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Additive Difference Scheme for Two-Dimensional Fractional in Time Diffusion Equation

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Abstract. An additive finite-difference scheme for numerical approximation of initial-boundary value problem for two-dimensional fractional in time diffusion equation is proposed. Its stability is investigated and a convergence rate estimate is obtained.

1. Introduction

There has been increasing interest in the description of physical and chemical processes by means of equations involving fractional derivatives and integrals over the last decade. Fractional partial differential equations are used for the description of large classes processes that occur in media with fractal geometry, disordered materials, viscoelastic media as well as in the mathematical modeling of economic, biological and social phenomena (see [8, 12, 13]).

The time fractional diffusion equation is obtained from the classical diffusion equation by replacing the first order time derivative by a fractional derivative of order α with $0 < \alpha < 1$. It represents anomalous sub-diffusion which has been investigated by many authors. In this article we consider the first initial-boundary value problem for two-dimensional fractional in time diffusion equation. The problem is approximated by additive finite difference scheme. Contrary to explicit scheme, which is unstable, and implicit scheme [16], which is not numerically efficient, additive scheme is absolutely stable and efficient. Locally one-dimensional difference schemes for the fractional order diffusion equation are investigated in [3, 9].

The paper is organized as follows. In Section 2 we introduce Riemann-Liouville fractional derivative. In Section 3 we define some function spaces containing functions with fractional derivatives. In Section 4 we expose the problem and prove existence and uniqueness of its weak solution. In Section 5 we define additive difference scheme approximating considered problem and prove its stability. In Section 6 we investigate the convergence of proposed difference scheme. One numerical example is presented in Section 7.

2. Fractional Derivatives

There are two most popular ways to express fractional derivative: Caputo and Riemann-Liouville definitions. We will use the Riemann-Liouville definition through the paper.

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Let u be a function that $\operatorname{supp} u \subset [a,b]$ and $k-1<\alpha<k$, $k\in\mathbb{N}$. Then the left Riemann-Liouville fractional derivative of order α is defined to be

$$D_{a+}^{\alpha}u(t)=\frac{1}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_a^t\frac{u(\tau)}{(t-\tau)^{\alpha+1-k}}\,d\tau \tag{1}$$

and the right Riemann-Liouville fractional derivative is defined analogously

$$D_{b-}^{\alpha}u(t)=\frac{(-1)^k}{\Gamma(k-\alpha)}\frac{d^k}{dt^k}\int_t^b\frac{u(\tau)}{(t-\tau)^{\alpha+1-k}}\,d\tau, \tag{2}$$

where the $\Gamma(\cdot)$ denotes the Gamma function. Notice that if function $u(t)$ has k-order continuous derivative in $[a,b]$, then as $\alpha\to k$ or $\alpha\to k-1$, the left (right) Riemann-Liouville derivative becomes a standard k- or $(k-1)$-order derivative of $u(t)$.

Because of the integral in the definition of the fractional order derivatives, it is apparent that these derivatives are nonlocal operators.

For the functions of many variables, the partial Riemann-Liouville fractional derivatives are defined in an analogous manner, for example

$$D_{t,a+}^{\alpha}u(x,t)=\frac{1}{\Gamma(k-\alpha)}\frac{\partial^k}{\partial t^k}\int_a^t\frac{u(x,\tau)}{(t-\tau)^{\alpha+1-k}}\,d\tau,\quad k-1<\alpha<k,\quad k\in\mathbb{N}.$$

3. Some Function Spaces

We define some function spaces, norms and inner products that we will use thereafter. Let Ω be an open domain in $\mathbb{R}^n$. With $C^k(\Omega)$ and $C^k(\bar\Omega)$ we denote the spaces of k-fold differentiable functions in Ω and $\bar\Omega$, respectively. In particular, $\dot C^\infty(\Omega)=C_0^\infty(\Omega)$ stand for the space of infinitely differentiable functions with compact support in Ω. As usual, the space of measurable functions whose square is Lebesgue integrable in Ω is denoted by $L^2(\Omega)$. The inner product and norm in that space are defined by

$$(u,v)_\Omega=(u,v)_{L^2(\Omega)}=\int_\Omega uv\,d\Omega,\qquad \|u\|_\Omega=\|u\|_{L^2(\Omega)}=(u,u)_\Omega^{1/2}.$$

We also use $H^\alpha(\Omega)$ and $\dot H^\alpha(\Omega)=H_0^\alpha(\Omega)$ to denote the usual Sobolev spaces [11] whose norms are denoted by $\|u\|_{H^\alpha(\Omega)}$.

