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Curvature Inequalities for $C-$Totally Real Doubly Warped Products of Locally Conformal Almost Cosymplectic Manifolds

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Abstract. In this paper, we establish some optimal inequalities for the squared mean curvature in terms warping functions of a $C-$totally real doubly warped product submanifold of a locally conformal almost cosymplectic manifold with a pointwise $\varphi-$sectional curvature c. The equality case in the statement of inequalities is also considered. Moreover, some applications of obtained results are derived.

Contents

1. Introduction

The idea of warped product manifolds was first introduced by Bishop and O'Neill (cf. [5]) to study manifolds of negative curvature. Later on, doubly warped product manifolds studied by Unal [29]. They defined these manifolds as follows:

Definition 1.1. *Let M_1 and M_2 be two Riemannian manifolds of dimensions n_1 and n_2 endowed with Riemannian matrics $g_1$and g_2 such that $f_1 : M_1 \to (0,\infty)$ and $f_2 : M_2 \to (0,\infty)$ be positive differentiable functions on M_1 and M_2, respectively. Thus, the doubly warped product $M =_{f_2} M_1 \times_{f_1} M_2$ is defined to be the product manifold $M_1 \times M_2$ with equipped metric $g = f_2^2\, g_1 + f_1^2\, g_2$. Moreover, If we consider $\gamma_1 : M_1 \times M_2 \to M_1$ and $\gamma_2 : M_1 \times M_2 \to M_2$ are the natural projections on M_1 and M_2, respectively then the metric g on doubly warped product is defined as*

$$g(X,Y) = (f_2 \circ \gamma_2)^2\, g_1(\gamma_{1\star}X, \gamma_{1\star}Y) + (f_1 \circ \gamma_1)^2 g_2(\gamma_{2\star}X, \gamma_{2\star}Y) \tag{1.1}$$

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for any vector fields X, Y tangent to M, where $\star$ is the symbol for the tangent maps. Thus, the functions f_1 and f_2 are called warping functions on M. If both $f_1 = 1$ and $f_2 = 1$, then M is called a simply Riemannian product manifold. If either $f_1 = 1$ or $f_2 = 1$, then M is called a (single) warped product manifold. If $f_1 \neq 1$ and $f_2 \neq 1$, then M is called a non-trivial doubly warped product manifold. Let $M = {}_{f_2} M_1 \times_{f_1} M_2$ be a non-trivial doubly warped product manifold of an arbitrary Riemannian manifold $\widetilde{M}$. Then

$$\nabla_X Z = \nabla_Z X = (Z \ln f_2)X + (X \ln f_1)Z, \tag{1.2}$$

$$\nabla_X Y = \nabla^1_X Y - \frac{{f_2}^2}{{f_1}^2} g_1(X, Y)(\ln f_2), \tag{1.3}$$

for any vector fields $X, Y \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$. Further, ∇^1 and ∇^2 are Levi-Citvita connections of the induced metrics on Riemannian manifolds M_1 and M_2, respectively.

The following well-known result of Chen in [8] obtained a sharp relationship between the squared norm of mean curvature and the warping function f of warped product $M \times_f M_2$ isometrically immersed into a real space form.

Theorem 1.2. *[8]. Let $x : M_1 \times_f M_2$ be an isometrically immersion of an n-dimensional warped product into a $2m$-dimensional real space form $\widetilde{M}(c)$ with constant sectional curvature c. Then*

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1.c$$

where $n_i = dimM_i$, $i = 1, 2$ and Δ is the Lapalcian operator of M_1. Moreover, the equality holds in the above inequality if and only if x is mixed totally geodesic and $n_1 H_1 = n_2 H_2$ such that H_1 and H_2 are partial mean curvature.

Motivated by Chen's result several inequalities have been obtained by many geometers for warped products and doubly warped products in different setting of the ambient manifolds [9, 18, 24, 25, 30, 34, 35]. In this paper, we study to $C-$totally real doubly warped product isometrically immersed into a locally conformal almost cosymplectic manifold. The inequalities which we obtain in this paper are very fascinating because we derive upper bound and lower bound for warping functions in terms of mean curvature, scalar curvature and pointwise constant $\varphi-$sectional curvature c. The obtained results generalise some other inequalities as special cases..

