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$\mathcal{I}_\lambda$-Statistically Convergent Functions of Order α

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Abstract. In present paper, we further generalize the recently introduced summability method of [22] and introduce the new notion namely, $\mathcal{I}_\lambda-$statistical convergence of order α for a real valued function which is measurable in the interval $(1,\infty)$. We mainly investigate certain properties of this convergence. The study leaves some interesting open problems.

1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [12] and later also by Schoenberg [28]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, measure theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with with summability theory by Fridy [13], Connor [5], Šalát [18], Cakalli [2] and many others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover statistical convergence is closely related to the concept of convergence in probability.

The statistical convergence depends on the density of subsets of $\mathbf{N}$. The natural density of a subset A of $\mathbf{N}$ is defined by $\delta(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \le n : k \in A\}|$ where the vertical bars indicate the total number of elements in the enclosed set. It is clear that any finite subset of N has zero natural density and $\delta(A^c) = 1-\delta(A)$, where $A^c = N - A$. A set A is said to be statistically dense if $\delta(A) = 1$. A subsequence of a sequence is said to be statistically dense if the set of all indices of its elements is statistically dense.

Definition 1.1. *A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$*

$$\delta(\{k \in \mathbf{N} : |x_k - L| \ge \varepsilon\}) = 0.$$

We write $st - \lim x_k = L$ in case $x = (x_k)$ is $st-$ statistically convergent to L.

The concept of $\mathcal{I}-$convergence was introduced by Kostyrko, Salat and Wilczyński in a metric space [14]. Later it was further studied by Dems [10], Das and Savas ([8],[25], [26]), Savas ([20], [21], [23], [24], [22]), Savas and Gumus [27] and many others. $\mathcal{I}-$convergence is a generalization form of statistical convergence

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and that is based on the notion of an ideal of the subset of positive integers $\mathbb{N}$. More applications of ideals can be found in [7] and [17].

Recently in [9, 25] we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence and $\mathcal{I}_\lambda$-statistical convergence and investigated their properties. Quite recently we also defined the notions of $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence of order α,(see, [8]).

On the other hand in [1, 3] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha < 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence. One can also see [4] for related works.

In present paper we introduce a new and further general notion, namely, $\mathcal{I}_\lambda-$statistical convergence of order α for a nonnegative real valued function which is measurable in the interval $(1, \infty)$. We mainly investigate some basic properties of this new summability method and also we leave some interesting open problems.

Throughout by a function $x(t)$ we shall mean a nonnegative real valued function which is measurable in the interval $(1, \infty)$.

2. Main Results

The following definitions and notions will be needed in the sequel.

Definition 2.1. *A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$ if the following conditions hold:*
(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(b) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$,

Definition 2.2. *A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be an filter of $\mathbb{N}$ if the following conditions hold:*
(a) $\phi \notin \mathcal{F}$,
(b) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(c) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$,

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 2.3. *A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.*

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 2.4. *(See [14]) Let $I \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$.*
The sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \epsilon\} \in \mathcal{I}$.

We now introduce our main definitions.

Definition 2.5. *Let $x(t)$ be a nonnegative real valued function which is measurable in the interval $(1, \infty)$. A function $x(t)$ is said to be $\mathcal{I}$-statistically convergent of order α to L or $S(I)^\alpha$-convergent to L, where $0 < \alpha \leq 1$, if for each $\epsilon > 0$ and $\delta > 0$*

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{t \leq n : |x(t) - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write $x(t) \to L(S(\mathcal{I})^\alpha)$. The class of all $\mathcal{I}$-statistically convergent of order α functions will be denoted by simply $S(\mathcal{I})^\alpha$.

Remark 2.6. *For $\mathcal{I} = \mathcal{I}_f in = \{A \subseteq \mathbf{N} : A$ is a finite subset $\}, S(\mathcal{I})^\alpha$-convergence coincides with statistical convergence of order α for functions which has not been study till now. For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$ it coincides with $\mathcal{I}$-statistical convergence of function, (see,[22]). When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only statistical convergence of functions,(see,[16]).*

Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1.$$

The collection of such sequences λ will be denoted by Δ.

