1 + (u^{*1})^2\right)^2\left(\cos^2 u^{*2} + \sin^2 u^{*2}\right) = \left(1 + (u^{*1})^2\right)^2$$
$$g^*_{12}(u^{*i}) = g_{11}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^1}{\partial u^{*1}}(u^{*i})\frac{\partial u^1}{\partial u^{*2}}(u^{*i}) + g_{22}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^2}{\partial u^{*1}}(u^{*i})\frac{\partial u^2}{\partial u^{*2}}(u^{*i})$$
$$= \left(1 + (u^{*1})^2\right)^2\left(-u^{*1}\sin u^{*2}\cos u^{*2} + u^{*1}\sin u^{*2}\cos u^{*2}\right) = 0$$
$$g^*_{22}(u^{*i}) = g_{11}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^1}{\partial u^{*2}}(u^{*i})\right)^2 + g_{22}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^2}{\partial u^{*2}}(u^{*i})\right)^2$$
$$= \left(1 + (u^{*1})^2\right)^2\left((u^{*1})^2\cos^2 u^{*2} + (u^{*1})^2\sin^2 u^{*2}\right) = (u^{*1})^2\left(1 + (u^{*1})^2\right)^2$$

and

$$g^*(u^{*i}) = g^*_{11}(u^{*i})g^*_{22}(u^{*i}) - g_{12}*(u^{*i}) = (u^{*1})^2\left(1 + (u^{*1})^2\right)^4.$$

Writing $\rho = u^{*1}$, we obtain the formulae in (19).

(b) Since $\text{sgn}(D) = 1$, the formulae of transformation for the second fundamental coefficients

$$L_{jk}(u^{*i}) = L_{\ell m}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^\ell}{\partial u^{*j}}(u^{*i})\frac{\partial u^m}{\partial u^{*j}}(u^{*i})sgn(D) \text{ for } j,k = 1,2$$

yield by Lemma 3.1 (b)

$$L^*_{11}(u^{*i}) = L_{11}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^1}{\partial u^{*1}}(u^{*i})\right)^2 + L_{22}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^2}{\partial u^{*1}}(u^{*i})\right)^2$$
$$= 2\left(\cos^2 u^{*2} - \sin^2 u^{*2}\right) = 2\cos(2u^{*2})$$

$$L^*_{12}(u^{*i}) = L_{11}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^1}{\partial u^{*1}}(u^{*i})\frac{\partial u^1}{\partial u^{*2}}(u^{*i}) + L_{22}(u^1(u^{*i}), u^2(u^{*i}))\frac{\partial u^2}{\partial u^{*1}}(u^{*i})\frac{\partial u^2}{\partial u^{*2}}(u^{*i})$$
$$= 2\left(-u^{*1}\sin u^{*2}\cos u^{*2} - u^{*1}\sin u^{*2}\cos u^{*2}\right) = -2u^{*1}\sin(2u^{*2})$$
$$L^*_{22}(u^{*i}) = L_{11}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^1}{\partial u^{*2}}(u^{*i})\right)^2 + L_{22}(u^1(u^{*i}), u^2(u^{*i}))\left(\frac{\partial u^2}{\partial u^{*2}}(u^{*i})\right)^2$$
$$= -2\left((u^{*1})^2\cos^2 u^{*2} - (u^{*1})^2\sin^2 u^{*2}\right) = -2(u^{*1})^2\cos(2u^{*2})$$

and

$$L^*(u^{*i}) = L^*_{11}(u^{*i})L^*_{22}(u^{*i}) - L_{12}*(u^{*i}) = -4(u^{*1})^2\cos^2 2u^{*2} - 4(u^{*1})^2\sin^2(2u^{*2}) = -4(u^{*1})^2.$$

Writing $\rho = u^{*1}$ and $\phi = u^{*2}$, we obtain the formulae in (20). □

Finally, we give the results of Theorem 3.2 in terms of the parameters ρ and ϕ of the parametric representation (2).

