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Torse-Forming η-Ricci Solitons in Almost Paracontact η-Einstein Geometry

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Abstract. Torse-forming η-Ricci solitons are studied in the framework of almost paracontact metric η-Einstein manifolds. By adding a technical condition, called regularity and concerning with the scalars provided by the two η-conditions, is obtained a reduction result for the parallel symmetric covariant tensor fields of order two.

1. Introduction

The problem of studying Ricci solitons in the context of metric paracontact geometry was initied by G. Calvaruso and D. Perrone [5]. More precisely, they study the case when the fundamental (Reeb-type) vector field ξ of the paracontact structure is harmonic, which is the impact of its harmonicity to the paracontact Ricci solitons. So they prove that any metric paracontact η-Einstein manifold is H-paracontact, which means that the vector field ξ is harmonic and provide a necessary and sufficient condition for a metric paracontact manifold to be H-paracontact, namely if ξ is an eigenvector of the Ricci tensor field Q; as example they obtain that the para-Sasakian manifolds are H-paracontact. The case of Ricci solitons provided by ξ in a 3-dimensional normal paracontact manifold was treated by C. L. Bejan and M. Crasmareanu in [2] respectively G. Calvaruso and A. Perrone in [6].

The objective of present paper is to study a generalization called *η-Ricci solitons* in almost paracontact metric manifolds. These structures were introduced by J. T. Cho and M. Kimura in [10] and studied in Hopf hypersurfaces of complex space forms in [7] as well as in [13].

With the framework of [12] we shall consider *almost paracontact metric manifolds* as data $(M, \varphi, \xi, \eta, g)$, for M a $(2n+1)$-dimensional smooth manifold, φ a tensor field of $(1,1)$-type, ξ a vector field, η a 1-form and g a pseudo-Riemannian metric of signature $(n+1, n)$, satisfying:
1) $\varphi^2 = I - \eta \otimes \xi$;
2) $\eta(\xi) = 1$;
3) $g(\varphi\cdot, \varphi\cdot) = -g + \eta \otimes \eta$.
From these conditions follows immediately that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, $i_\xi g = \eta$, $g(\xi, \xi) = 1$ which means that ξ is a space-like vector field and $g(\varphi X, Y) = -g(X, \varphi Y)$ for any $X, Y \in \mathfrak{X}(M)$. For other aspects concerning this type of structures please see [4]-[5] while very interesting generalizations appear in [3] and [16].

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Due to the complexity of computations for the general case we restrict our work to a first technical condition called *η-Einstein* which concerns with a simplified expression of the Ricci curvature; two scalars a, b are then introduced. This situation is sufficient of general since it is a generalization of the Einstein classical condition regarding the Ricci tensor as multiple of the metric. Let us point out that from the definition of η-Ricci soliton other two scalars appear: λ and μ. A second condition is then introducing with respect to the covariant derivative of ξ; more precisely ξ is assumed to be torse-forming. Then we derive relationships between all four scalars before.

A second problem studied here is about conditions which assure the reduction of a parallel symmetric tensor field of $(0,2)$-type to a scalar multiple of metric tensor. This problem implies another condition called regularity as in [7, p. 58] and is expressed in terms of only two scalars. The last section is devoted to this study.

2. η-Ricci Solitons on Almost Paracontact η-Einstein Manifolds

The framework of this paper is specified by a condition on the Ricci tensor field considered firstly in [15]:

Definition 2.1. *A pseudo-Riemannian manifold (M, g) is called η-Einstein if there exist two real constants a and b such that the Ricci curvature tensor field S is:*

$$S = ag + b\eta \otimes \eta. \tag{1}$$

Assume from now that the almost paracontact metric manifold $(M, \varphi, \xi, \eta, g)$ is η-Einstein; an important consequence is that S is a symmetric tensor field. With respect to φ it is skew-symmetric:

$$S(\varphi X, Y) = -S(X, \varphi Y) = ag(\varphi X, Y), \quad S(\varphi X, \varphi Y) = -S(X, Y) + (a + b)\eta(X)\eta(Y), \tag{2}$$

$$S(X, \xi) = (a + b)\eta(X), \quad S(\xi, \xi) = a + b. \tag{3}$$

The Ricci $(1,1)$-tensor field Q defined by $S(X, Y) = g(QX, Y)$ is given by $Q = aI + b\eta\otimes\xi$ and ξ is an eigenvector of Q corresponding to the eigenvalue $a + b$.