We define the generalized de la Vallée-Pousin mean of order α by

$$t_n(x) = \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$. More applications of de la Vallée-Pousin mean can be found in ([11],[15] and [19]).

We now have

Definition 2.7. *A function $x(t)$ is said to be $[V, \lambda]^\alpha (\mathcal{I})-$ summable to L, if for any $\delta > 0$,*

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \int_{n-\lambda_n+1}^{n} |x(t) - L| \geq \delta\} \in \mathcal{I}.$$

If $\mathcal{I} = \mathcal{I}_{fin}$, $[V, \lambda]^\alpha (\mathcal{I})-$ summablility becomes $[V, \lambda]^\alpha$ summability.

Definition 2.8. *A function $x(t)$ is said to be $\mathcal{I}_\lambda-$statistically convergent of order α or $S_\lambda^\alpha(\mathcal{I})$- convergent to L, if for every $\epsilon > 0$ and $\delta > 0$,*

$$\left\{n \in N : \frac{1}{\lambda_n^\alpha} | \{t \in I_n : |x(t) - L| \geq \epsilon\} | \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $S_\lambda^\alpha(\mathcal{I}) - \lim x(t) = L$ or $x(t) \to L\ (S_\lambda^\alpha)(\mathcal{I})$.

Remark 2.9. *For $\mathcal{I} = \mathcal{I}_{fin}$, $S_\lambda^\alpha(\mathcal{I})$-convergence again coincides with $\lambda-$ statistical convergence of order α for function. For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$ it coincides with $\mathcal{I}_\lambda$-statistical convergence (see,[22]). Finally for $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes λ-statistical convergence of function, [16].*

Also note that taking $\lambda_n = n$ we get definition 5 from definition 6.

We shall denote by $S_\lambda (\mathcal{I})^\alpha$ and $[V, \lambda]^\alpha (\mathcal{I})$ the collections of all $S_\lambda(\mathcal{I})$-convergent of order α and $[V, \lambda] (\mathcal{I})$ -convergent of order α for a real valued functions which are measurable in the interval $(1, \infty)$.

Theorem 2.10. *Let $0 < \alpha \leq \beta \leq 1$. Then $S_\lambda(\mathcal{I})^\alpha \subset S_\lambda(\mathcal{I})^\beta$.*

Proof. Let $0 < \alpha \leq \beta \leq 1$. Then

$$\frac{|\{t \in I_n : |x(t) - L| \geq \epsilon\}|}{\lambda_n^\beta} \leq \frac{|\{t \in I_n : |x(t) - L| \geq \epsilon\}|}{\lambda_n^\alpha}$$

and so for any $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{|\{t \in I_n : |x(t) - L| \geq \epsilon\}|}{\lambda_n^\beta} \geq \delta\} \subset \{n \in \mathbb{N} : \frac{|\{t \in I_n : |x(t) - L| \geq \epsilon\}|}{\lambda_n^\alpha} \geq \delta\}.$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S_\lambda(\mathcal{I})^\alpha \subset S_\lambda(\mathcal{I})^\beta$. □

Corollary 2.11. *If a function $x(t)$ is $\mathcal{I}_\lambda$-statistically convergent of order α to L for some $0 < \alpha \leq 1$ then it is $\mathcal{I}_\lambda$-statistically convergent function to L i.e. $S_\lambda(\mathcal{I})^\alpha \subset S_\lambda(\mathcal{I})$.*

Similarly we can show that

Theorem 2.12. *Let $0 < \alpha \leq \beta \leq 1$. Then*
(i) $S(\mathcal{I})^\alpha \subset S(\mathcal{I})^\beta$.
(ii) $[V, \lambda]^\alpha (\mathcal{I}) \subset [V, \lambda]^\beta (\mathcal{I})$.
(iii) In particular $S(\mathcal{I})^\alpha \subset S(\mathcal{I})$ and $[V, \lambda]^\alpha (\mathcal{I}) \subset [V, \lambda] (\mathcal{I})$.