For non-integer $\alpha>0$ we set

$$|u|_{C_+^\alpha[a,b]}=\|D_{a+}^\alpha u\|_{C[a,b]},\qquad |u|_{C_-^\alpha[a,b]}=\|D_{b-}^\alpha u\|_{C[a,b]},$$
$$\|u\|_{C_\pm^\alpha[a,b]}=\Big(\|u\|^2_{C^{[\alpha]}[a,b]}+|u|^2_{C_\pm^\alpha[a,b]}\Big)^{1/2},$$
$$|u|_{H_+^\alpha(a,b)}=\|D_{a+}^\alpha u\|_{L^2(a,b)},\qquad |u|_{H_-^\alpha(a,b)}=\|D_{b-}^\alpha u\|_{L^2(a,b)}$$

and

$$\|u\|_{H_\pm^\alpha(a,b)}=\Big(\|u\|^2_{H^{[\alpha]}(a,b)}+|u|^2_{H_\pm^\alpha(a,b)}\Big)^{1/2},$$

where $[\alpha]$ denotes the largest integer $\le\alpha$. Then we define $C_\pm^\alpha[a,b]$ as the space of functions $u\in C^{[\alpha]}[a,b]$ with the finite norm $\|u\|_{C_\pm^\alpha[a,b]}$. The space $H_\pm^\alpha(a,b)$ is defined analogously, while the space $\dot H_\pm^\alpha(a,b)$ is defined as the closure of $\dot C^\infty(a,b)$ with respect to the norm $\|\cdot\|_{H_\pm^\alpha(a,b)}$. Because for $\alpha=k\in\mathbb{N}\cup\{0\}$ fractional derivative reduces to standard k-th derivative, we set $C_\pm^k[a,b]=C^k[a,b]$ and $H_\pm^k(a,b)=H^k(a,b)$.

Lemma 3.1. *(See [10]) Let $0<\alpha<1$, $u\in H_+^\alpha(a,b)$ and $v\in H_-^\alpha(a,b)$. Then*

$$(D_{a+}^\alpha u,v)_{L^2(a,b)}=(u,D_{b-}^\alpha v)_{L^2(a,b)}.$$

Lemma 3.2. *(See [6]) Let $\alpha > 0$, $u \in \dot{C}^\infty(\mathbb{R})$ and $\operatorname{supp} u \subset (a,b)$. Then*

$$(D^\alpha_{a+}u, D^\alpha_{b-}u)_{L^2(a,b)} = \cos\pi\alpha \, \|D^\alpha_{a+}u\|^2_{L^2(a,\infty)}.$$

For $\alpha > 0$, $\alpha \neq n + 1/2$, $n \in \mathbb{N}$, we set

$$|u|_{H^\alpha_c} = |(D^\alpha_{a+}u, D^\alpha_{b-}u)_{L^2(a,b)}|^{1/2}, \qquad \|u\|_{H^\alpha_c} = \left(\|u\|^2_{L^2(a,b)} + |u|^2_{H^\alpha_c}\right)^{1/2}.$$

Lemma 3.3. *(See [10]) For $\alpha > 0$, $\alpha \neq k + 1/2$, $k \in \mathbb{N}$, the spaces $\dot{H}^\alpha_+(a,b)$, $\dot{H}^\alpha_-(a,b)$, $\dot{H}^\alpha_c(a,b)$ and $\dot{H}^\alpha(a,b)$ are equal and their seminorms as well as norms are equivalent.*

For the vector valued functions mapping real interval $(0,T)$ (or $[0,T]$) into Banach space X we introduce the spaces $C^k([0,T],X)$, $k \in \mathbb{N} \cup \{0\}$ and $H^\alpha((0,T),X)$, $\alpha \geq 0$, in the usual way [11]. In analogous manner we define the spaces $C^\alpha_\pm([0,T],X)$ and $H^\alpha_\pm((0,T),X)$.

Through the paper by C we will denote positive generic constant which not depend on the solution of the problem and discretization parameters.

4. Problem Formulation

Let $0 < \alpha < 1$, $\Omega = (0,1) \times (0,1)$ and $Q = \Omega \times (0,T)$. We shall consider the time fractional diffusion equation

$$D^\alpha_{t,0+}u - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y,t), \quad (x,y,t) \in Q, \tag{3}$$

subject to homogeneous initial and boundary conditions

$$u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \in (0,T) \tag{4}$$

$$u(x,y,0) = 0, \quad (x,y) \in \bar{\Omega} \tag{5}$$

Initial-boundary value problem (3)-(5) is often called sub-diffusion problem.

Taking inner product of equation (3) with test function v and using Lemma 1 and properties of fractional derivatives one obtains the following weak formulation of the problem (3)-(5) (see [10]): find $u \in \dot{H}^{1,\alpha/2}(Q)$ such that

$$a(u,v) = l(v), \quad \forall v \in \dot{H}^{1,\alpha/2}(Q), \tag{6}$$

where

$$\dot{H}^{1,\alpha/2}(Q) = L^2((0,T), \dot{H}^1(\Omega)) \cap \dot{H}^{\alpha/2}((0,T), L^2(\Omega)),$$

the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \left(D^{\alpha/2}_{t,0+}u, D^{\alpha/2}_{t,T-}v\right)_Q + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_Q + \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_Q,$$

and the linear functional $l(\cdot)$ is given by

$$l(v) = (f,v)_Q.$$

Now, from lemmas 3.1-3.3 and the Lax-Milgram lemma, we immediately obtain the following result (see [10]):