Theorem 3.4. *Let Enneper's surface be given by the parametric representation*

$$\vec{x}(\rho,\phi) = \left\{\rho\cos\phi - \frac{\rho^3}{3}\cos(3\phi), \rho\sin\phi + \frac{\rho^3}{3}\sin(3\phi), \rho^2\cos(2\phi)\right\} \quad ((\rho,\phi) \in (0,\infty)\times(0,2\pi)).$$

(a) Then the Gaussian curvature is given by

$$K(\rho,\phi) = -\frac{4}{(1+\rho^2)^4}.$$

(b) The mean curvature is $H(\rho,\phi) = 0$.
(c) The principal curvatures are

$$\kappa_{1,2} = \pm\frac{1}{(1+\rho^2)^2}.$$

(d) The asymptotic lines are given by

$$\rho^{(1)}(\phi) = c^{(1)}\sqrt{\left|\frac{\tan(\phi+\pi/4)}{\cos(2\phi)}\right|} = c^{(1)}\frac{1}{|\cos\phi - \sin\phi|} \quad \text{for } \phi \neq \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \tag{21}$$

$$\rho^{(2)}(\phi) = c^{(2)}\sqrt{\left|\frac{1}{\tan(\phi+\pi/4)\cos(2\phi)}\right|} = c^{(2)}\frac{1}{|\cos\phi + \sin\phi|} \quad \text{for } \phi \neq \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \tag{22}$$

where $c^{(1)}$ *and* $c^{(2)}$ *are positive constants, and the ρ–lines that correspond to* $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$ *(Figure 6).*
(e) The lines of curvature are given by

$$\rho^{(1)}(\phi) = \frac{c^{(1)}}{|\cos\phi|} \quad \text{and} \quad \rho^{(2)}(\phi) = \frac{c^{(1)}}{|\sin\phi|} \quad \text{for } \phi \neq \pi/2, \pi, 3\pi/2,$$

where $c^{(1)}$ *and* $c^{(2)}$ *are positive constants, and the ρ–lines that correspond to* $\phi = \pi/2, \pi, 3\pi/2$ *(Figure 7).*
(f) Let $(\rho_0, \phi_0) \in I_\rho \times I_\phi$ *and* $\Theta_0 \in (-\pi/2, \pi/2)$ *be given. We put*

$$c = \rho_0(1+\rho_0^2)\cos\Theta_0$$

and

$$\rho_1 = 2\sqrt{\frac{1}{3}}\sinh\left(\frac{1}{3}\log\left(\frac{3\sqrt{3}|c|}{2} + \sqrt{\frac{27c^2}{4}+1}\right)\right).$$

Then the geodesic line through (ρ_0, ϕ_0) *with an angle* Θ_0 *to the* ϕ*–line through* ρ_0 *is given by*

$$\phi(\rho) = c\int_{\rho_0}^{\rho}\frac{dt}{t\sqrt{t^2(1+t^2)^2 - c^2}} + \phi_0 \quad \textit{for } \rho > \rho_1 \ \ \textit{(Figure 8).}$$