Theorem 2.13. *Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \in \Delta$. Then $x(t) \to L\, [V, \lambda]^\alpha (\mathcal{I}) \Rightarrow x(t) \to L\, (S_\lambda (\mathcal{I})^\alpha)$ and the inclusion $[V, \lambda]^\alpha (\mathcal{I}) \subset S_\lambda (\mathcal{I})^\alpha$ is proper for every ideal $\mathcal{I}$.*

Proof. Let $\epsilon > 0$ and $x(t) \to L\, [V, \lambda]^\alpha (\mathcal{I})$. We have

$$\int_{t\epsilon I_n} |x(t) - L| \geq \sum_{t\in I_n \& |x(t)-L|>\varepsilon} |x(t) - L| \geq \epsilon. |\{t \in I_n : |x(t) - L| \geq \epsilon\}|.$$

So for a given $\delta > 0$,

$$\frac{1}{\lambda_n^\alpha} |\{t \in I_n : |x(t) - L| \geq \epsilon\}| \geq \delta \Rightarrow \frac{1}{\lambda_n^\alpha} \int_{k\epsilon I_n} |x(t) - L| \geq \epsilon\delta$$

i.e. $\left\{n \in N : \frac{1}{\lambda_n^\alpha} |\{t \in I_n : |x(t) - L| \geq \epsilon\}| \geq \delta\right\} \subset \left\{n \in N : \frac{1}{\lambda_n^\alpha} \left\{\sum_{t\in I_n} |x(t) - L| \geq \epsilon\right\} \geq \epsilon\delta\right\}$.
Since $x(t) \to L\, [V, \lambda]^\alpha (\mathcal{I})$, so the set on the right hand side belongs to $\mathcal{I}$ and so it follows that $x(t) \to L\, (S_\lambda (\mathcal{I}))^\alpha$.

To show that $S_\lambda (\mathcal{I})^\alpha \subsetneq [V, \lambda]^\alpha (\mathcal{I})$, take a fixed $A \in \mathcal{I}$. Define $x(t)$ by

$$x_k(t) = \begin{cases} t & \text{for } n - [\sqrt{\lambda_n^\alpha}] + 1 \leq k \leq n, n \notin A \\ t & \text{for } n - \lambda_n + 1 \leq k \leq n, n \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $\epsilon > 0$ $(0 < \epsilon < 1)$ since

$$\frac{1}{\lambda_n^\alpha} |\{t \in I_n : |x(t)| \geq \epsilon\}| = \frac{\left[\sqrt{\lambda_n^\alpha}\right]}{\lambda_n^\alpha} \to 0$$

as $n \to \infty$ and $n \notin A$, so for every $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} |\{t \in I_n : |x(t)| \geq \epsilon\}| \geq \delta\right\} \subset A \cup \{1, 2, ..., m\}$$

for some $m \in N$. Since $\mathcal{I}$ is admissible so it follows that $x_k \to 0\,(S_\lambda\,(\mathcal{I}))^\alpha$. Obviously

$$\frac{1}{\lambda_n^\alpha}\int_{t\in I_n}|x(t)| \to \infty\,(n\to\infty)$$

i.e.$x(t) \nrightarrow 0\,[V,\lambda]^\alpha\,(\mathcal{I})$. Note that if $A \in \mathcal{I}$ is infinite then $x(t) \nrightarrow \theta\,(S_\lambda)^\alpha$. This example also shows that $\mathcal{I}_\lambda-$ statistical convergence of order α is more general than λ-statistical convergence of order α. □

Theorem 2.14. *$S(\mathcal{I})^\alpha \subset S_\lambda\,(\mathcal{I})^\alpha$ if $\liminf_n \frac{\lambda_n^\alpha}{n^\alpha} > 0$.*

