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Asymetric Fuglede Putnam's Theorem for Operators Reduced by their Eigenspaces

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Abstract. Fuglede-Putnam Theorem has been proved for a considerably large number of class of operators. In this paper by using the spectral theory, we obtain a theoretical and general framework from which Fuglede-Putnam theorem may be promptly established for many classes of operators.

1. Introduction and Basic Definitions

Fuglede-Putnam Theorem has been studied in the last two decades by several authors and most of them have essentially proved such theorem for special classes of operators. Many times the arguments used, to prove Fuglede-Putnam Theorem are similar, but in this paper we show that it is possible to bring back up this theorem from some general common properties. We use the spectral theory to obtain a theoretical and general framework from which Fuglede-Putnam theorem is established, and we can deduce that Fuglede-Putnam Theorem hold for many classes of operators. Let H be an infinite complex Hilbert space and consider two bounded linear operators $A, B \in L(H)$. Let $L_A \in L(L(H))$ and $R_B \in L(L(H))$ be the left and the right multiplication operators, respectively, and denote by $d_{A,B} \in L(L(H))$ either the elementary operator $\Delta_{A,B}(X) = AXB - X$ or the generalized derivation $\delta_{A,B}(X) = AX - XB$.
Given $T \in L(H)$, $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$ and $\sigma_p(T)$ will stand for the null space, the range of T, the spectrum of T and the point spectrum of T. Recall that if M, N are linear subspaces of a normed linear space V, then M is orthogonal to N in the sense of Birkhoff, $M \perp N$ for short, if $\|m\| \leq \|m+n\|$ for all $m \in M$ and $n \in N$.

It is known that if A, $B^* \in L(H)$ are hyponormal operators, then $d_{A,B}$ satisfies the asymmetric Putnam Fuglede commutativity property $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$, hence $\ker d_{A,B} \perp \mathcal{R}(d_{A,B})$. From the fact that hyponormal operators are closed under translation and multiplication by scalars, B. P. Duggal in [12] deduced that if A and B^* are hyponormal, then

$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I), \forall \lambda \in \mathbb{C}, \tag{1}$$

where $\overline{\lambda}$ is the conjugate of the complex number λ. An operator $T \in L(H)$ is said to be p-hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$, where $|T| = (T^*T)^{\frac{1}{2}}$. An invertible operator $T \in L(H)$ is log-hyponormal if $\log|T^*|^2 \leq \log|T|^2$. In [12] B. P. Duggal proved that if A and B^* are p-hyponormal or log-hyponormal, then

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$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I)$ and $\ker(d_{A,B} - \lambda I) \perp \mathcal{R}(d_{A,B} - \lambda I)$, for every complex number λ. An operator $T \in L(H)$ is said to be w-hyponormal if $(|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*|$, see [19]. It is shown in [2, 3] that the class of w-hyponormal properly contains the class of p-hyponormal ($0 < p \leq 1$), and log-hyponormal. T. Furuta, M. Ito and T. Yamazaki [15] introduced a very interesting class $\mathcal{A}$ operators defined by $|T^2| - |T|^2 \geq 0$, and they showed that class $\mathcal{A}$ is a subclass of paranormal operators (i.e., $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in H$) and contains w-hyponormal operators. An operator $T \in L(H)$ is said to be class $\mathcal{A}(s,t)$, where s and t are strictly positive integers, if $|T^*|^{2t} \leq (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{t+s}}$. Then $T \in \mathcal{A}(\frac{1}{2},\frac{1}{2})$ if and only if T is w-hyponormal and $T \in \mathcal{A}(1,1)$ if and only if T is class $\mathcal{A}$.

I. H. Jeon and I. H. Kim [20] introduced quasi-class $\mathcal{A}$ operators defined by $T^*(|T^2| - |T|^2)T \geq 0$, as an extension of the notion of class $\mathcal{A}$ operators. K. Tanahash, I. H. Jeon, I, H. Kim and A. Uchiyama [26] introduced k-quasi-class $\mathcal{A}$ operators defined by $T^{*k}(|T^2| - |T|^2)T^k \geq 0$, for a positive integer k as an extension of the notion of quasi-class $\mathcal{A}$ operators, for interesting properties of k-quasi-class $\mathcal{A}$ operators, called also quasi-class ($\mathcal{A}$, k), see [17, 26].