On the almost paracontact metric η-Einstein manifold $(M, \varphi, \xi, \eta, g, a, b)$ we consider the *paracontact η-Ricci soliton* [10], that is a data (ξ, λ, μ) satisfying the equation:

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0 \tag{4}$$

for λ and μ real constants.

Replacing the expression (1) of the Ricci curvature tensor field in the previous equation and writing $L_\xi g$ as usually in terms of the Levi-Civita connection ∇ we get:

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + 2[(a + \lambda)g(X, Y) + (b + \mu)\eta(X)\eta(Y)] = 0 \tag{5}$$

who yields the first main result of this note:

Proposition 2.2. *An η-Ricci soliton on the almost paracontact metric η-Einstein manifold $(M, \varphi, \xi, \eta, g, a, b)$ satisfies: 1) $a + b + \lambda + \mu = 0$; 2) ξ is a geodesic vector field: $\nabla_\xi\xi = 0$; 3) $(\nabla_\xi\varphi)\xi = 0$ and $\nabla_\xi\eta = 0$ with the consequences: $\nabla_\xi S = 0$ and $\nabla_\xi Q = 0$.*

Proof. Replacing $X = Y = \xi$ in (5) we obtain $g(\nabla_\xi\xi, \xi) = -(a + b + \lambda + \mu)$ but $g(\nabla_X\xi, \xi) = 0$ for any $X \in \mathfrak{X}(M)$ since ξ has a constant norm; it follows 1). The equation (5) becomes:

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) + 2(a + \lambda)[g(X, Y) - \eta(X)\eta(Y)] = 0. \tag{6}$$

With $Y = \xi$ one obtains $g(X, \nabla_\xi\xi) = 0$ for any $X \in \mathfrak{X}(M)$ and so we have 2). The first two relations of 3) are straightforward consequences of 2). More precisely, the general expression for ∇S and ∇Q is:

$$(\nabla_X S)(Y, Z) = b[\eta(Y)g(Z, \nabla_X\xi) + \eta(Z)g(Y, \nabla_X\xi)], \quad (\nabla_X Q)Y = b[\eta(Y)\nabla_X\xi + g(Y, \nabla_X\xi)\xi]$$

and we get 3). $\square$

An important case arises when the vector field ξ is *torse-forming* [17]:

$$\nabla_X \xi = fX + \gamma(X)\xi \tag{7}$$

for a smooth function $f \in C^\infty(M)$ and a 1-form $\gamma \in \Omega^1(M)$. Note that torse-forming vector fields appear in many areas of differential geometry and physics as is point out in [14]. For our setting we derive:

Proposition 2.3. *If ξ is a torse-forming η-Ricci soliton on the almost paracontact metric η-Einstein manifold $(M, \varphi, \xi, \eta, g, a, b)$ then f is a constant, η is closed and:*

$$b = -a - 2n(a+\lambda)^2, \quad \mu = 2n(a+\lambda)^2 - \lambda. \tag{8}$$

With X an unitary space-like or time-like vector field and orthogonal on ξ we have the sectional curvature: $K(X,\xi) = -f^2 = -(a+\lambda)^2$.

Proof. We have $g(\nabla_X \xi, \xi) = (f\eta + \gamma)X$ hence we get $\gamma = -f\eta$ and so $\nabla_X \xi = f[X - \eta(X)\xi]$. Then the equation (5) becomes:

$$(f + a + \lambda)[g(X,Y) - \eta(X)\eta(Y)] = 0, \tag{9}$$

for all vector fields X, Y. It follows that $f = -a - \lambda$ and:

$$\nabla_X \xi = -(a+\lambda)[X - \eta(X)\xi] \tag{10}$$

which means that $\nabla_X \xi$ is collinear to $\varphi^2 X$ for any $X \in \mathfrak{X}(M)$; hence we get that $d\eta = 0$. It follows the curvature tensor field:

$$R(X,Y)\xi = (a+\lambda)^2(\eta \otimes I - I \otimes \eta)(X,Y) \tag{11}$$

which yields the claimed expression of the sectional curvature and:

$$S(X,\xi) = -2n(a+\lambda)^2\eta(X). \tag{12}$$

Comparing this equation with the first equation of (3) it results the first part of (8). The second part is consequence of the property 1) of proposition 2.2. $\square$