Theorem 4.1. *For all $\alpha \in (0,1)$, and $f \in L^2(Q)$, the problem (6) is well posed and its solution satisfies a priori estimate*

$$\|u\|_{H^{1,\alpha/2}(Q)} \leq C\|f\|_{L^2(Q)}.$$

5. Finite Difference Approximation

In the area $\bar{Q} = [0,1]\times[0,1]\times[0,T]$, we define the uniform mesh $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$, where $\bar{\omega}_h = \{(x_i, y_j) = (ih, jh)\,|\, i, j = 0, 1, ..., n;\ h = 1/n\}$ and $\bar{\omega}_\tau = \{t_k = k\tau\,|\,k = 0, 1, ..., 2m;\ \tau = T/2m\}$. We also define $\omega_h = \bar{\omega}_h \cap \Omega$, $\gamma_h = \bar{\omega}_h \setminus \omega_h$, $\omega_{1h} = \bar{\omega}_h \cap (0,1]\times(0,1)$, $\omega_{2h} = \bar{\omega}_h \cap (0,1)\times(0,1]$, $\omega_\tau = \bar{\omega}_\tau \cap (0,T)$, $\omega_\tau^- = \bar{\omega}_\tau \cap [0,T)$ and $\omega_\tau^+ = \bar{\omega}_\tau \cap (0,T]$. We will use standard notation from the theory of the finite difference schemes (see [14]):

$$v = v(x,y,t), \quad \hat{v} = v(x,y,t+\tau), \quad v^k = v(x,y,t_k), \quad (x,y)\in\bar{\omega}_h,$$

$$v_x = \frac{v(x+h,y,t)-v(x,y,t)}{h}, \quad v_{\bar{x}} = \frac{v(x,y,t)-v(x-h,y,t)}{h},$$

$$v_y = \frac{v(x,y+h,t)-v(x,y,t)}{h}, \quad v_{\bar{y}} = \frac{v(x,y,t)-v(x,y-h,t)}{h}.$$

For a function u defined on $\bar{Q}$ which satisfies zero initial condition, we approximate the left Riemann-Liouville fractional derivative $D^\alpha_{t,0+}u(x,y,t_k)$ by (see [5]):

$$(\Delta^\alpha_{t,0+,\tau}u)^k = \frac{1}{\Gamma(2-\alpha)}\sum_{l=0}^{k-1}(t^{1-\alpha}_{k-l} - t^{1-\alpha}_{k-l-1})u^l_t.$$

We approximate the initial-boundary value problem (3)-(5) with the following additive finite difference scheme:

$$(\Delta^\alpha_{t,0+,\tau}v)^{2k-1} - 2v^{2k-1}_{x\bar{x}} = \bar{f}^{2k-1}, \quad (x,y)\in\omega_h, \quad k = 1,2,\dots,m, \tag{7}$$

$$(\Delta^\alpha_{t,0+,\tau}v)^{2k} - 2v^{2k}_{y\bar{y}} = \bar{f}^{2k}, \quad (x,y)\in\omega_h, \quad k = 1,2,\dots,m, \tag{8}$$

subject to zero boundary and initial conditions:

$$v(x,y,t) = 0, \quad (x,y,t)\in\gamma_h\times\omega_\tau^+, \tag{9}$$

$$v(x,y,0) = 0, \quad (x,y)\in\bar{\omega}_h. \tag{10}$$

When the right-hand side f is continuous function, we set $\bar{f} = f$, otherwise we must use some averaged value, for example $\bar{f} = T_1T_2f$, where T_1 and T_2 are Steklov averaging operators:

$$T_1 f(x,y,t) = \int_{-1/2}^{1/2} f(x+hs,y,t)\,ds, \quad T_2 f(x,y,t) = \int_{-1/2}^{1/2} f(x,y+hs,t)\,ds.$$

Notice that on each time level the finite difference scheme (7)-(10) is one-dimensional problem, which reduces to n independent linear systems with three-diagonal matrices. On the other hand, the solution v^k on time level t_k explicitly depends on the solutions at all previous time levels. Thus, numerical effort is $O(n^2m^2)$.

We define the following discrete inner products and norms:

$$(v,w)_h = (v,w)_{L^2(\omega_h)} = h^2\sum_{(x,y)\in\omega_h} vw, \qquad \|v\|_h = \|v\|_{L^2(\omega_h)} = (v,v)_h^{1/2},$$

$$(v,w)_{ih} = (v,w)_{L^2(\omega_{ih})} = h^2\sum_{(x,y)\in\omega_{ih}} vw, \qquad \|v\|_{ih} = \|v\|_{L^2(\omega_{ih})} = (v,v)_{ih}^{1/2}, \qquad i = 1,2,$$

$$\|v\|_{L^2(Q_{h\tau})} = \left(\tau\sum_{j=1}^{2m}\|v^j\|_h^2\right)^{1/2},$$

$$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})} = \left[\tau\sum_{k=1}^{m}\left(\|v^{2k-1}_{\bar{x}}\|^2_{L^2(\omega_{1h})} + \|v^{2k}_{\bar{y}}\|^2_{L^2(\omega_{2h})}\right) + \tau\sum_{k=1}^{2m}\left(\Delta^\alpha_{t,0+,\tau}\left(\|v\|_h^2\right)\right)^k\right]^{1/2}.$$