2. Preliminaries

A $(2m + 1)$-dimensional smooth manifold $\widetilde{M}$ is called *locally conformal almost cosymplectic* manifold, if it is consisting an endomorphism φ of its tangent bundle $T\widetilde{M}$, a structure vector field ξ and a 1-form η which satisfies the following:

$$\varphi^2 = -I + \eta \oplus \xi, \quad \eta(\xi) = 1, \quad \eta o \varphi = 0, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi), \tag{2.2}$$

$$(\widetilde{\nabla}_U \varphi)V = \vartheta\{g(\varphi U, V) - \eta(V)\varphi U\}, \tag{2.3}$$

$$\widetilde{\nabla}_U \xi = \vartheta\{U - \eta(U)\xi\}, \tag{2.4}$$

for any U, V tangent to $\widetilde{M}$ and $\omega = \vartheta\eta$ (see [26]). Let us we consider that the function $\vartheta = 0$ and $\vartheta = 1$, then $\widetilde{M}$ becomes cosympelctic manifold and Kenmotsu manifold, respectively (see [14, 34]). An almost contact metric manifold $\widetilde{M}$, a plane section σ in $T_p\widetilde{M}$ of $\widetilde{M}$ is said to be a $\varphi -$ *section* if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. The sectional curvature $\widetilde{K}(\sigma)$ does not depend on the choice of the $\varphi-$*scetion* σ of $T_p\widetilde{M}$ at each point $p \in \widetilde{M}$, then $\widetilde{M}$ is called a manifold with pointwise constant $\varphi -$ *sectional* curvature. In this case for any $p \in \widetilde{M}$ and for $\varphi-$*scetion* σ of $T_p\widetilde{M}$, the fuction c defined by $c(p) = \widetilde{K}(p)$ is said to be $\varphi -$ *sectional* curvature of $\widetilde{M}$. That is, for a locally conformal almost cosymplectic manifold $\widetilde{M}$ of *dimension* ≥ 5 with pointwise $\varphi-$sectional curvature c, its curvature tensor $\widetilde{R}$ is defined as

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = &\frac{c-3\vartheta^2}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\\ &+\frac{c+\vartheta^2}{4}\{g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)\\ &-2g(X,\varphi Y)g(Z,\varphi W)\}\\ &-\{\frac{c+\vartheta^2}{4}+\vartheta'\}\{g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(X)\eta(W)\\ &+g(Y,Z)\eta(X)\eta(W)g(Y,W)(X)\eta(Z)\},\end{aligned} \tag{2.5}$$

for any X,Y,Z,W are tangent to $\widetilde{M}(c)$, where ϑ is the conformal function such that $\omega=\vartheta\eta$ and $\vartheta'=\xi\vartheta$. Moreover, c is the function of constant $\varphi-$sectional curvature of $\widetilde{M}$. If we consider the fuction $\vartheta=0$ and $\vartheta=1$, then $\widetilde{M}(c)$ becomes cosympelctic space form and Kenmotsu space form, respectively (see [18, 34]). Let us consider that M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then Gauss and Weingarten formulas are given by

$$(i)\ \widetilde{\nabla}_U V=\nabla_U V+h(U,V),\quad (ii)\ \widetilde{\nabla}_U N=-A_N U+\nabla^{\perp}_U N, \tag{2.6}$$

for each $U,V\in\Gamma(TM)$ and $N\in\Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\widetilde{M}$. They are related as

$$g(h(U,V),N)=g(A_N U,V), \tag{2.7}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Now, for any $U\in\Gamma(TM)$ and $N\in\Gamma(T^{\perp}M)$, we have

$$(i)\ \varphi U=TU+FU,\qquad (ii)\ \varphi N=tN+fN, \tag{2.8}$$

where TU (tN) and FU (fN) are tangential and normal components of φU (φN), respectively. From (2.1) and (2.5) (i), it is easy to observe that for each $U,V\in\Gamma(TM)$, we have

$$(i)\ g(TU,V)=-g(U,TV)\quad (ii)\ \|T\|^2=\sum_{i,j=1}^{n}g^2(Te_i,e_j). \tag{2.9}$$

