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Color Hypergroup and Join Space Obtained by the Vertex Coloring of a Graph

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Abstract. In this paper, some new classes of hyperoperations, extracting from the vertex coloring of a graph, are presented. By these hyperoperations, we define a color hypergroup and a color join space on the vertex set of a graph. Also we give some examples to clarify these structures. Finally, we investigate the connection between the color join space and Graph Theory.

1. Introduction

Hyperstructure Theory was born in 1934, when Marty, at the 8th Congress of Scandinavian Mathematicians, gave the definition of a hypergroup. After that the join spaces introduced by Prenowitz in the 40's and were utilized by him to reconstruct several kinds of geometries. Nowadays we know that the join spaces have many applications in different fields of mathematics including Graph Theory. The correlation between hyperstructures and Graph Theory also has been investigated by several researches, such as Ashrafi [1], Corsini [3], Davvaz [7], Kalampakas [8-10], Leoreanu [11], Rosenberg [12, 13], Spartalis [14], and so on.

The purpose of the present paper is to construct a new kind of commutative hypergroups and join spaces by considering the concept of vertex coloring of a graph.

In section 2, basic concepts of hyperstructures and graphs are presented. In section 3, first, by taking idea from the vertex coloring of a graph $G = (V, E)$, we define a new hyperoperation $'' *_{\chi} ''$ on $V(G)$. Then by this hyperoperation we could construct a commutative hypergroup $(V, *_{\chi})$ on the vertex set of a graph G. We call $(V, *_{\chi})$ a color hypergroup. Also by an example we show that the color hypergroup $(V, *_{\chi})$ is not a join space. In the next section, we define a hyperoperation $'' *_{\chi'} ''$ on $V(G)$ and we prove that $(V, *_{\chi'})$ is a commutative hypergroup. Moreover, we show that this hypergroup is a join space and we call it, a color join space. Then we give an example to clarify this structure. Finally, we investigate the correlation between subhypergroups of $(V, *_{\chi'})$ and subgraphs of graph G.

2. Preliminaries

Let H be a non-empty set and $o : H \times H \longrightarrow \mathcal{P}^*(H)$ be a hyperoperation, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple (H, o) is called a hypergroupoid.

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Definition 2.1 (6). *A hypergroupoid* (H, o) *is called a semihypergroup if for all* a, b, c *of* H *we have* $(aob)oc = ao(boc)$*, which means that*

$$\bigcup_{u \in aob} uoc = \bigcup_{v \in boc} aov\cdot$$

Also (H, o) *is called a quasihypergroup if for all* a *of* H *we have:*

$$aoH = Hoa = H$$

Definition 2.2 (6). *A hypergroupoid* (H, o) *which is both a semihypergroup and a quasihypergroup is called a hypergroup. Furthermore, a hypergroup* (H, o) *is called commutative if for all* $a, b \in H$*, it holds* $aob = boa$*. An element* $e \in H$ *is called an identity if for all* $a \in H$*:*

$$a \in eoa \cap aoe\cdot$$

Also, an element x' *is called an inverse of* x *if an identity* e *exists such that*

$$e \in xox' \cap x'ox\cdot$$

Definition 2.3 (6). *A regular hypergroup* H *is a hypergroup which it has at least one identity and every element has at least one inverse.*

Definition 2.4 (6). *A non-empty subset* K *of a hypergroup* (H, o) *is called a subhypergroup if it is a hypergroup.*

In order to define a join space, we recall the following notation:
If a, b are elements of a hypergroup (H, o), then we denote $a/b = \{x \in H \mid a \in xob\}$.

Definition 2.5 (6). *A commutative hypergroup* (H, o) *is called a join space if the following condition holds for all elements* a, b, c, d *of* H*:*

$$a/b \cap c/d \neq \emptyset \implies aod \cap boc \neq \emptyset\cdot$$

In the present paper, we will define a commutative hypergroup and a join space, extracting from a graph, called color hypergroup and color join space. For this we will need the following definitions of a graph:
Formally, a graph G is a pair (V, E) where:
- V is a finite set, the elements of which we call vertices and
- $E \subseteq V \times V$ is a set of pairs of V, the elements of which we call edges.

We draw a graph on paper by placing each vertex at a point and representing each edge by a curve joining the locations of its endpoints.
A graph $G' = (V', E')$ is a subgraph of the graph $G = (V, E)$ if it holds $V' \subseteq V$ and $E' \subseteq E$.