In [9, Lemma 2.4], [5, Theorem 3.6] and [13, Lemma 2.4] the authors proved that if $A, B^* \in L(H)$ are w-hyponormal operators with $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then $d_{A,B}$ satisfies the asymmetric Putnam Fuglede commutativity property $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$. Recently B.P. Duggal, C. S. Kubruslly and I. H. Kim in [13, Theorem 2.5] have proved that if $A \in \mathcal{A}(s_1,t_1)$ and $B^* \in \mathcal{A}(s_2,t_2)$, where $0 < s_1, s_2, t_1, t_2 \leq 1$ are such that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then $\delta_{A,B}$ satisfies the asymmetric Putnam Fuglede commutativity property $\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}$. Starting from the fact that all class of operators satisfying (1) share the following conditions

i) A and B^* are reduced by each of its eigenspaces,
ii) A and B^* are polaroid,
iii) A and B^* have property (β).

In this paper we prove that if A and B^* satisfy the conditions i), ii) and iii), then property (1) holds. Which give a generalization of all results obtained before. Now we recall some definitions

Definition 1.1. *An operator $T \in L(H)$ has Bishop's property (β) if for every open set $U \subset \mathbb{C}$ and every sequence of analytic functions $f_n : U \to X$, with the property that $(T - \lambda I)f_n(\lambda) \to 0$ uniformly on every compact subset of U, it follows that $f_n \to 0$, again locally uniformly on U.*

For more information on Bishop's property (β) we refer the interested reader to [22]. Recall that the ascent $p(T)$ of an operator T, is defined by $p(T) = \inf\{n \in \mathbb{N} : \ker T^n = \ker T^{n+1}\}$ and the descent $q(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$, with $\inf \emptyset = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$. We denote by $\Pi(T) = \{\lambda \in \mathbb{C} : p(T - \lambda I) = q(T - \lambda I) < \infty\}$ the set of poles of the resolvent. An operator $T \in L(H)$ is called Drazin invertible if and only if it has finite ascent and descent. The Drazin spectrum of an operator T is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

In the sequel we shall denote by *accS* and *isoS*, the set of accumulation points and the set of isolated points of $S \subset \mathbb{C}$, respectively

Definition 1.2. *An operator $T \in L(H)$ is said to be polaroid if*

$$iso\sigma(T) \subseteq \Pi(T).$$

It is easily seen that, if $T \in L(H)$ is polaroid, then $\Pi(T) = E(T)$, where $E(T)$ is the set of eigenvalues of T which are isolated in the spectrum of T

An important subspace in local spectral theory is the the quasinilpotent part of T, given by

$$H_0(T) = \{x \in H : \lim_{n\to\infty} \|T^n(x)\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $\ker T^n \subset H_0(T)$ for every $n \in \mathbb{N}$, see [1] for information on $H_0(T)$.

The range-kernel orthogonality of $d_{A,B}$ in the sense of G. Birkhoff, $\|X\| \leq \|X - (d_{A,B} - \lambda I)Y\|$, for all $X \in \ker(d_{A,B} - \lambda I)$ and $Y \in L(H)$, was studied by numerous mathematicians, see [4, 5, 10, 21, 27] and the references therein. A sufficient condition guaranteeing the range-kernel orthogonality of $d_{A,B}$ is that $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$ [10]. The main objective of this paper is to give sufficient conditions to have $\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I)$, for every complex number λ. After section one where several basic definitions are assembled, in section two, we prove that if A and B^* satisfy the conditions i), ii) and iii), then $\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I)$ for every complex number λ and that the elementary operator $d_{A,B}$ satisfies the range-kernel orthogonality. We apply the results obtained to k-quasi-class $\mathcal{A}$ operators. We prove that $d_{A,B}$ satisfies the asymmetric Putnam Fuglede commutativity property $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$, if A and B^* are k-quasi-class $\mathcal{A}$ operator with $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$. Our results generalizes the ones given by B. P. Duggal in [12, Theorem 2.3], A. Bachir and F. Lombarkia in [5, Theorem 3.6], B. P. Duggal in [13, Lemma 2.4] and B. P. Duggal in [13, Theorem 2.5].