For a subamnifold M, the Gauss equation is:

$$\begin{aligned}\widetilde{R}(U,V,Z,W)=&R(U,V,Z,W)+g(h(U,Z),h(V,W))\\ &-g(h(U,W),h(V,Z)),\end{aligned} \tag{2.10}$$

for any $U,V,Z,W\in\Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}$ and M, respectively. The mean curvature vector H for an orthonormal frame $\{e_1,\cdots,e_n\}$ of tangent space TM on M is defined by

$$H=\frac{1}{n}trace(h)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i), \tag{2.11}$$

where $n=\dim M$. Also, for any $r\in\{e_{n+1},\cdots,e_{2m+1}\}$, we set

$$h^r_{ij}=g(h(e_i,e_j),e_r)\ \ and\ \ \|h\|^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j)). \tag{2.12}$$

The scalar curvature ρ for a submanifold M of an almost contact manifold $\widetilde{M}$ is given by

$$\rho = \sum_{1\leq i\neq j\leq n} K(e_i \wedge e_j), \tag{2.13}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by e_i and e_j. Let G_r be a r-plane section on TM and $\{e_1, e_2, \cdots, e_r\}$ any orthonormal basis of G_r. Then the scalar curvature $\rho(G_r)$ of G_r is given by

$$\rho(G_r) = \sum_{1\leq i\neq j\leq r} K(e_i \wedge e_j). \tag{2.14}$$

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *totally umbilical* and *totally geodesic* if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, respectively, for any $U, V \in \Gamma(TM)$, where H is the mean curvature vector of M. Furthermore, if $H = 0$, then M is *minimal* in $\widetilde{M}$. On the oher hand, a submanifold M is called totally real submanifold if T is identically zero, i.e. $\varphi U \in \Gamma(T_p^{\perp}M)$ for any $U \in \Gamma(T_pM)$ for each $p \in M$.

Moreover, if the structure vector field ξ is normal to submanifold M, then M is said to be a $C-$*totally real submanifold* [19] of an almost contact manifold.

Let $\phi : M =_{f_2} M_1 \times_f M_2 \to \widetilde{M}$ be isometric immersion of a doubly warped product ${}_{f_2}M_1 \times_f M_2$ into a Riemannian manifold of $\widetilde{M}$ of constant sectional curvature c. Suppose that n_1, n_2 and n be the dimensions of M_1, M_2 and $M_1 \times_f M_2$, respectively. Then for unit vector fields X and Z tangent to M_1 and M_2 respectively, we have

$$\begin{aligned} K(X \wedge Z) &= g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) \\ &= \frac{1}{f_1}\{(\nabla^1_X X) f_1 - X^2 f_1\} + \frac{1}{f_2}\{(\nabla^2_Z Z) f_2 - Z^2 f_2\}. \end{aligned} \tag{2.15}$$

If we consider the local orthonormal frame $\{e_1, e_2, \cdots, e_n)$ such that $e_1, e_2, \cdots, e_{n_1}$ tangent to M_1 and $e_{n_1+1}, \cdots, e_n$ are tangent to M_2, then the sectional curvatre in terms of general doubly warped product is defined by

$$\sum_{1\leq i\leq n_1} \sum_{n_1+1\leq j\leq n} K(e_i \wedge e_j) = \frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \tag{2.16}$$

for each $j = n_1 + 1, \cdots, n$. Now, we have the following useful lemma.

Lemma 2.1. *[8]. Let $a_1, a_2, \cdots, a_n, a_{n+1}$ be $n + 1$ $(n \geq 2)$ be real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + a_{n+1}). \tag{2.17}$$

Then $2a_1a_2 \geq a_3$ with the equality holds if and only if $a_1 + a_2 = a_3 = \cdots, a_n$.

On the other hand, we analyze general doubly warped products in locally conformal almost cosymplectic manifold. That is, let $M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}$ be an isometric immersion from a doubly warped product ${}_{f_2}M_1 \times_{f_1} M_2$ into a locally conformal almost cosymplectic manifold $\widetilde{M}$. Assume that $\xi \in \Gamma(TM_1)$ and $X \in \Gamma(TM_2)$, thus, from (2.4), we obtain

$$\widetilde{\nabla}_X \xi = \vartheta\{X - \eta(X)\xi\},$$

which implies by using (2.6) (i) and $\eta(X) = 0$, that

$$\nabla_X \xi = \vartheta X, \quad h(X, \xi) = 0.$$

Using (1.3) in the first relation of above equation, we find

$$(X \ln f_2)\xi + (\xi \ln f_1)X = \vartheta X. \tag{2.18}$$

Now, taking the inner product with ξ in (2.18), we obtain $X \ln f_2 = 0$, i.e., f_2 is constant on M_2. Hence, there is no doubly warped product in a locally conformal almost cosymplectic manifold, if ξ is tangent to M_1. Moreover, if $\xi \in \Gamma(TM_2)$ and $Z \in \Gamma(TM_1)$, then again from (2.4), we have