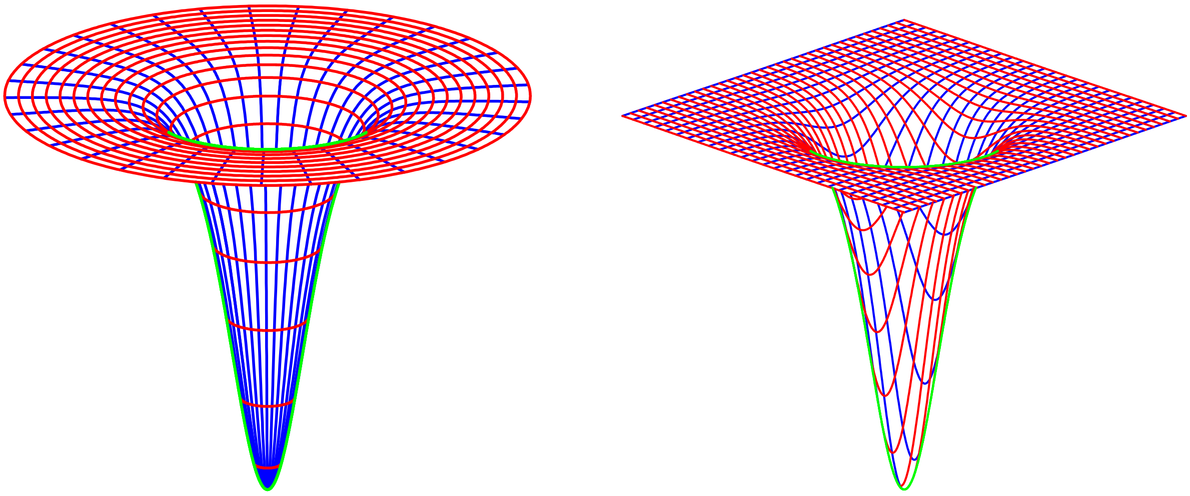

Figure 5: The Gaussian curvature of Enneper's surface represented as a
Left: screw surface $\vec{x}(\rho,\phi) = \{\rho\cos\phi, \rho\sin\phi, K(\rho,\phi)\}$
Right: explicit surface $\vec{x}(u^i) = \{u^1, u^2, K(u^1, u^2)\}$.

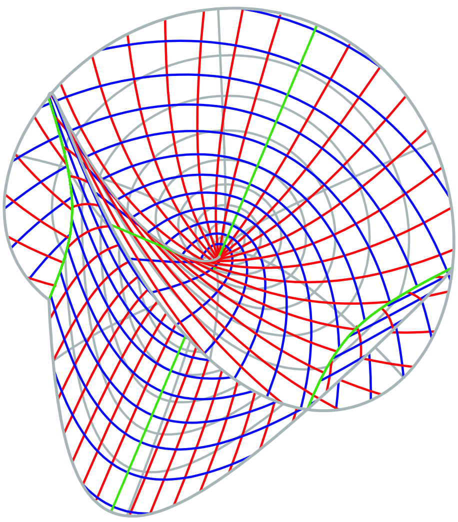

Figure 6: Asymptotic lines on Enneper's surface.

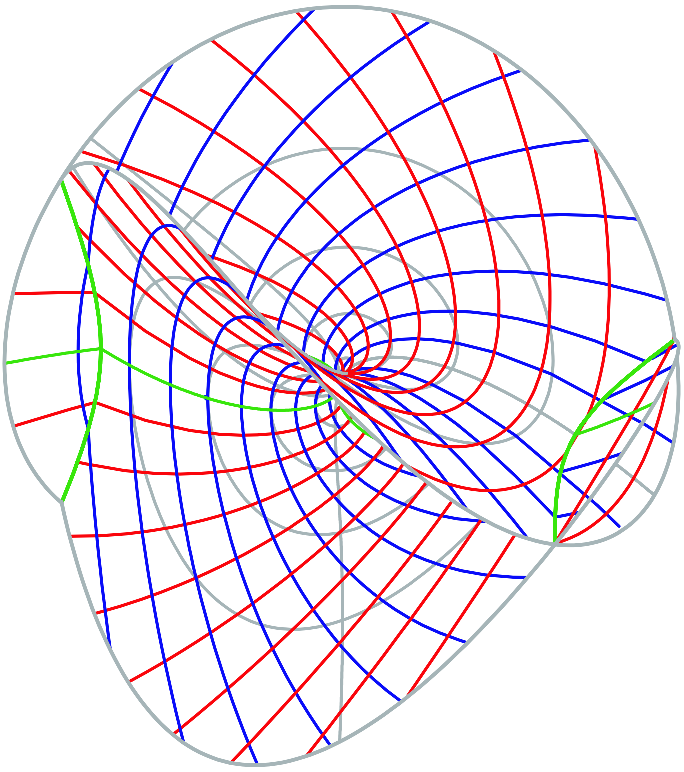

Figure 7: Lines of curvature on Enneper's surface.