2. Main Results

Let $T \in L(H)$ be reduced by each of its eigenspaces. If we let $M = \bigvee\{\ker(T - \mu I), \mu \in \sigma_p(T)\}$ (where $\bigvee(.)$ denotes the closed linear span), it follows that M reduces T. Let $T_1 = T|_M$ and $T_2 = T|_{M^\perp}$. By [7, Proposition 4.1] we have

- T_1 is normal with pure point spectrum,
- $\sigma_p(T_1) = \sigma_p(T)$,
- $\sigma(T_1) = cl\sigma_p(T_1)$ (here cl denotes the closure),
- $\sigma_p(T_2) = \emptyset$.

The classical and most known form of the Fuglede Putnam theorem is the following

Theorem 2.1. *[16, 25] If X, A and B are bounded operators acting on complex Hilbert space H such that A and B are normal, then*

$$AX = XB \Longrightarrow A^*X = XB^*.$$

Now we give our main results

Theorem 2.2. *Suppose that $A, B^* \in L(H)$ are reduced by each of its eigenspaces, polaroid and have Bishop's property (β), then*

$$\ker(\delta_{A,B} - \lambda I) \subseteq \ker(\delta_{A^*,B^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

Proof. Since A and B^* are reduced by each of its eigenspaces, then there exists

$$M_1 = \bigvee\{\ker(A - \beta I), \beta \in \sigma_p(A)\} \text{ and } M_2 = H \ominus M_1$$

on the one hand and

$$N_1 = \bigvee\{\ker(B^* - \overline{\alpha} I), \overline{\alpha} \in \sigma_p(B^*)\} \text{ and } N_2 = H \ominus N_1$$

on the other hand such that A and B have the representations

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } H = M_1 \oplus M_2,$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ on } H = N_1 \oplus N_2.$$

Recall from [14] that $\sigma(\delta_{A,B}) = \sigma(A) - \sigma(B)$. We consider the following cases:

Case 1: If $\lambda \in \mathbb{C} \backslash \sigma(\delta_{A,B})$, then $\ker(\delta_{A,B} - \lambda I) = \{0\}$ and hence

$$\ker(\delta_{A,B} - \lambda I) \subseteq \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

Case 2: If $\lambda \in iso\sigma(\delta_{A,B})$, then there exists finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$, where $\mu_i \in iso\sigma(A)$ and $\nu_i \in iso\sigma(B)$ such that

$$\lambda = \mu_i - \nu_i, \text{ for all } 1 \leq i \leq n$$

Since the spectrum of A_2 and the spectrum of B_2 does not contains isolated points, then $\lambda \notin \sigma(\delta_{A_i,B_j})$ for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Consider $X \in \ker(\delta_{A,B} - \lambda I)$ such that $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$ have the representation $X = [X_{kl}]_{k,l=1}^2$. Hence

$$(\delta_{A,B} - \lambda I)(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)(X_{11}) & (\delta_{A_1,B_2} - \lambda I)(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)(X_{21}) & (\delta_{A_2,B_2} - \lambda I)(X_{22}) \end{pmatrix} = 0.$$

Observe that $\delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Hence $X_{22} = X_{21} = X_{12} = 0$. Since $A_1 - \mu_i$ and $B_1 - \nu_i$ are normal, it follows from Fuglede-Putnam theorem that

$$(A_1^* - \overline{\mu_i})X_{11} - X_{11}(B_1^* - \overline{\nu_i}) = 0,$$

consequently

$$X = X_{11} \oplus 0 \in \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

Case 3: If $\lambda \in acc\sigma(\delta_{A,B})$, it follows from [24, Lemma 3.1] that $\lambda \in (\sigma(A) - acc\sigma(B)) \cup (acc\sigma(A) - \sigma(B))$, then there exists $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ such that $\lambda = \mu - \nu \in (\sigma(A) - acc\sigma(B))$ or $\lambda = \mu - \nu \in (acc\sigma(A) - \sigma(B))$. Since A and B are polaroid, then $acc\sigma(A) = \sigma_D(A)$ and $acc\sigma(B) = \sigma_D(B)$, it is easy to see that