$$\widetilde{\nabla_Z}\xi = \vartheta Z,$$

From (2.6) (i), we get

$$\nabla_Z \xi = \vartheta Z, \quad h(Z, \xi) = 0. \tag{2.19}$$

Again, using (1.3) in (2.19) and then taking the inner product with ξ, it is easy to see f_1 is also constant function on M_1. Hence, in both the cases, any one of the warping function is constant. Thus, we conclude that there do not exist doubly warped product submanifold in a locally conformal almost cosymplectic manifold such that ξ is tangent to the submanifold. Therefore, we consider ξ is normal to submanifold M and there is a non-trivial doubly warped product in a locally conformal almost cosymplectic manifold which is called $C-$totally real doubly warped product. In the next section, we obtain some geometric inequalities for such type doubly warped product immersions.

3. Main inequalities of $C-$totally real doubly warped products

Theorem 3.1. *Let $\widetilde{M}(c)$ be a $(2m+1)-$dimensional locally conformal almost cosymplectic manifold and $\phi :_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real doubly warped product into $\widetilde{M}(c)$ such that c is pointwise constant $\varphi-$sectional curvature. Then*

(i) The relation between warping functions and the squared norm of mean curvature is given by

$$\frac{n_2 \Delta_1 f_1}{f_1} + \frac{n_1 \Delta_2 f_2}{f_2} \leq \frac{n^2}{4}\|H\|^2 + \frac{c - 3\vartheta^2}{4} n_1 n_2, \tag{3.1}$$

where $n_i = \dim M_i$, $i = 1, 2$ and Δ^i is the Laplacian operator on M_i, $i - 1, 2$.

(ii) The equality sign holds in the above inequality if and only if ϕ is mixed totally geodesic immersion and $n_1.H_1 = n_2.H_2$, where H_1 and H_2 are the partial mean curvature vectors on M_1 and M_2, respectively.

Proof. Suppose that ${}_{f_2}M_1 \times_f M_2$ be a $C-$totally real doubly warped product submanifold in a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with pointwise constant $\varphi-$sectional curvature c. Then from Gauss equation (2.10) and (2.5), we derive

$$2\rho = \frac{c - 3\vartheta^2}{4} n(n-1) + n^2\|H\|^2 - \|h\|^2. \tag{3.2}$$

Let us consider that

$$\delta = 2\rho - \frac{c - 3\vartheta^2}{4} n(n-1) - \frac{n^2}{2}\|H\|^2. \tag{3.3}$$

Then from (3.2) and (3.3), it follows that

$$n^2\|H\|^2 = 2(\delta - \|h\|^2). \tag{3.4}$$

Thus from the orthonormal frame field $\{e_1, e_2, \cdots, e_n\}$, the above equation takes the form

$$\Big(\sum_{r=n+1}^{2m+1} \sum_{i=1}^{n} h_{ii}^r\Big)^2 = 2\Big(\delta + \sum_{r=n+1}^{2m+1} \sum_{i=1}^{n} (h_{ii}^r)^2 + \sum_{r=n+1}^{2m+1} \sum_{i<j=1}^{n} (h_{ij}^r)^2 + \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2\Big),$$

which simplifies as

$$\frac{1}{2}\Big(h_{11}^{n+1}+\sum_{i=2}^{n_1}h_{ii}^{n+1}+\sum_{t=n_1+1}^{n}h_{tt}^{n+1}\Big)^2=\delta+(h_{11}^{n+1})^2+\sum_{i=2}^{n_1}(h_{ii}^{n+1})^2+\sum_{t=n_1+1}^{n}(\delta_{tt}^{n+1})^2$$
$$-\sum_{2\le j\neq l\le n_1}h_{jj}^{n+1}h_{ll}^{n+1}-\sum_{N_1+1\le t\neq s\le n}h_{tt}^{n+1}h_{ss}^{n+1}$$
$$+\sum_{i<j=1}^{n}(h_{ij}^{n+1})^2+\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2. \tag{3.5}$$