Proof. For given $\epsilon > 0$,

$$\begin{aligned}\frac{1}{n^\alpha}\left|\{t\le n : |x(t)-L|\ge\epsilon\}\right| &\ge \frac{1}{n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right| \\ &\ge \frac{\lambda_n^\alpha}{n^\alpha}\frac{1}{\lambda_n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|.\end{aligned}$$

İf $\liminf_{n\to\infty}\frac{\lambda_n^\alpha}{n^\alpha} = a$ then from definition $\left\{n\in N : \frac{\lambda_n^\alpha}{n^\alpha} < \frac{a}{2}\right\}$is finite. For $\delta > 0$,

$$\begin{aligned}&\left\{n\in N : \frac{1}{\lambda_n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|\ge\delta\right\}\\ \subset\ &\left\{n\in N : \frac{1}{n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|\ge\frac{a}{2}\delta\right\}\cup\\ &\left\{n\in N : \frac{\lambda_n^\alpha}{n^\alpha} < \frac{a}{2}\right\}.\end{aligned}$$

Since $\mathcal{I}$ is admissible, the set on the right hand side belongs to $\mathcal{I}$ and this completes the proof. □

Theorem 2.15. *If $\lambda \in \triangle$ be such that for a particular $\alpha, 0 < \alpha < 1$, $\lim_n \frac{n-\lambda_n}{n^\alpha} = 0$ then $S_\lambda(\mathcal{I})^\alpha \subset S(\mathcal{I})^\alpha$.*

Proof. Let $\delta > 0$ be given. Since $lim_n \frac{n-\lambda_n}{n^\alpha} = 0$, we can choose $m \in N$ such that $\frac{n-\lambda_n}{n^\alpha} < \frac{\delta}{2}$, for all $n \ge m$. Now observe that , for $\varepsilon > 0$

$$\begin{aligned}&\frac{1}{n^\alpha}\left|\{t\le n : |x(t)-L|\ge\epsilon\}\right| = \frac{1}{n^\alpha}\left|\{t\le n-\lambda_n : |x(t)-L|\ge\epsilon\}\right| + \frac{1}{n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|\\ \le\ &\frac{n-\lambda_n}{n^\alpha} + \frac{1}{n^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|\\ \le\ &\frac{\delta}{2} + \frac{1}{{\lambda_n}^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|,\end{aligned}$$

for all $n \ge m$. Hence

$$\begin{aligned}&\left\{n\in N : \frac{1}{n^\alpha}\left|\{t\le n : |x(t)-L|\ge\epsilon\}\right|\ge\delta\right\}\\ \subset\ &\left\{n\in N : \frac{1}{{\lambda_n}^\alpha}\left|\{t\in I_n : |x(t)-L|\ge\epsilon\}\right|\ge\frac{\delta}{2}\right\}\cup\{1,2,3,...,m\}.\end{aligned}$$

If $S_\lambda^\alpha(\mathcal{I}) - \lim x(t) = L$ then the set on the right hand side belongs to $\mathcal{I}$ and so the set on the left hand side also belongs to $\mathcal{I}$. This shows that $x(t)$ is $\mathcal{I}$- statistically convergent of order α to L. □

Remark 2.16. *For $\alpha = 1$ the sufficient condition for the validity of this result is different which is given in Theorem 2.3 [22]. We do not know whether the conditions in the above theorem as well as in Theorem 2.3 [22] are necessary and leave them as open problems.*