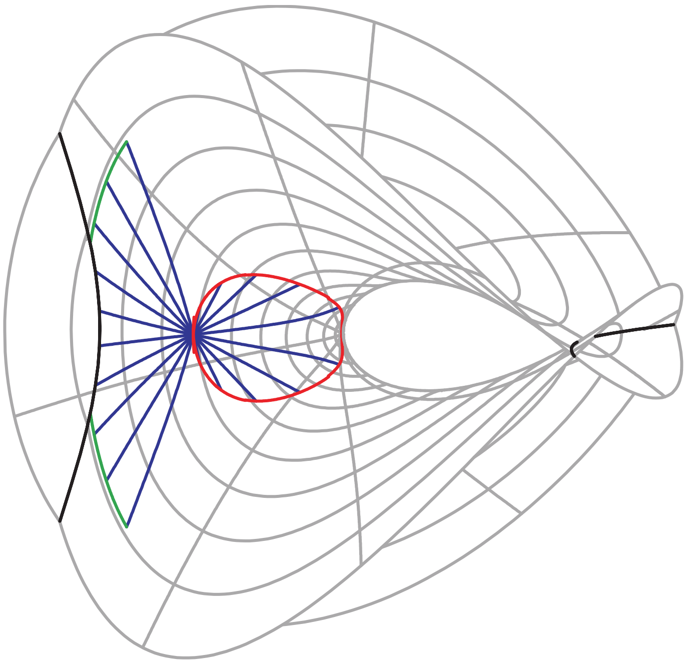

Figure 8: Geodesic lines on Enneper's surface.

Proof. Parts (a), (b) and (c) are immediate consequences of Theorem 3.2 (a), (b) and (c).

(d) The differential equation for the asymptotic lines is

$$\begin{aligned} &L_{11}(\rho,\phi)(d\rho)^2 + 2L_{12}(\rho,\phi)d\rho d\phi + L_{22}(\rho,\phi)(d\phi)^2 \\ &= 2\cos(2\phi)(d\phi)^2 - 4\rho\sin(2\phi)d\rho d\phi - 2\rho^2\cos(2\phi)(d\phi)^2 = 0. \end{aligned}$$

If $\cos(2\phi) = 0$, that is $\phi = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, then the ρ–lines that correspond to these values of ϕ are asymptotic line. Otherwise, we have

$$\left(\frac{d\rho}{d\phi}\right)^2 - 2\rho\tan(2\phi)\frac{d\rho}{d\phi} - \rho^2 = 0,$$

or

$$\begin{aligned} \frac{d\rho}{d\phi} &= \rho\tan(2\phi) \pm \sqrt{\rho^2\tan^2(2\phi)+1} = \rho\left(\tan(2\phi) \pm \sqrt{\tan^2+1}\right) \\ &= \rho\left(\frac{\sin(2\phi)}{\cos(2\phi)} \pm \frac{1}{|\cos(2\phi)|}\right). \end{aligned}$$

This implies

$$\begin{aligned} \log\rho &= \int\frac{\sin(2\phi)}{\cos(2\phi)}\,d\phi \pm \int\frac{1}{|\cos(2\phi)|}\,d\phi \\ &= -\frac{1}{2}\log(\cos(2\phi)) \pm \frac{1}{2}\log\left(\tan\left(\phi+\pi/4\right)\right) + \log\delta, \end{aligned}$$

where $\delta > 0$ is a constant, and the first identities in (21) and (22) are an immediate consequence. Furthermore, we have

$$\begin{aligned} \frac{\tan\left(\phi+\pi/4\right)}{\cos(2\phi)} &= \frac{\sin\left(\phi+\pi/4\right)}{\cos\left(\phi+\pi/4\right)\left(\cos^2\phi-\sin^2\phi\right)} \\ &= \frac{\frac{1}{\sqrt{2}}\left(\sin\phi+\cos\phi\right)}{\frac{1}{\sqrt{2}}\left(\sin\phi-\cos\phi\right)\left(\cos^2\phi-\sin^2\phi\right)} = \frac{1}{\left(\cos\phi-\sin\phi\right)^2} \end{aligned}$$

and similarly

$$\begin{aligned} \frac{1}{\tan\left(\phi+\pi/4\right)\cos(2\phi)} &= \frac{\cos\left(\phi+\pi/4\right)}{\sin\left(\phi+\pi/4\right)\left(\cos^2\phi-\sin^2\phi\right)} \\ &= \frac{\frac{1}{\sqrt{2}}\left(\sin\phi-\cos\phi\right)}{\frac{1}{\sqrt{2}}\left(\sin\phi+\cos\phi\right)\left(\cos^2\phi-\sin^2\phi\right)} = \frac{1}{\left(\cos\phi+\sin\phi\right)^2}. \end{aligned}$$

Thus the second identities in (21) and (22) also hold.