Remark 2.4. *The closedness of η means that our "almost" paracontact framework is different to that of "metric" from [5] where an essential axiom is: $d\eta(\cdot,\cdot) = g(\cdot, \varphi\cdot)$.*

A straightforward computation gives:

$$(\nabla_X S)(Y,Z) = -b(a+\lambda)[\eta(Z)g(X,Y) + \eta(Y)g(Z,X) - 2\eta(X)\eta(Y)\eta(Z)] \tag{13}$$

$$(\nabla_X Q)Y = -b(a+\lambda)[\eta(Y)X + g(X,Y)\xi - 2\eta(X)\eta(Y)\xi] \tag{14}$$

The important particular cases are the following:
I) $f := -1$ i.e. ξ is a *irrotational* vector field, $\lambda = 1 - a$ and $\mu = -1 - b$. In this case:

$$R(X,Y)\xi = (\eta \otimes I - I \otimes \eta)(X,Y). \tag{15}$$

Let us remark that from $\gamma = \eta \neq 0$ it follows that ξ is not a concurrent vector field as studied in [9].
II) $f := 0$ i.e. ξ is a *recurrent* vector field, $\lambda = -a$ and $\mu = -b$. In this case ξ is parallel (hence Killing) and:

$$R(X,Y)\xi = 0 \tag{16}$$

and S respectively Q are also parallel tensor fields.

A consequence of proposition 2.3 gives a type of Ricci solitons in the particular case of η-Einstein paracontact geometry:

Corollary 2.5. *A torse-forming Ricci soliton i.e.* $\mu = 0$ *on the non-Ricci flat almost paracontact* η*-Einstein manifold* M *with* $a = 0$ *is expanding* $\lambda = \frac{1}{2n} > 0$ *with* $b = -\frac{1}{2n}$*.*

Proof. From the second part of (8) we have: $\lambda = 2n\lambda^2$. A first solution $\lambda_1 = 0$ means that $b = 0$ and then M is Ricci flat. It follows the unique compatible solution $\lambda_2 = \frac{1}{2n}$. □

Example 2.6. *The case of dimension* $2n + 1 = 3$ *is studied in [2] by means of the functions:* $\alpha = \frac{1}{2}div\xi$ *and* $\beta = \frac{1}{2}trace(\varphi\nabla\xi)$*. By adding the normality condition it follows the Ricci tensor:*

$$S(X,Y) = \left[\alpha^2 + \beta^2 + \xi(\alpha) + \frac{r}{2}\right]g(X,Y) - \left[3(\alpha^2 + \beta^2) + \xi(\alpha) + \frac{r}{2}\right]\eta(X)\eta(Y) + [\varphi X(\beta) - X(\alpha)]\eta(Y) +$$
$$+[\varphi Y(\beta) - Y(\alpha)]\eta(X) \tag{17}$$

and it results that (M^3, g) *is an* η*-Einstein manifold if and only if* α, β *and* r *are constants where* r *is the scalar curvature. The expanding character obtained above corresponds to the para-Kenmotsu case of the cited paper with* $\alpha = -\frac{1}{2} = r$, $\beta = 0$ *as well as in the Example 3.7 of [1, p. 240].*

Remark 2.7. *Expanding torse-forming Ricci solitons are obtained in [11, p. 368] and general expanding Ricci solitons in 3D paracontact geometry are provided by theorem 3.4 of [6].*

3. Parallel Symmetric (0, 2)-Tensor Fields on Almost Paracontact η-Einstein Manifolds

According to the last particular case of torse-forming vector fields we study now symmetric $(0, 2)$-tensor fields which are parallel with respect to the Levi-Civita connection having as model the paper [8].