References

[1] S. Bhunia, Pratulananda Das, S. Pal, Restricting statistical convergenge, Acta Math. Hungar. 134(1-2) (2012) 153 –161.
[2] H. Cakalli, A study on statistical convergence, Funct. Anal. Approx. Comput., 1(2)(2009) 19–24, MR2662887.
[3] R. Colak, Statistical convergence of order α, Modern methods in Analysis and its Applications, New Delhi, India, Anamaya Pub. (2010) 121–129.
[4] R. Colak, C. A. Bektas, λ-statistical convergence of order α, Acta Math. Scientia 31B (3) (2011) 953–959.
[5] J. Connor, A topological and functional analytic approach to statistical convergence , in: Analysis of Divergence,Orono,ME. 1997, Applied and Numerical Harmonic Analysis, Birkhauser Boston, Boston, MA, (1999) pp.403-413.
[6] J. Connor, E. Savas, Lacunary statistical and sliding window convergence for measurable functions. Acta Math. Hungar. 145(2) (2015) 416-432.
[7] P. Das, S. Ghosal, Some further results on I-Cauchy sequences and condition (AP), Comp. Math. Appl. 59 (2010) 2597–2600.
[8] P. Das, E. Savas, On I-statistical and I-lacunary statistical convergence of order α, Bull. Iranian Math. Soc. 40(2) (2014) 459-472.
[9] P. Das, E. Savas, S. K. Ghosal, On generalizations of certain summability methods using ideals, Appl.Math. Letters 24 (2011) 1509 –1514.
[10] K. Dems, On I-Cauchy sequences, Real Anal. Exchance, 30 (2004-2005) 123–128.
[11] M. Et, S. Abdul Mohiuddine, A. Alotaibi, On λ-statistical convergence and strongly λ-summable functions of order α. J. Inequal. Appl. 2013(469) (2013) 8 pp.
[12] H. Fast, Sur la convergence statistique, Colloq. Math., 2 (1951) 241–244.
[13] J. A. Fridy, On statistical convergence, Analysis, 5 (1985) 301–313.
[14] P. Kostyrko, T. Šalát, W.Wilczynki, I-convergence, Real Anal. Exchange, 26 (2) (2000/2001) 669–685.
[15] M. Mursaleen, λ-statistical convergence, Math. Slovaca, 50 (2000) 111 –115.
[16] F. Nuray, λ- strongly summable and λ statistically convergent functions, Iranian Jour. Science and Tech. Trans.A, 34(A4) (2010) 335–339.
[17] A. Sahiner, M.Gurdal, S. Saltan and H. Gunawan, Ideal Convergence in 2-normed spaces, Taiwanese J. Math., 11(5) (2007) 1477–1484.
[18] T. Šalát, On Statistically convergent sequences of real numbers, Math. Slovaca, 30 (1980) 139–150.
[19] E. Savaş, Strong almost convergence and almost λ-statistical convergence, Hokkaido Math. J., 29 (3)(2000) 531–536.
[20] E.Savaş, $\triangle^m$ -strongly summable sequences spaces in 2-Normed Spaces defined by Ideal Convergence and an Orlicz Function, App. Math. Comp. 217(2010) 271–276.
[21] E.Savaş, A-sequence spaces in 2-normed space defined by ideal convergence and an Orlicz function, Abst. Appl. Analy., 2011 (2011), Article ID 741382.
[22] E. Savaş, On Generalized Statistically Convergent Functions Via Ideals, (preprint).
[23] E. Savaş, On some new sequence spaces in 2-normed spaces using Ideal convergence and an Orlicz function, J. Ineq. Appl. Article Number: 482392 DOI: 10.1155/2010/482392 Published: 2010.
[24] E. Savaş, On $\mathcal{I}$-Asymptotically Lacunary Statistical Equivalent Sequences, Advances in Difference Equations 2013(111) (2013) (2013) (18 April 2013).
[25] E. Savaş, P. Das, A generalized statistical convergence via ideals, Appl. Math. Letters, 24 (2011) 826– 830.
[26] E. Savaş, P. Das, S. Dutta, A note on strong matrix summability via ideals, Appl. Math Letters, 25 (4) (2012) 733– 738.
[27] E.Savaş and H.Gumuş, A Generalization on $\mathcal{I}-$Asymptotically Lacunary Statistical Equivalent Sequences J.Ineq. Appl. 2013(270) (30 May 2013).
[28] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66 (1959) 361–375.