Lemma 5.1. *([1]; see also generalized result in [2]) For* $0 < \alpha < 1$ *and any function* $v(t)$ *defined on mesh* $\bar{\omega}_\tau$ *one has equality*

$$v^k(\Delta^\alpha_{t,0+,\tau}v)^k = \frac{1}{2}\left(\Delta^\alpha_{t,0+,\tau}(v^2)\right)^k + \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[\sum_{l=1}^{k-1}\left(a^{-1}_{k-l+1} - a^{-1}_{k-l}\right)(w^l_k)^2 + a^{-1}_1(w^k_k)^2\right] \tag{11}$$

where $a_l = l^{1-\alpha} - (l-1)^{1-\alpha}$ *and*

$$w^l_k = \tau\sum_{s=0}^{l-1} a_{k-s}\, v^s_t, \quad l = 1, 2, \ldots, k.$$

Lemma 5.2. *For* $0 < \alpha < 1$ *and any function* $v(t)$ *defined on mesh* $\bar{\omega}_\tau$ *the following inequality is valid*

$$v^k\left(\Delta^\alpha_{t,0+,\tau}v\right)^k \geq \frac{1}{2}\left(\Delta^\alpha_{t,0+,\tau}(v^2)\right)^k + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}(v^{k-1}_t)^2. \tag{12}$$

Proof. For $k \geq 2$, using Lemma 5.1 and obvious inequality $a^{-1}_{l+1} > a^{-1}_l$, we obtain

$$v^k(\Delta^\alpha_{t,0+,\tau}v)^k - \frac{1}{2}\left(\Delta^\alpha_{t,0+,\tau}(v^2)\right)^k \geq \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[\left(a^{-1}_2 - a^{-1}_1\right)\left(w^{k-1}_k\right)^2 + a^{-1}_1\left(w^k_k\right)^2\right]$$

$$= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[\left(\frac{1}{2^{1-\alpha}-1} - 1\right)\left(w^{k-1}_k\right)^2 + \left(w^{k-1}_k + \tau\, v^{k-1}_t\right)^2\right]$$

$$= \frac{\tau^{-\alpha}}{2\Gamma(2-\alpha)}\left[\left(\frac{w^{k-1}_k}{\sqrt{2^{1-\alpha}-1}} + \tau\sqrt{2^{1-\alpha}-1}\, v^{k-1}_t\right)^2 + 2(1-2^{-\alpha})\tau^2\left(v^{k-1}_t\right)^2\right]$$

whereby it follows result. For $k = 1$ result is obvious because the sum on the right-hand side of (11) reduces to $a^{-1}_1\,(w^1_1)^2 = \tau^2(v^0_t)^2$. □

Lemma 5.3. *(See [5]) For every function* $v(t)$ *defined on the mesh* $\bar{\omega}_\tau$ *which satisfies* $v(0) = 0$ *the following equality is valid*

$$\tau\sum_{k=1}^{2m}\left(\Delta^\alpha_{t,0+,\tau}(v^2)\right)^k = \frac{1}{\Gamma(2-\alpha)}\sum_{k=1}^{2m}\left(t^{1-\alpha}_{2m-k+1} - t^{1-\alpha}_{2m-k}\right)(v^k)^2.$$

In particular, from Lemma 5.3 follows that the norm $\|v\|_{B^{1,\alpha/2}(Q_{h\tau})}$ is well defined.

Theorem 5.4. *Let* $\alpha \in (0,1)$. *Then the finite difference scheme (7)-(10) is absolutely stable and its solution satisfies the following a priori estimate:*

$$\|v\|_{B^{1,\alpha/2}(Q_{h\tau})} \leq C\|\bar{f}\|_{L^2(Q_{h\tau})} \tag{13}$$

Proof. Taking inner products of equations (7) and (8) with v^{2k-1} and v^{2k}, respectively, we obtain

$$\left(v^{2k-1}, \left(\Delta^\alpha_{t,0+,\tau}v\right)^{2k-1}\right)_h - 2\left(v^{2k-1}_{x\bar{x}}, v^{2k-1}\right)_h = \left(\bar{f}^{2k-1}, v^{2k-1}\right)_h,$$

$$\left(v^{2k}, \left(\Delta^\alpha_{t,0+,\tau}v\right)^{2k}\right)_h - 2\left(v^{2k}_{y\bar{y}}, v^{2k}\right)_h = \left(\bar{f}^{2k}, v^{2k}\right).$$

Using Lemma 5.1, partial summation, Cauchy-Schwarz and ε inequalities we further obtain

$$\frac{1}{2}\left(\Delta^\alpha_{t,0+,\tau}\left(\|v\|^2_h\right)\right)^{2k-1} + 2\left\|v^{2k-1}_{\bar{x}}\right\|^2_{1h} \leq \varepsilon\left\|v^{2k-1}\right\|^2_h + \frac{1}{4\varepsilon}\left\|\bar{f}^{2k-1}\right\|^2_h,$$

$$\frac{1}{2}\left(\Delta^{\alpha}_{t,0+,\tau}\Big(\|v\|^2_h\Big)\right)^{2k} + 2\left\|v^{2k}_{\bar{y}}\right\|^2_{2h} \le \varepsilon\left\|v^{2k}\right\|^2_h + \frac{1}{4\varepsilon}\left\|\bar{f}^{2k}\right\|^2_h.$$