$$\sigma_D(A) = \sigma_D(A_1) \cup \sigma_D(A_2) \text{ and } \sigma_D(B) = \sigma_D(B_1) \cup \sigma_D(B_2),$$

since $\sigma_p(A_2) = \sigma_p(B_2^*) = \emptyset$, then $\sigma_D(A_2) = \sigma(A_2)$ and $\sigma_D(B_2) = \sigma(B_2)$. Hence we have

$$\mu \in \sigma_D(A_1) \cup \sigma(A_2) \text{ and } \nu \in \sigma(B_1) \cup \sigma(B_2).$$

Or

$$\mu \in \sigma(A_1) \cup \sigma(A_2) \text{ and } \nu \in \sigma_D(B_1) \cup \sigma(B_2).$$

Let $X \in \ker(\delta_{A,B} - \lambda I)$ such that

$$X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2 \text{ have the representation } X = [X_{kl}]_{k,l=1}^2.$$

Hence

$$(\delta_{A,B} - \lambda I)(X) = \begin{pmatrix} (\delta_{A_1,B_1} - \lambda I)(X_{11}) & (\delta_{A_1,B_2} - \lambda I)(X_{12}) \\ (\delta_{A_2,B_1} - \lambda I)(X_{21}) & (\delta_{A_2,B_2} - \lambda I)(X_{22}) \end{pmatrix} = 0.$$

We consider the following cases

- $\mu \in \sigma(A_1)$ and $\nu \in \sigma_D(B_1)$, or
- $\mu \in \sigma(A_1)$ and $\nu \in \sigma(B_2)$, or
- $\mu \in \sigma(A_2)$ and $\nu \in \sigma_D(B_1)$, or
- $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_2)$.

or

- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_1)$, or
- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_2)$, or
- $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_1)$.