Assume that $a_1=h_{11}^{n+1}$, $a_2=\sum_{i=2}^{n_1}h_{ii}^{n+1}$ and $a_3=\sum_{t=n_1+1}^{n}h_{tt}^{n+1}$. Then applying the Lemma 2 in (3.5), it is easily seen that

$$\frac{\delta}{2}+\sum_{i<j=1}^{n}(h_{ij}^{n+1})^2+\frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\le\sum_{2\le j\neq l\le n_1}h_{jj}^{n+1}h_{ll}^{n+1}+\sum_{n_1+1\le t\neq s\le n}h_{tt}^{n+1}h_{ss}^{n+1}. \tag{3.6}$$

The equality holds in (3.6) if and only if

$$\sum_{i=1}^{n_1}h_{ii}^{n+1}=\sum_{t=n_1+1}^{n}h_{tt}^{n+1}. \tag{3.7}$$

On the other hand, from (2.13) and (2.16), we find that

$$\frac{n_2\Delta_1 f_1}{f_1}+\frac{n_1\Delta_2 f_2}{f_2}=\rho-\sum_{1\le j\neq k\le n_1}K(e_i\wedge e_k)-\sum_{n_1+1\le t\neq s\le n}K(e_t\wedge e_s).$$

From (2.5) and (2.10), it follows that

$$\frac{n_2\Delta_1 f_1}{f_1}+\frac{n_1\Delta_2 f_2}{f_2}=\rho-\frac{c-3\vartheta^2}{8}n_1(n_1-1)-\sum_{r=1}^{2m+1}\sum_{2\le j\neq k\le n_1}(h_{jj}^{r}h_{kk}^{r}-(h_{jk}^{r})^2)$$
$$-\frac{c-3\vartheta^2}{8}n_2(n_2-1)-\sum_{r=1}^{2m+1}\sum_{n_1+1\le t\neq s\le n}(h_{tt}^{r}h_{ss}^{r}-(h_{ts}^{r})^2). \tag{3.8}$$

Thus combining (3.6) and (3.8), it is easily seen that

$$\frac{n_2\Delta_1 f_1}{f_1}+\frac{n_1\Delta_2 f_2}{f_2}\le\rho-\frac{c-3\vartheta^2}{8}n(n-1)+\frac{c-3\vartheta^2}{4}n_1n_2-\frac{\delta}{2}. \tag{3.9}$$

Hence, from (3.3), the inequalities (3.9) reduce to

$$\frac{n_2\Delta_1 f_1}{f_1}+\frac{n_1\Delta_2 f_2}{f_2}\le\frac{n^2}{4}\|H\|^2+\frac{c-3\vartheta^2}{4}n_1n_2 \tag{3.10}$$

which is the inequality (3.1). On the other hand, the equality sign holds in (3.10) if and only if from (3.7), we get $n_1H_1=n_2H_2$. Moreover, from (3.6), we find that $h_{ij}^{r}=0$, for each $1\le i\le n_1$, $n_1+1\le j\le n$ and $n+1\le r\le 2m+1$, which means that ϕ is a mixed totally geodesic immersion. The converse part is straightforward. Thus, the proof is complete. □

Now, we have the following applications of Theorem 3.1

Remark 3.1. *If we substitute either $f_1 = 1$ or $f_2 = 1$ in Theorem 3.1, then Theorem 3 turns into $C-$totally real warped product.*

Corollary 3.1. *Let $\widetilde{M}(c)$ be a $(2m+1)-$dimensional locally conformal almost cosymplectic manifold and $\phi : M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real warped product into $\widetilde{M}(c)$ such that c is pointwise constant $\varphi-$sectional curvature. Then*

(i) *The relation between warping function and the squared norm of mean curvature is given by*

$$\frac{n_2 \Delta f}{f} \leq \frac{n^2}{4}||H||^2 + \frac{c - 3\vartheta^2}{4} n_1 n_2,$$

where $n_i = \dim M_i$, $i = 1, 2$ and Δ is the Laplacian operator on M_1.