Using inequalities $\|v\|^2_h \le \frac{1}{8}\|v_{\bar{x}}\|^2_{1h}$ and $\|v\|^2_h \le \frac{1}{8}\|v_{\bar{y}}\|^2_{2h}$ (see [14]), we have

$$\left(\Delta^{\alpha}_{t,0+,\tau}\Big(\|v\|^2_h\Big)\right)^{2k-1} + \Big(4 - \frac{2}{8}\,\varepsilon\Big)\left\|v^{2k-1}_{\bar{x}}\right\|^2_{1h} \le \frac{1}{2\varepsilon}\left\|\bar{f}^{2k+1}\right\|^2_h,$$

$$\left(\Delta^{\alpha}_{t,0+,\tau}\Big(\|v\|^2_h\Big)\right)^{2k} + \Big(4 - \frac{2}{8}\,\varepsilon\Big)\left\|v^{2k}_{\bar{y}}\right\|^2_{2h} \le \frac{1}{2\varepsilon}\left\|\bar{f}^{2k}\right\|^2_h.$$

Now, for suitable ε, multiplying the last inequalities by τ and summing for $k = 1, ..., m$ we obtain the a priori estimate (13). □

Lemma 5.5. *(See [15]) Suppose that* $v \in C^2[0,t]$, $t \in \omega^+_\tau$. *Then*

$$|D^{\alpha}_{t,0+}v - \Delta^{\alpha}_{t,0+,\tau}v| \le \tau^{2-\alpha}\frac{1}{1-\alpha}\left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha})\right]\max_{0\le s\le t}|v''(s)|.$$

6. Convergence of the Difference Scheme

Let u be the solution of the initial-boundary-value problem (3)-(5) and v the solution of the difference problem (7)-(10) with $\bar{f} = T_1T_2f$. The error $z = u - v$ is defined on the mesh $\bar{\omega}_h \times \bar{\omega}_\tau$. Putting $v = -z + u$ into (7)-(10) it follows that error satisfies

$$\Big(\Delta^{\alpha}_{t,0+,\tau}z\Big)^{2k-1} - 2z^{2k-1}_{x\bar{x}} = \psi^{2k-1}_1, \quad (x,y) \in \omega_h, \quad k = 1,2,\dots,m, \tag{14}$$

$$\Big(\Delta^{\alpha}_{t,0+,\tau}z\Big)^{2k} - 2z^{2k}_{y\bar{y}} = \psi^{2k}_2, \quad (x,y) \in \omega_h, \quad k = 1,2,\dots,m, \tag{15}$$

$$z = 0, \quad (x,y) \in \gamma_h, \quad t \in \bar{\omega}_\tau, \tag{16}$$

$$z^0 = z(x,y,0) = 0, \quad (x,y) \in \omega_h, \tag{17}$$

where

$$\psi^{2k-1}_1 = \Big(\Delta^{\alpha}_{t,0+,\tau}u\Big)^{2k-1} - 2u^{2k-1}_{x\bar{x}} - T_1T_2f^{2k-1}$$
$$= \Big(\Delta^{\alpha}_{t,0+,\tau}u - T_1T_2D^{\alpha}_{t,0+}u\Big)^{2k-1} + 2\left(T_1T_2\frac{\partial^2 u}{\partial x^2} - u_{x\bar{x}}\right)^{2k-1} + T_1T_2\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2}\right)^{2k-1} = \xi^{2k-1} + \eta^{2k-1}_1 + \chi^{2k-1}$$

and

$$\psi^{2k}_2 = \Big(\Delta^{\alpha}_{t,0+,\tau}u\Big)^{2k} - 2u^{2k}_{y\bar{y}} - T_1T_2f^{2k}$$
$$= \Big(\Delta^{\alpha}_{t,0+,\tau}u - T_1T_2D^{\alpha}_{t,0+}u\Big)^{2k} + 2\Big(T_1T_2\tfrac{\partial^2 u}{\partial y^2} - u_{y\bar{y}}\Big)^{2k} - T_1T_2\Big(\tfrac{\partial^2 u}{\partial y^2} - \tfrac{\partial^2 u}{\partial x^2}\Big)^{2k} = \xi^{2k} + \eta^{2k}_2 - \chi^{2k}.$$

Further, using properties of Steklov averaging operators, we obtain

$$\eta_1 = \zeta_{1,x}, \qquad \eta_2 = \zeta_{2,y},$$

where

$$\zeta_1(x,y,t) = 2\Big(T_2\frac{\partial u}{\partial x}(x-h/2,y,t) - u_{\bar{x}}(x,y,t)\Big), \qquad \zeta_2(x,y,t) = 2\Big(T_1\frac{\partial u}{\partial y}(x,y-h/2,t) - u_{\bar{y}}(x,y,t)\Big).$$

Lemma 6.1. *The finite difference scheme (14)-(17) is absolutely stable and the following a priori estimate holds:*

$$\|z\|_{B^{1,\alpha/2}(Q_{h\tau})} \leq C\left(\tau\sum_{k=1}^{2m}\|\xi^k\|_h^2 + \tau\sum_{k=1}^{m}\|\zeta_1^{2k-1}\|_{1h}^2 + \tau\sum_{k=1}^{m}\|\zeta_2^{2k}\|_{2h}^2 + \tau^3\sum_{k=1}^{m}\|\chi_t^{2k-1}\|_h^2 + \tau^{1+\alpha}\sum_{k=1}^{m}\|\chi^{2k}\|_h^2\right)^{1/2}. \quad (18)$$