We start by studying these cases

- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma_D(B_1)$. Since $\mu \notin \sigma(A_2)$ and $\nu \notin \sigma(B_2)$, then $\delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Let $X \in \ker(\delta_{A,B} - \lambda I)$. Hence $X_{12} = X_{21} = X_{22} = 0$. Since $A_1 - \mu$ and $B_1 - \nu$ are normal, it follows from Fuglede-Putnam theorem that
$$(A_1^* - \overline{\mu_i})X_{11} - X_{11}(B_1^* - \overline{\nu_i}) = 0,$$
consequently
$$X = X_{11} \oplus 0 \in \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma(B_2)$, let $X \in \ker(\delta_{A,B} - \lambda I)$, then $X_{22} = X_{21} = X_{11} = 0$ and
$$A_1X_{12} = X_{12}(B_2 - \lambda). \tag{2}$$
Let x is any vector in N_2, then $X_{12}x \in M_1$, hence $X_{12}x = \sum a_n\varphi_n$, where φ_n are the eigenvector associated to the eigenvalue μ_n of the normal operator A_1 ($A_1\varphi_n = \mu_n\varphi_n$). Form equation (2) we get
$$(B_2^* - \overline{\lambda} - \overline{\mu_n})X_{12}^*\varphi_n = X_{12}^*(A_1^* - \overline{\mu_n})\varphi_n = 0,$$
which implies that $X_{12}^*\varphi_n = 0$, since $B_2^* - (\overline{\lambda} + \overline{\mu_n})$ is injective.
Thus $\|X_{12}x\|^2 = \langle X_{12}x, \sum a_n\varphi_n\rangle = \langle x, \sum a_nX_{12}^*\varphi_n\rangle = 0$, whence $X_{12} = 0$ and
$$X = 0 \in \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma_D(B_1)$, let $X \in \ker(\delta_{A,B} - \lambda I)$, then $X_{11} = X_{22} = X_{12} = 0$ and $(A_2 - \lambda)X_{21} = X_{21}B_1$. Let x is any vector in M_2, then $X_{21}^*x \in N_1$, hence $X_{21}^*x = \sum a_n\varphi_n$, where φ_n are the eigenvector associated to the eigenvalue ν_n of the normal operator B_1 ($B_1\varphi_n = \nu_n\varphi_n$). Similarly as in the precedent case we obtain $X_{21} = 0$, hence
$$X = 0 \in \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_2)$. Since A has Bishop's property (β) it follows from [6, Remarks 3.2] that A_2 has property (β), applying [1, Theorem 2.20] we get $H_0(A_2 - \mu)$ is closed and from [22, Proposition 1.2.20] that $\sigma(A_2|_{H_0(A_2-\mu)}) \subseteq \{\mu\}$. If $\sigma(A_2|_{H_0(A_2-\mu)}) = \emptyset$, then $H_0(A_2 - \mu) = \{0\}$, the case $\sigma(A_2|_{H_0(A_2-\mu)}) = \{\mu\}$ is not possible, since the spectrum of the operator A_2 does not contains isolated points. Hence $H_0(A_2 - \mu) = \{0\}$. let $X \in \ker(\delta_{A,B} - \lambda I)$, then $X_{21} = X_{12} = X_{11} = 0$ and $(A_2 - \mu)X_{22} = X_{22}(B_2 - \nu)$, this implies that, if $t \in H_0(B_2 - \nu)$, then $X_{22}t \in H_0(A_2 - \mu) = \{0\}$. Hence $X_{22}t = 0$. Since $t \in H_0(B_2 - \nu)$, using properties of quasinilpotent part, we get $(B_2 - \nu)(t) \in H_0(B_2 - \nu)$, consequently $N_2 = \overline{H_0(B_2 - \nu)}$. So $X_{22} = 0$, hence
$$X = 0 \in \ker(\delta_{A^*,B^*} - \overline{\lambda} I).$$

The cases

- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_1)$, or
- $\mu \in \sigma_D(A_1)$ and $\nu \in \sigma(B_2)$, or

- $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_1)$,

can be proved similarly. □

Theorem 2.3. *Let $A, B \in L(H)$. If all the eigenvalues of A, B^* are reduced by each of its eigenspaces, polaroid and have Bishop's property (β), then*

$$\ker(\Delta_{A,B} - \lambda I) \subseteq \ker(\Delta_{A^*,B^*} - \overline{\lambda} I), \quad \forall \lambda \in \mathbb{C}.$$

Proof. Since A and B^* are reduced by each of its eigenspaces, then then there exists

$$M_1 = \bigvee \{\ker(A - \beta I), \beta \in \sigma_p(A)\} \text{ and } M_2 = H \ominus M_1$$

on the one hand and

$$N_1 = \bigvee \{\ker(B^* - \overline{\alpha} I), \overline{\alpha} \in \sigma_p(B^*)\} \text{ and } N_2 = H \ominus N_1$$

on the other hand such that A and B have the representations

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } H = M_1 \oplus M_2$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ on } H = N_1 \oplus N_2.$$

Recall from [14] that $\sigma(\Delta_{A,B}) = \sigma(A)\sigma(B) - \{1\}$. We consider the following cases.

Case 1: If $\lambda \in \mathbb{C} \backslash \sigma(\Delta_{A,B})$, the result is immediate.

Case 2: If $\lambda \in iso\sigma(\Delta_{A,B})$ and $\lambda \neq -1$, then there exists finite sequences $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$, where $\mu_i \in iso\sigma(A)$ and $\nu_i \in iso\sigma(B)$ such that

$$\lambda = \mu_i \nu_i - 1, \text{ for all } 1 \leq i \leq n$$

Since the spectrum of A_2 and the spectrum of B_2 does not contains isolated points, then $\lambda \notin \sigma(\Delta_{A_i,B_j})$ for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Let $X \in \ker(\Delta_{A,B} - \lambda I)$ such that

$$X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2 \text{ have the representation } X = [X_{kl}]_{k,l=1}^2.$$