(ii) *The equality sign holds in the above inequality if and only if ϕ is mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where H_1 and H_2 are the partial mean curvature vectors on M_1 and M_2, respectively.*

Remark 3.2. *If we put either $f_1 = 1$ or $f_2 = 1$ and $\vartheta = 0$ in Theorem 3.1, then it is the same inequality of Theorem 3.2 in [34].*

Remark 3.3. *If we consider either $f_1 = 1$ or $f_2 = 1$ and $\vartheta = 1$ in Theorem 3.1, then the Theorem 3.1 is exactly the Lemma 3.1 of [18].*

Corollary 3.2. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant $\varphi-$sectional curvature such that the warping functions are harmonics. Then, M is not a minimal submanifold of $\widetilde{M}$ with inequality*

$$\vartheta > \sqrt{\frac{c}{3}}.$$

Corollary 3.3. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant $\varphi-$sectional curvature. Suppose that the warping functions f_1 and f_2 of M are eigenfunctions of Laplacian on M_1 and M_2 with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Then M is not a minimal submanifold of $\widetilde{M}$ with inequality*

$$\vartheta \geq \sqrt{\frac{c}{3}}.$$

Corollary 3.4. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant $\varphi-$sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of Laplacian with corresponding eigenvalue $\lambda > 0$. Then M is not minimal in $\widetilde{M}$ with inequality*

$$\vartheta \geq \sqrt{\frac{c}{3}}.$$

Now, motivated by the Chen's paper [9], we establish the following sharp relationship for the squared norm of the mean curvature vector in terms of intrinsic invariants.

Theorem 3.2. *Let $\widetilde{M}(c)$ be a $(2m+1)-$dimensional locally conformal almost cosymplectic manifold and $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real doubly warped product into $\widetilde{M}(c)$ such that c is pointwise constant $\varphi-$sectional curvature. Then*

(i)

$$\Big(\frac{\Delta_1 f_1}{n_1 f_1}\Big)+\Big(\frac{\Delta_2 f_2}{n_2 f_2}\Big)\geq \rho-\frac{n^2(n-2)}{2(n-1)}\|H\|^2-\Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2), \tag{3.11}$$

where $n_i=\dim M_i$, $i=1,2$ and Δ^i is the Laplacian operator on M_i, $i=1,2$.

(ii) If the equality sign holds in (3.11), then equality sign in (3.23) holds automatically.

(iii) If $n=2$, then equality sign in (3.11) holds identically.

Proof. Let us consider that ${}_{f_2}M_1\times_f M_2$ be a $C-$totally real doubly warped product in a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with pointwise constant $\varphi-$scetional curvature c. Then from Gauss equation, we find

$$2\rho=\Big(\frac{c-3\vartheta^2}{4}\Big)n(n-1)+n^2\|H\|^2-\|h\|^2. \tag{3.12}$$

Now, we consider that

$$\delta=2\rho-\Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2)-\frac{n^2(n-2)}{n-1}\|H\|^2. \tag{3.13}$$

Then from (3.12) and (3.13), it follows that

$$n^2\|H\|^2=(n-1)\left[\|h\|^2+\delta-\Big(\frac{c-3\vartheta^2}{2}\Big)\right]. \tag{3.14}$$

Let $\{e_1,e_2,\cdots,e_n\}$ be an orthonormal frame, the above equation takes the following form

$$\left[\sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}h^r_{ii}\right]^2=(n-1)\left[\delta+\sum_{r=n+1}^{2m+1}\sum_{i=1}^{n}(h^r_{ii})^2+\sum_{r=n+1}^{2m+1}\sum_{i<j=1}^{n}(h^r_{ij})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h^r_{ij})^2-\Big(\frac{c-3\vartheta^2}{2}\Big)\right],$$

which implies that

$$\left[h^{n+1}_{11}+\sum_{i=2}^{n_1}h^{n+1}_{ii}+\sum_{t=n_1+1}^{n}h^{n+1}_{tt}\right]^2=\delta+(h^{n+1}_{11})^2+\sum_{i=2}^{n_1}(h^{n+1}_{ii})^2+\sum_{t=n_1+1}^{n}(h^{n+1}_{tt})^2-\sum_{2\leq j\neq l\leq n_1}h^{n+1}_{jj}h^{n+1}_{ll}$$

$$-\sum_{n_1+1\leq t\neq s\leq n}h^{n+1}_{tt}h^{n+1}_{ss}+\sum_{i<j=1}^{n}(h^{n+1}_{ij})^2+\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h^r_{ij})^2-\Big(\frac{c-3\vartheta^2}{2}\Big). \tag{3.15}$$