Proof. Taking inner products of (14) and (15) with z^{2k+1} and z^{2k}, respectively, and performing summation by part, we obtain

$$\left(z^{2k-1},\left(\Delta_{t,0+,\tau}^{\alpha}z\right)^{2k-1}\right)_h + 2\left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 = \left(\psi_1^{2k-1},z^{2k-1}\right)_h = \left(\xi^{2k-1},z^{2k-1}\right)_h - \left(\zeta_1^{2k-1},z_{\bar{x}}^{2k-1}\right)_{1h} + \left(\chi^{2k-1},z^{2k-1}\right)_h$$

and

$$\left(z^{2k},\left(\Delta_{t,0+,\tau}^{\alpha}z\right)^{2k}\right)_h + 2\left\|z_{\bar{y}}^{2k}\right\|_{2h}^2 = \left(\psi_2^{2k},z^{2k}\right)_h = \left(\xi^{2k},z^{2k}\right)_h - \left(\zeta_2^{2k},z_{\bar{y}}^{2k}\right)_{2h} - \left(\chi^{2k},z^{2k}\right)_h.$$

By summing the last two equalities and using (12) it follows that

$$\frac{1}{2}\left(\Delta_{t,0+,\tau}^{\alpha}(\|z\|_h^2)\right)^{2k-1} + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\left\|z_t^{2k-1}\right\|_h^2 + 2\left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 + \frac{1}{2}\left(\Delta_{t,0+,\tau}^{\alpha}(\|z\|_h^2)\right)^{2k} + \frac{\tau^{2-\alpha}(1-2^{-\alpha})}{\Gamma(2-\alpha)}\left\|z_t^{2k}\right\|_h^2 + 2\left\|z_{\bar{y}}^{2k}\right\|_h^2$$
$$\leq \left(\xi^{2k-1},z^{2k-1}\right)_h + \left(\xi^{2k},z^{2k}\right)_h - \left(\zeta_1^{2k-1},z_{\bar{x}}^{2k-1}\right)_{1h} - \left(\zeta_2^{2k},z_{\bar{y}}^{2k}\right)_{2h} - \left(\chi^{2k}-\chi^{2k-1},z^{2k-1}\right)_h - \left(\chi^{2k},z^{2k}-z^{2k-1}\right)_h.$$

Next, we estimate the terms at the right-hand side:

$$\left|\left(\xi^{2k-1},z^{2k-1}\right)_h\right| \leq \frac{\varepsilon}{2}\left\|z^{2k-1}\right\|_h^2 + \frac{1}{2\varepsilon}\left\|\xi^{2k-1}\right\|_h^2 \leq \frac{\varepsilon}{16}\left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 + \frac{1}{2\varepsilon}\left\|\xi^{2k-1}\right\|_h^2,$$

$$\left|\left(\xi^{2k},z^{2k}\right)_h\right| \leq \frac{\varepsilon}{2}\left\|z^{2k}\right\|_h^2 + \frac{1}{2\varepsilon}\left\|\xi^{2k}\right\|_h^2 \leq \frac{\varepsilon}{16}\left\|z_{\bar{y}}^{2k}\right\|_{2h}^2 + \frac{1}{2\varepsilon}\left\|\xi^{2k}\right\|_h^2,$$

$$\left|\left(\zeta_1^{2k-1},z_{\bar{x}}^{2k-1}\right)_{1h}\right| \leq \frac{\varepsilon}{2}\left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 + \frac{1}{2\varepsilon}\left\|\zeta_1^{2k-1}\right\|_{1h}^2,$$

$$\left|\left(\zeta_2^{2k},z_{\bar{y}}^{2k}\right)_{2h}\right| \leq \frac{\varepsilon}{2}\left\|z_{\bar{y}}^{2k}\right\|_{2h}^2 + \frac{1}{2\varepsilon}\left\|\zeta_2^{2k}\right\|_{2h}^2,$$

$$\left|\left(\chi^{2k}-\chi^{2k-1},z^{2k-1}\right)_h\right| = \left|\left(\tau\chi_t^{2k-1},z^{2k-1}\right)_h\right| \leq \frac{\varepsilon}{16}\left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 + \frac{\tau^2}{2\varepsilon}\left\|\chi_t^{2k-1}\right\|_h^2,$$

$$\left|\left(\chi^{2k},z^{2k}-z^{2k-1}\right)_h\right| = \tau\left|\left(\chi^{2k},z_t^{2k-1}\right)_h\right| \leq \frac{c}{2}\tau^{2-\alpha}\|z_t^{2k-1}\|_h^2 + \frac{\tau^{\alpha}}{2c}\|\chi^{2k}\|_h^2.$$