Hence

$$(\Delta_{A,B} - \lambda I)(X) = \begin{pmatrix} (\Delta_{A_1,B_1} - \lambda I)(X_{11}) & (\Delta_{A_1,B_2} - \lambda I)(X_{12}) \\ (\Delta_{A_2,B_1} - \lambda I)(X_{21}) & (\Delta_{A_2,B_2} - \lambda I)(X_{22}) \end{pmatrix} = 0.$$

Observe that $\Delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \leq i, j \leq 2$ other than $i = j = 1$. Hence $X_{22} = X_{21} = X_{12} = 0$. Since A_1 and B_1 are normal, it follows from Fuglede-Putnam theorem and [11, Theorem 2] that

$$\frac{1}{1 + \overline{\lambda}} A_1^* X_{11} B_1^* - X_{11} = 0,$$

consequently

$$X = X_{11} \oplus 0 \in \ker(\Delta_{A^*,B^*} - \overline{\lambda} I).$$

If $\lambda = -1$, we consider the case $-1 \in iso\sigma(\Delta_{A,B})$, that is $0 \in iso\sigma(L_A R_B)$, hence either $0 \in iso\sigma(A)$ and $0 \in iso\sigma(B)$ or $0 \in iso\sigma(A)$ and $0 \notin \sigma(B)$ or $0 \in iso\sigma(B)$ and $0 \notin \sigma(A)$. If $0 \in iso\sigma(A)$ and $0 \in iso\sigma(B)$. Let $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$, have the matrix representation $X = [X_{kl}]_{k,l=1}^2$. If $X \in \ker(L_A R_B)$, then $\begin{pmatrix} A_1 X_{11} B_1 & A_1 X_{12} B_2 \\ A_2 X_{21} B_1 & A_2 X_{22} B_2 \end{pmatrix} = 0$, it follows that $X_{22} = X_{21} = X_{12} = 0$, and $A_1 X_{11} B_1 = 0$, hence $X_{11} \in \ker L_{A_1^*} R_{B_1^*}$. Thus $X \in \ker(L_{A^*} R_{B^*})$.

If $0 \in iso\sigma(A)$ and $0 \notin \sigma(B)$, then B is invertible. Let $X : N_1 \oplus N_2 \longrightarrow M_1 \oplus M_2$, have the matrix representation $X = [X_{kl}]_{k,l=1}^2$. If $X \in \ker(L_A R_B)$, then $\begin{pmatrix} A_1X_{11}B_1 & A_1X_{12}B_2 \\ A_2X_{21}B_1 & A_2X_{22}B_2 \end{pmatrix} = 0$, it follows that $X_{22} = X_{21} = X_{12} = 0$, and $A_1X_{11}B_1 = 0$, hence $X_{11} \in \ker L_{A_1^*}R_{B_1^*}$. Thus $X \in \ker(L_{A^*}R_{B^*})$.

The proof of the other remaining case can be proved similarly.

Case 3: If $\lambda \in acc\sigma(\Delta_{A,B}) = (acc\sigma(A)\sigma(B) - 1) \cup (\sigma(A)acc\sigma(B) - 1)$, and $\lambda = -1$ then $0 \in \sigma(A)acc\sigma(B)$ or $0 \in acc\sigma(A)\sigma(B)$. Since A and B are polaroid, then $0 \in acc\sigma(A) = \sigma_D(A)$ and $0 \in acc\sigma(B) = \sigma_D(B)$.