Let us consider that $a_1=h^{n+1}_{11}$, $a_2=\sum_{i=2}^{n_1}h^{n+1}_{ii}$ and $a_3=\sum_{t=n_1+1}^{n}h^{n+1}_{tt}$. Then from Lemma 2.1 and equation (3.15), we get

$$\frac{\delta}{2}-\Big(\frac{c-3\vartheta^2}{2}\Big)+\sum_{i<j=1}^{n}(h^{n+1}_{ij})^2+\frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}(h^r_{ij})^2\leq\sum_{2\leq j\neq l\leq n_1}h^{n+1}_{jj}h^{n+1}_{ll}+\sum_{n_1+1\leq t\neq s\leq n}h^{n+1}_{tt}h^{n+1}_{ss}. \tag{3.16}$$

with equality holds in (3.16) if and only if

$$\sum_{i=1}^{n_1}h^{n+1}_{ii}=\sum_{t=n_1+1}^{n}h^{n+1}_{tt}. \tag{3.17}$$

On the other hand, from(3.16) and (2.13), we have

$$K(e_1\wedge e_{n_1+1})\geq\sum_{r=n+1}^{2m+1}\sum_{j\in P_{1n_1+1}}(h^r_{1j})^2+\frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{j\in P_{1n_1+1}}^{i\neq j}(h^r_{ij})^2+\sum_{r=n+1}^{2m+1}\sum_{j\in P_{1n_1+1}}(h^r_{n_1+1j})^2$$

$$+\frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{i,j\in P_{1n_1+1}}(h_{ij}^r)^2+\frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n_1+1}(h_{ij}^r)^2+\frac{\delta}{2},$$

where $P_{1n_1+1} = \{1, \cdots, n\} - \{1, n_1 + 1\}$. Thus, it implies that

$$K(e_1 \wedge e_{n_1+1}) \geq \frac{\delta}{2}, \tag{3.18}$$

Since, $M =_{f_2} M_1 \times_{f_1} M_2$ is a $C-$totally real doubly warped product submanifold, we have $\nabla_X Z = \nabla_Z X = (X \ln f_1)Z + (Z \ln f_2)X$, for any unit vector fields X and Z tangent to M_1 and M_2, respectively. Then from (2.15), (3.13) and (3.18), the scalar curvature derives as;

$$\rho \leq \frac{1}{f_1}\{(\nabla_{e_1}e_1)f_1 - e_1^2 f_1\} + \frac{1}{f_2}\{(\nabla_{e_2}e_2)f_2 - e_2^2 f_2\} + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2). \tag{3.19}$$

Let the equality holds in (3.19), then all leaving terms in (3.16) and (3.18), we obtain the follwing conditions, i.e.,

$$h_{1j}^r = 0,\ \ h_{jn_1+1}^r = 0,\ \ h_{ij}^r = 0,\ \textit{where } i \neq j,\ \textit{and } r \in \{n+1, \cdots, 2m+1\};$$

$$h_{1j}^r = h_{jn_1+1}^r = h_{ij}^r = 0,\ \ \textit{and } h_{11}^r + h_{n_1+1n_1+1}^r = 0,\ \ i, j \in P_{1n_1+1},\ r = n+2, \cdots, 2m+1. \tag{3.20}$$

Similarly, we extend the relation (3.19) as follows

$$\rho \leq \frac{1}{f_1}\{(\nabla_{e_\alpha}e_\alpha)f_1 - e_\alpha^2 f_1\} + \frac{1}{f_2}\{(\nabla_{e_\beta}e_\beta)f_2 - e_\beta^2 f_2\} + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2), \tag{3.21}$$

for any $\alpha = 1, \cdots, n_1$ *and* $\beta = n_1 + 1, \cdots n$. Taking the summing up α from 1 to n_1 and up β from $n_1 + 1$ to n_2 repectively, we arrive at

$$n_1.n_2.\rho \leq \frac{n_2.\Delta_1 f_1}{f_1} + \frac{n_1.\Delta_2 f_2}{f_2} + \frac{n^2.n_1.n_2(n-2)}{2(n-1)}\|H\|^2 + \Big(\frac{c-3\vartheta^2}{4}\Big)n_1.n_2(n+1)(n-2). \tag{3.22}$$