For $c = \frac{2(1-2^{-\alpha})}{\Gamma(2-\alpha)}$ and sufficiently small ε, from previous inequalities one obtains

$$\left(\Delta_{t,0+,\tau}^{\alpha}\left(\|z\|_h^2\right)\right)^{2k-1} + \left(\Delta_{t,0+,\tau}^{\alpha}\left(\|z\|_h^2\right)\right)^{2k} + \left\|z_{\bar{x}}^{2k-1}\right\|_{1h}^2 + \left\|z_{\bar{y}}^{2k}\right\|_{2h}^2$$
$$\leq C\left(\left\|\xi^{2k-1}\right\|_h^2 + \left\|\xi^{2k}\right\|_h^2 + \left\|\zeta_1^{2k-1}\right\|_{1h}^2 + \left\|\zeta_2^{2k}\right\|_{2h}^2 + \tau^2\left\|\chi_t^{2k-1}\right\|_h^2 + \tau^{\alpha}\left\|\chi^{2k}\right\|_h^2\right).$$

Result follows after summation for $k = 1,2,\ldots,m$. □

In such a way, to obtain the error bound of finite difference scheme (7)-(10) it is sufficient to estimate the right-hand side terms in (18).

Theorem 6.2. *Let the solution u of initial-boundary value problem (3)-(5) belongs to the space $C^2([0,T],C(\bar{\Omega})) \cap C^1([0,T],H^2(\Omega)) \cap C([0,T],H^3(\Omega))$. Then the solution v of finite difference scheme (7)-(10) with $\bar{f} = T_1T_2f$ converges to u and the following convergence rate estimate holds:*

$$\|u-v\|_{B^{1,\alpha/2}(Q_{h\tau})} = O(h^2+\tau^{\alpha/2}).$$

Proof. Let us set $\xi = \xi_1 + \xi_2$, where

$$\xi_1 = \Delta^{\alpha}_{t,0+,\tau} u - D^{\alpha}_{t,0+} u, \qquad \xi_2 = D^{\alpha}_{t,0+} u - T_1 T_2 D^{\alpha}_{t,0+} u = D^{\alpha}_{t,0+}(u - T_1 T_2 u).$$

From Lemma 5.5 immediately follows

$$\left(\tau \sum_{k=1}^{2m} \|\xi_1^k\|_h^2\right)^{1/2} \leq C\tau^{2-\alpha}\, \|u\|_{C^2([0,T],C(\bar{\Omega}))}. \tag{19}$$

From integral representation

$$u(x,y) - T_1T_2u(x,y) = \frac{1}{h^2}\int_{x-h/2}^{x+h/2}\int_{y-h/2}^{y+h/2}\left(\int_{x'}^{x}\int_{y'}^{y}\frac{\partial^2 u}{\partial x \partial y}(x'',y'')\,dy''dx'' \right.$$
$$\left. - \int_{x'}^{x}\int_{x''}^{x}\frac{\partial^2 u}{\partial x^2}(x''',y')\,dx'''dx'' - \int_{y'}^{y}\int_{y''}^{y}\frac{\partial^2 u}{\partial y^2}(x',y''')\,dy'''dy''\right)dy'dx'$$

one easily obtains

$$\left(\tau \sum_{k=1}^{2m} \|\xi_2^k\|_h^2\right)^{1/2} \leq Ch^2\left(\tau \sum_{k=1}^{2m} \|D^{\alpha}_{t,0+}u(\cdot,\cdot,t_k)\|^2_{H^2(\Omega)}\right)^{1/2} \leq Ch^2\, \|u\|_{C^{\alpha}_{+}([0,T],H^2(\Omega))}. \tag{20}$$

Using Bramble-Hilbert lemma [4] and methodology presented in [7] one obtains

$$\left(\tau \sum_{k=1}^{m} \|\zeta_1^{2k-1}\|^2_{1h} + \tau \sum_{k=1}^{m} \|\zeta_2^{2k}\|^2_{2h}\right)^{1/2} \leq Ch^2\, \|u\|_{C([0,T],H^3(\Omega))}. \tag{21}$$

Term χ can be estimated directly: $\|\chi(\cdot,\cdot,t)\|_h \leq C\,\|u(\cdot,\cdot,t)\|_{H^2(\Omega)}$, whereby it follows that

$$\left(\tau^{1+\alpha} \sum_{k=1}^{m} \|\chi^{2k}\|_h^2\right)^{1/2} \leq C\tau^{\alpha/2}\, \|u\|_{C([0,T],H^2(\Omega))} \quad \text{and} \quad \left(\tau^{3} \sum_{k=1}^{m} \|\chi_t^{2k-1}\|_h^2\right)^{1/2} \leq C\tau\, \|u\|_{C^1([0,T],H^2(\Omega))}. \tag{22}$$

Result follows from (18)-(22). □

7. Numerical Example

We considered initial-boundary value problem (3)-(5) with

$$f(x,y,t) = \sin(\pi x)\sin(\pi y)t^3\left(\frac{t^{-\alpha}\Gamma(4)}{\Gamma(4-\alpha)} + 2\pi^2\right),$$

whose exact solution is $u = \sin(\pi x)\sin(\pi y)t^3$. For different values of α, the problem is solved by using additive difference scheme (7)-(10).

The errors and convergence rates, in space and time directions, in L^2 and $B^{1,\alpha/2}$ norm, are presented in Tables 1-4. The obtained results suggests that the convergence in t direction is even faster than predicted $O(\tau^{\alpha/2})$. This will be subject of our future investigations.