- If $0 \in \sigma(A_1)$ and $0 \in \sigma_D(B_1)$. Observe that $L_{A_i}R_{B_j}$ is invertible for all $1 \le i, j \le 2$ other than $i = j = 1$, let $X \in \ker(L_A R_B)$. Hence $X_{12} = X_{21} = X_{22} = 0$. Since A_1 and B_1 are normal, it follows from Fuglede-Putnam theorem that
$$A_1^* X_{11} B_1^* = 0,$$
consequently
$$X = X_{11} \oplus 0 \in \ker(L_{A^*}R_{B^*}).$$

- If $0 \in \sigma(A_1)$ and $0 \in \sigma(B_2)$. Let $X \in \ker(L_A R_B)$, then $X_{22} = X_{21} = X_{11} = 0$. We have $A_1X_{12}B_2 = 0$ this implies that $A_1X_{12} = 0$. Since A_1 is normal, then $A_1^*X_{12} = 0$, consequently $A_1^*X_{12}B_2^* = 0$. Hence
$$X = \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix} \in \ker(L_{A^*}R_{B^*}).$$

- If $0 \in \sigma(A_2)$ and $0 \in \sigma_D(B_1)$, this case can be proved similarly as the precedent one.

- If $0 \in \sigma(A_2)$ and $0 \in \sigma(B_2)$. Let $X \in \ker(L_A R_B)$, then $X_{12} = X_{21} = X_{11} = 0$. Since A_2 and B_2^* are injective, then $X_{22} = 0$. Hence
$$X = 0 \in \ker(L_{A^*}R_{B^*}).$$

Case 4: If $\lambda \in acc\sigma(\Delta_{A,B}) = (acc\sigma(A)\sigma(B) - 1) \cup (\sigma(A)acc\sigma(B) - 1)$ and $\lambda \neq -1$, then there exists $\mu \in \sigma(A)$ and $\nu \in \sigma(B)$ such that $\lambda = \mu\nu \in (\sigma(A)acc\sigma(B) - 1)$ or $\lambda = \mu\nu \in (acc\sigma(A)\sigma(B) - 1)$.

- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma_D(B_1)$. Observe that $\Delta_{A_i,B_j} - \lambda I$ is invertible for all $1 \le i, j \le 2$ other than $i = j = 1$, let $X \in \ker(\Delta_{A,B} - \lambda I)$. Hence $X_{12} = X_{21} = X_{22} = 0$. Since A_1 and B_1 are normal, it follows from Fuglede-Putnam theorem and [11, Theorem 2] that
$$\frac{1}{1+\overline{\lambda}} A_1^* X_{11} B_1^* - X_{11} = 0,$$
consequently
$$X = X_{11} \oplus 0 \in \ker(\Delta_{A^*,B^*} - \overline{\lambda} I).$$

- If $\mu \in \sigma(A_1)$ and $\nu \in \sigma(B_2)$. Let $X \in \ker(\Delta_{A,B} - \lambda I)$, then $X_{22} = X_{21} = X_{11} = 0$ and $A_1X_{12}B_2 - (1+\lambda)X_{12} = 0$. Let x is any vector in N_2, then $X_{12}x \in M_1$, hence $X_{12}x = \sum a_n\varphi_n$, where φ_n are the eigenvector associated to the eigenvalue μ_n of the normal operator A_1 ($A_1\varphi_n = \mu_n\varphi_n$). Note that
$$B_2^* X_{12}^* (A_1^* - \overline{\mu_n})\varphi_n - (1 + \overline{\lambda} - \overline{\mu_n} B_2^*) X_{12}^* \varphi_n = 0,$$
which implies that $X_{12}^*\varphi_n = 0$, since $\sigma_p(B_2^*) = \emptyset$. Thus $\|X_{12}x\|^2 = \langle X_{12}x, \sum a_n\varphi_n\rangle = \langle x, \sum a_n X_{12}^*\varphi_n\rangle = 0$, whence $X_{12} = 0$ and
$$X = 0 \in \ker(\Delta_{A^*,B^*} - \overline{\lambda} I).$$

- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma_D(B_1)$, this case can be proved similarly as the precedent one.