Let α be such a symmetric $(0, 2)$-tensor field which is parallel i.e. $\nabla\alpha = 0$. From the Ricci identity $\nabla^2\alpha(X, Y; Z, W) - \nabla^2\alpha(X, Y; W, Z) = 0$ one obtains similar to [2]:

$$\alpha(R(X,Y)Z, W) + \alpha(Z, R(X,Y)W) = 0. \tag{18}$$

With $Z = W = \xi$ and the symmetry of α it follows:

$$\alpha(R(X,Y)\xi, \xi) = 0. \tag{19}$$

Suppose now that ξ is a torse-forming vector field and replace the expression (10) in α to get:

$$(a + \lambda)^2[\eta(X)\alpha(Y, \xi) - \eta(Y)\alpha(X, \xi)] = 0. \tag{20}$$

With $X = \xi$ and $Y = \varphi^2 Z$ it follows:

$$0 = (a + \lambda)^2\alpha(\varphi^2 Z, \xi) = (a + \lambda)^2[\alpha(Z, \xi) - \eta(Z)\alpha(\xi, \xi)], \tag{21}$$

for any $Z \in \mathfrak{X}(M)$. We must introduce a special type of torse-forming η-Ricci solitons in η-Einstein paracontact geometry:

Definition 3.1. *The paracontact* η*-Ricci soliton* $(M, \varphi, \xi, \eta, g, a, b, \lambda, \mu)$ *is regular if* $a + \lambda \neq 0$*.*

The main consequence of this type of Ricci solitons is:

Proposition 3.2. *If the paracontact* η*-Ricci soliton* $(M, \varphi, \xi, \eta, g, a, b, \lambda, \mu)$ *is regular and torse-forming then any parallel symmetric* $(0, 2)$*-tensor field is a constant multiple of the metric.*

Proof. From regularity we have:

$$\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi) = 0, \tag{22}$$

for any $Y \in \mathfrak{X}(M)$. Differentiating this equation covariantly with respect to $X \in \mathfrak{X}(M)$ we obtain:

$$\alpha(Y,\varphi^2 X) = \alpha(\xi,\xi)g(Y,\varphi^2 X)$$

and substituting the expression of φ^2 we get:

$$\alpha(Y,X) = \alpha(\xi,\xi)g(Y,X) \tag{23}$$

for any X, $Y \in \mathfrak{X}(M)$. Since α is ∇-parallel it follows that $\alpha(\xi,\xi)$ is a constant and the proof is complete. □

Remark 3.3. *The case of recurrent η-Ricci solitons does not belongs to the proposition above since we do not have the regularity.*

Example 3.4. *Returning to the Example 2.6 we have from [2]: $\nabla_X\xi = \alpha(X - \eta(X)\xi) + \beta\varphi(X)$ and then ξ is torse-forming only for $\beta = 0$. Hence, with* (17)+(10) *we derive:*

$$a = \alpha^2 + \xi(\alpha) + \frac{r}{2}, \quad a + \lambda = -\alpha \tag{24}$$

and therefore M is regular if and only if it is not quasi-para-Sasakian. In particular, M is not para-Sasakian and not H-paracontact. Also, from the second equation above it results that α is constant which means that the scalar curvature is constant:

$$r = 2(a - \alpha^2). \tag{25}$$

A last reduction result for regular eta-Ricci solitons is provided by the Codazzi condition in Ricci tensor:

Proposition 3.5. *A regular paracontact η-Ricci soliton $(M,\varphi,\xi,\eta,g,a,b,\lambda,\mu)$ with Codazzi-Ricci tensor is an Einstein manifold with $b = 0$ and ξ a Killing vector field.*

Proof. Recall that the Ricci tensor field is a Codazzi one if the following commutation formula holds:

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{26}$$

for all vector fields X, Y. With the expression of $\nabla_X Q$ from the last equation in the proof of proposition 2.2 and supposing $b \neq 0$ we get:

$$\eta(Y)\nabla_X\xi + g(Y,\nabla_X\xi)\xi = \eta(X)\nabla_Y\xi + g(X,\nabla_Y\xi)\xi. \tag{27}$$

For $Y = \xi$ one obtains $\nabla_X\xi = 0$ yielding the Killing conclusion and which plugged in (5) gives:

$$(a+\lambda)(g(X,Y) - \eta(X)\eta(Y)) = 0. \tag{28}$$

The regularity yields $\eta \otimes \eta = 0$ which the condition 3) from the definition of almost paracontact structures from Introduction gives $g(\varphi\cdot,\varphi\cdot) = 0$. This relation is impossible since the $2n$-dimensional structural distribution $Ker\eta$ have a basis with eigenvectors corresponding to the eigenvalues ± 1 of φ. Hence $b = 0$ and (M,g) is Einstein. □

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