Table 1: The error and convergence rate in space direction with fixed $\tau = 2^{-12}$ (in L^2 norm)

α	h	$\|z\|_{L^2(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h,\tau})}}{\|z\|_{L^2(Q_{h/2,\tau})}}$
0.5	2^{-2}	$9.07810 \cdot 10^{-3}$	2.03
	2^{-3}	$2.22514 \cdot 10^{-3}$	2.01
	2^{-4}	$5.53610 \cdot 10^{-4}$	2.00
	2^{-5}	$1.38200 \cdot 10^{-4}$	no data
0.9	2^{-2}	$8.54922 \cdot 10^{-3}$	2.02
	2^{-3}	$2.10267 \cdot 10^{-3}$	2.00
	2^{-4}	$5.25980 \cdot 10^{-4}$	1.97
	2^{-5}	$1.33970 \cdot 10^{-4}$	no data

Table 2: The error and convergence rate in time direction with fixed $h = 2^{-11}$ (in L^2 norm)

α	τ	$\|z\|_{L^2(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{L^2(Q_{h,\tau})}}{\|z\|_{L^2(Q_{h,\tau/2})}}$
0.5	2^{-3}	$7.57410 \cdot 10^{-4}$	1.47
	2^{-4}	$2.74110 \cdot 10^{-4}$	1.47
	2^{-5}	$9.90400 \cdot 10^{-5}$	1.47
	2^{-6}	$3.56710 \cdot 10^{-5}$	no data
0.9	2^{-3}	$3.68682 \cdot 10^{-3}$	1.12
	2^{-4}	$1.70209 \cdot 10^{-3}$	1.11
	2^{-5}	$7.90220 \cdot 10^{-4}$	1.10
	2^{-6}	$3.67880 \cdot 10^{-4}$	no data

Table 3: The error and convergence rate in space direction with fixed $\tau = 2^{-12}$ (in $B^{1,\alpha/2}$ norm)

α	h	$\|z\|_{B^{1,\alpha/2}(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{B^{1,\alpha/2}(Q_{h,\tau})}}{\|z\|_{B^{1,\alpha/2}(Q_{h/2,\tau})}}$
0.5	2^{-2}	$3.15568 \cdot 10^{-2}$	2.01
	2^{-3}	$7.85253 \cdot 10^{-3}$	2.00
	2^{-4}	$1.96106 \cdot 10^{-3}$	2.00
	2^{-5}	$4.90300 \cdot 10^{-4}$	no data
0.9	2^{-2}	$3.35193 \cdot 10^{-2}$	2.01
	2^{-3}	$8.33999 \cdot 10^{-3}$	2.00
	2^{-4}	$2.09138 \cdot 10^{-3}$	1.98
	2^{-5}	$5.32060 \cdot 10^{-4}$	no data

Table 4: The error and convergence rate in time direction with fixed $h = 2^{-11}$ (in $B^{1,\alpha/2}$ norm)

α	τ	$\|z\|_{B^{1,\alpha/2}(Q_{h,\tau})}$	$\log_2 \frac{\|z\|_{B^{1,\alpha/2}(Q_{h,\tau})}}{\|z\|_{B^{1,\alpha/2}(Q_{h,\tau/2})}}$
0.5	2^{-3}	$2.58562 \cdot 10^{-3}$	1.46
	2^{-4}	$9.37210 \cdot 10^{-4}$	1.47
	2^{-5}	$3.38800 \cdot 10^{-4}$	1.47
	2^{-6}	$1.22040 \cdot 10^{-4}$	no data
0.9	2^{-3}	$1.29852 \cdot 10^{-2}$	1.11
	2^{-4}	$6.03312 \cdot 10^{-3}$	1.11
	2^{-5}	$2.80974 \cdot 10^{-3}$	1.10
	2^{-6}	$1.31002 \cdot 10^{-3}$	no data

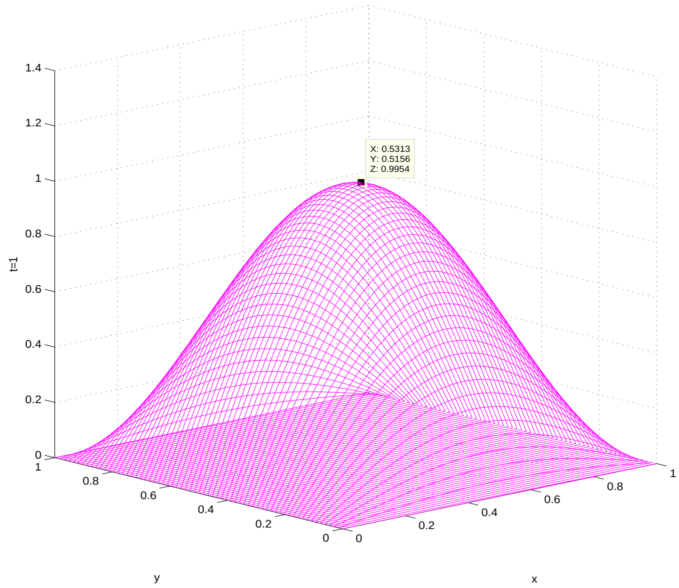

Fig.1 Numerical solution at time level t=1, with $\alpha = 0.9$, $h = 2^{-6}$, $\tau = 2^{-6}$

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