- If $\mu \in \sigma(A_2)$ and $\nu \in \sigma(B_2)$. Since A has property (β) it follows from [6, Remarks 3.2] that A_2 has property (β), applying [1, Theorem 2.20] we get $H_0(A_2-\mu)$ is closed and from [22, Proposition 1.2.20] That $\sigma(A_2|_{H_0(A_2-\mu)}) \subseteq \{\mu\}$. If $\sigma(A_2|_{H_0(A_2-\mu)}) = \emptyset$, then $H_0(A_2-\mu) = \{0\}$, the case $\sigma(A_2|_{H_0(A_2-\mu)}) = \{\mu\}$ is not possible, since the spectrum of the operator A_2 does not contains isolated points. Hence $H_0(A_2-\mu) = \{0\}$, we have $X \in \ker(\Delta_{A,B} - \lambda I)$, then $X_{21} = X_{12} = X_{11} = 0$ and $A_2X_{22}B_2 - (1+\lambda)X_{22} = 0$, this implies that
$$(A_2-\mu)X_{22}(B_2-\nu) + \nu(A_2-\mu)X_{22} + \mu X_{22}(B_2-\nu) = 0.$$
if $t \in H_0(B_2-\nu)$, then $X_{22}t \in H_0(A_2-\mu) = \{0\}$. Hence $X_{22}t = 0$. Since $t \in \overline{H_0(B_2-\nu)}$, using properties of quasinilpotent part, we get $(B_2-\nu)(t) \in H_0(B_2-\nu)$, consequently $N_2 = \overline{H_0(B_2-\nu)}$. So $X_{22} = 0$, hence
$$X = 0 \in \ker(\Delta_{A^*,B^*} - \overline{\lambda} I).$$

The other cases can be proved similarly. □

Theorem 2.4. *Suppose that $A, B^* \in L(H)$ are reduced by each of its eigenspaces, polaroid and have Bishop's property (β), then $\mathcal{R}(d_{A,B} - \lambda I)$ is orthogonal to $\ker(d_{A,B} - \lambda I)$, for all $\lambda \in \mathbb{C}$.*

Proof. Follows from [10, Lemma 4] □

Corollary 2.5. *[12, Lemma 2.1] Suppose that $A, B^* \in L(H)$ are p-hyponormal or log-hyponormal, then*
$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I),\ \forall \lambda \in \mathbb{C}$$

Corollary 2.6. *[9, Lemma 2.4] Let $A, B^* \in L(H)$ be w-hyponormal operators such that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then*

$$\ker d_{A,B} \subseteq \ker d_{A^*,B^*}.$$

Corollary 2.7. *[5, Theorem 3.6],[13, Lemma 2.4] Let $A, B^* \in L(H)$ be w-hyponormal operators such that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then*

$$\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}.$$

Corollary 2.8. *[13, Theorem 2.5] Let $A, B^* \in L(H)$. If $A \in \mathcal{A}(s_1,t_1)$ and $B^* \in \mathcal{A}(s_2,t_2)$, $0 < s_1, s_2, t_1, t_2 \leq 1$ are such that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then*

$$\ker \delta_{A,B} \subseteq \ker \delta_{A^*,B^*}.$$

As a nice application of our main results the Fuglede Putnam theorem for k-quasi-class $\mathcal{A}$ operators which contains all the precedent classes of operators.

Theorem 2.9. *Let $A, B^* \in LH)$ be k-quasi-class A operators, then*
$$\ker(d_{A,B} - \lambda I) \subseteq \ker(d_{A^*,B^*} - \overline{\lambda} I),$$
for all non null complex number λ.

Proof. We know from [18, Theorem 2.4] that k-quasi-class $\mathcal{A}$ operators are polaroid and from [26, Lemma 11] that k-quasi-class $\mathcal{A}$ operators have Bishop's property (β). Since by [26, Lemma 13], A and B^* are reduced by each of its eigenspaces, then the conclusion follows from Theorem 2.2 and Theorem 2.3. □

Theorem 2.10. *Let $A, B^* \in L(H)$ be k-quasi-class $\mathcal{A}$ operators such that $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$, then*

$$\ker d_{A,B} \subseteq \ker d_{A^*,B^*}.$$

Proof. The conditions $\ker A \subseteq \ker A^*$ and $\ker B^* \subseteq \ker B$ imply that 0 is normal eigenvalue of both A and B^*. It follows from Theorem 2.2 and Theorem 2.3 that $\ker d_{A,B} \subseteq \ker d_{A^*,B^*}$. □

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