(e) The differential equation for the lines of curvature is

$$\det\begin{pmatrix} L_{11}(\rho,\phi)d\rho + L_{12}(\rho,\phi)d\phi & g_{11}(\rho,\phi)d\rho + g_{12}(\rho,\phi)d\phi \\ L_{12}(\rho,\phi)d\rho + L_{22}(\rho,\phi)d\phi & g_{12}(\rho,\phi)d\rho + g_{22}(\rho,\phi)d\phi \end{pmatrix} =$$

$$L_{11}(\rho,\phi)g_{22}(\rho,\phi)d\rho d\phi + L_{12}(\rho,\phi)g_{22}(\rho,\phi)(d\rho)^2$$
$$- L_{12}(\rho,\phi)g_{11}(\rho,\phi)(d\rho)^2 - L_{22}(\rho,\phi)g_{11}(\rho,\phi)d\rho d\phi =$$
$$2\rho(1+\rho^2)^2\sin(2\phi)(d\rho)^2 + 4\rho^2(1+\rho^2)^2\cos(2\phi)d\rho d\phi - 2\rho^3(1+\rho^2)^2\sin(2\phi)(d\phi)^2$$

If $\phi = \pi/2, \pi, 3\pi/2$ then the ρ–lines corresponding to $\phi = \pi/2, \pi, 3\pi/2$ are lines of curvature. Otherwise, we have

$$(d\rho)^2 + 2\rho\cot(2\phi)d\rho d\phi - \rho^2(d\phi)^2 = 0$$

or

$$\frac{d\rho}{d\phi} = -\rho\cot(2\phi) \pm \sqrt{\rho^2\cot^2(2\phi)+1} = \rho\left(\frac{\cos(2\phi)}{\sin(2\phi)} \pm \frac{1}{|\sin(2\phi)|}\right).$$

This implies

$$\log\rho = -\frac{1}{2}\log|\sin(2\phi)| \pm \frac{1}{2}\log|\tan\phi| + \log\delta,$$

where $\delta > 0$ is a constant. We obtain

$$\rho^{(1)}(\phi) = c_1\sqrt{\frac{\tan\phi}{\sin(2\phi)}} = c_1\sqrt{\frac{\sin\phi}{2\cos^2\phi\sin\phi}} = \frac{c^{(1)}}{|\cos\phi|}$$

and

$$\rho^{(2)}(\phi) = c_2\sqrt{\frac{1}{\tan\phi\sin(2\phi)}} = c_2\sqrt{\frac{\cos\phi}{2\sin^2\phi\cos\phi}} = \frac{c^{(2)}}{|\sin\phi|}$$

where $c^{(k)} = c_k/\sqrt{2}$ for $k = 1, 2$.

(f) Since $g_{11}(\rho,\phi) = g_{11}(\rho) = (1+\rho^2)^2$, $g_{12}(\rho,\phi) = 0$ and $g_{22}(\rho,\phi) = g_{22}(\rho) = \rho^2(1+\rho^2)^2$ by (19) in Lemma 3.3, [10, Satz 4.3.1, p. 359] yields for the geodesic line through (ρ_0,ϕ_0) at an angle of Θ_0 to the ϕ–line through ρ_0

$$\phi(\rho) = c\int_{\rho_0}^{\rho} \frac{\sqrt{g_{11}(t)}}{\sqrt{g_{22}(t)}\sqrt{g_{22}(t)-c^2}}\,dt + \phi_0$$
$$= c\int_{\rho_0}^{\rho} \frac{(1+t^2)\,dt}{t(1+t^2)\sqrt{t^2(1+t^2)^2-c^2}} + \phi_0 = c\int_{\rho_0}^{\rho} \frac{dt}{t\sqrt{t^2(1+t^2)^2-c^2}} + \phi_0,$$

where

$$c = \sqrt{g_{22}(\rho_0)}\cos\Theta_0 = \rho_0(1+\rho_0^2)\cos\Theta_0,$$

and the integral exists for all $\rho \in I_\rho$ for which $\rho^2(1+\rho^2)^2 > c^2$, that is

$$\rho(1+\rho^2) > |c| \text{ which is equivalent to } \rho^3 + \rho - |c| > 0.$$

As in the proof of Lemma 2.1, we obtain $\rho > \rho_1$. □

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