Similarly, the equality sign holds in (3.22) identically. Thus the equality sign in (3.19) holds for each $\alpha \in \{1, \cdots, n_1\}$ and $\beta \in \{n_1 + 1, \cdots, n\}$. Then we get the following;

$$h_{\alpha j}^r = 0,\ \ h_{ij}^r = 0,\ \ h_{ij}^r = 0,\ \textit{where } i \neq j,\ \textit{and } r \in \{n+1, \cdots, 2m+1\};$$

$$h_{\alpha j}^r = h_{ij}^r = h_{ij}^r = 0,\ \ \textit{and } h_{\alpha\alpha}^r + h_{\beta\beta}^r = 0,\ \ i, j \in P_{1n_1+1},\ r = n+2, \cdots, 2m+1. \tag{3.23}$$

Moreover, If $n = 2$. Then $n_1 = n_2 = 1$. thus from (2.15), we get $\rho = \Delta_1 f_1 + \Delta_2 f_2$. Hence the equality in (3.11) holds, which proves the theorem completely. □

Now, we also have the following applications of Theorem 3.2.

Remark 3.4. *If either $f_1 = 1$ or $f_2 = 1$ in Theorem 3.2, then we get following corollary for a $C-$totally real warped product.*

Corollary 3.5. *Let $\widetilde{M}(c)$ be a $(2m+1)-$dimensional locally conformal almost cosymplectic manifold and $\phi : M = M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional $C-$totally real warped product into $\widetilde{M}(c)$ such that c is pointwise constant $\varphi-$sectional curvature. Then*

(i)

$$\Big(\frac{\Delta f}{n_1 f}\Big) \geq \rho - \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2), \tag{3.24}$$

where $n_i = \dim M_i$, $i = 1, 2$ and Δ is the Laplacian operator on M_1.

(ii) If the equality sign holds in (3.24), then equality sign in (3.23) holds automatically.
(iii) If $n = 2$, then equality sign in (3.11) holds identically.

We, also have the following special cases of Theorem 3.2.

Corollary 3.6. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an n-dimensional C−totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant φ−sectional curvature such that the warping functions are harmonics. Then M is not minimal in $\widetilde{M}$ with inequality*

$$c > \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2.$$

Corollary 3.7. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an n-diminesional C−totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant φ−sectional curvature. Suppose that the warping functions f_1 and f_2 of M are the eigenfunctions of Laplacians on M_1 and M_2 with corresponding eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, respectively. Then M is not minimal in $\widetilde{M}$ with inequality*

$$c \geq \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2.$$

Corollary 3.8. *Let $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an n-dimensional C−totally real doubly warped product into a locally conformal almost cosympelctic manifold $\widetilde{M}(c)$ with c, a pointwise constant φ−sectional curvature. Suppose that one of the warping function is harmonic and other one is eigenfunction of the Laplacian with corresponding eigenvalue $\lambda > 0$. Then M is not minimal in $\widetilde{M}$ with inequality*

$$c \geq \frac{4\rho}{(n+1)(n-2)} + 3\vartheta^2.$$

If we combine both Theorem 3.1 and Theorem 3.2, then we get the following result.

Theorem 3.3. *Assume that $\phi : M =_{f_2} M_1 \times_{f_1} M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional C−totally real doubly warped product into a $(2m + 1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ such that c is pointwise constant φ−sectional curvature. Then*

$$\rho - \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2) \leq \Big(\frac{\Delta_1 f_1}{n_1 f_1}\Big) + \Big(\frac{\Delta_2 f_2}{n_2 f_2}\Big) \leq \frac{n^2}{4n_1 n_2}\|H\|^2 + \Big(\frac{c-3\vartheta^2}{4}\Big).$$

Remark 3.5. *Theorem 3.3 represent an upper and lower bounds for warping functions of a C−totally real doubly warped product.*

Remark 3.6. *If either $f_1 = 1$ or $f_2 = 1$ in Theorem 3.3, then we get following corollary for a C−totally real warped product.*

Corollary 3.9. *Assume that $\phi : M = M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of an n-diminesional C−totally real doubly warped product into a $(2m + 1)$−dimensional locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ such that c is pointwise constant φ−sectional curvature. Then*

$$\rho - \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \Big(\frac{c-3\vartheta^2}{4}\Big)(n+1)(n-2) \leq \Big(\frac{\Delta f}{n_1 f}\Big) \leq \frac{n^2}{4n_1 n_2}\|H\|^2 + \Big(\frac{c-3\vartheta^2}{4}\Big).$$

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