Definition 2.6 (16). *A k-coloring of a graph* G *is a labeling* $f : V(G) \longrightarrow S$*, where* $|S| = k$*. The labels are colors; the vertices of one color form a color class. A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number* $\chi(G)$ *is the least k such that* G *is k-colorable.*

In a proper coloring, each color class is an independent set, so G is k-colorable if and only if $V(G)$ is the union of k independent sets. Note that, we say that a graph G has an optimal coloring if we pay attention to the following points when we want to color the graph G:
- The coloring must be a proper coloring,
- If $\chi(G) = k$, we should use only k colors $(\{1, \cdots, k\})$ to color the graph.

In this paper, when we say a colored graph, we mean that this graph has an optimal coloring.

Remark 2.7 (16). *We know that an optimal coloring of a graph* G *has an edge with endpoints of color* i *and* j *for each pair* i, j *of colors.*

3. Color Hypergroup and Graph Theory

In this section, we construct a color hypergroup, i.e., a hypergroup extracting from a colored graph. To achieve this goal, we need the following definitions:

Definition 3.1. *Let $G = (V, E)$ be a graph with $\mathcal{X}(G) = k$. Then the color class $\bar{i}$ is defined for every $i \in \{1, \cdots, k\}$ by:*

$$\bar{i} = \{u \in V(G) | u \text{ is colored by color } i\}$$

Definition 3.2. *Consider two color classes $\bar{i}$ and $\bar{j}$ of a colored graph G and $i, j \in \{1, \cdots, k\}$. The set $A_{i,j}$ is defined in the following way:*

$$A_{i,j} =$$

$$\{u \in V(G) \ | \ u \text{ is an endpoint of an edge in which, that edge has endpoints of the colors } i \text{ and } j\}$$

By remark 2.7, it is obvious that $A_{i,j} \neq \emptyset$ and also $A_{i,j} = A_{j,i}$ for all $i, j \in \{1, \cdots, k\}$.

Definition 3.3. *Let $G = (V, E)$ be a graph with $\mathcal{X}(G) = k$. Then we define the hyperoperation $*_{\mathcal{X}} : V \times V \to \mathcal{P}^*(V)$ as follows:*
For every $u, w \in V(G)$ and $i, j \in \{1, \cdots, k\}$

$$u *_{\mathcal{X}} w = \begin{cases} A_{i,j} \cup \{u, w\} & \text{if } u \in \bar{i}, w \in \bar{j} \text{ and } i \neq j \\ \{u, w\} & if u, w \in \bar{i} \end{cases}$$

Proposition 3.4. *Let $G = (V, E)$ be a colored graph. Then the hypergroupoid $(V, *_{\mathcal{X}})$ is commutative.*

Proof. By definitions of the hyperoperation $*_{\mathcal{X}}$ and $A_{i,j}$, it is easy to see that the hypergroupoid $(V, *_{\mathcal{X}})$ is commutative. □

Theorem 3.5. *Let $G = (V, E)$ be a colored graph with $\mathcal{X}(G) = k$. Then the hypergroupoid $(V, *_{\mathcal{X}})$ is a semihypergroup.*

Proof. It is enough to check the associativity of " $*_{\mathcal{X}}$ " i.e. $(u *_{\mathcal{X}} w) *_{\mathcal{X}} v = u *_{\mathcal{X}} (w *_{\mathcal{X}} v)$ For all $u, w, v \in V(G)$. To achieve this aim we check the following situations for every $u, w, v \in V(G)$ and $i, j, f \in \{1, \cdots, k\}$:

- $u \in \bar{i}, w \in \bar{j}, v \in \bar{f}, i \neq j \neq f \neq i$

$$(u *_{\mathcal{X}} w) *_{\mathcal{X}} v = (A_{i,j} \cup \{u, w\}) *_{\mathcal{X}} v = A_{i,j} \cup A_{i,f} \cup A_{j,f} \cup \{u, w, v\}$$

$$u *_{\mathcal{X}} (w *_{\mathcal{X}} v) = u *_{\mathcal{X}} (A_{j,f} \cup \{w, v\}) = A_{i,j} \cup A_{i,f} \cup A_{j,f} \cup \{u, w, v\}$$

- $u, w \in \bar{i}, v \in \bar{j}, i \neq j$

$$(u *_{\mathcal{X}} w) *_{\mathcal{X}} v = \{u, w\} *_{\mathcal{X}} v = A_{i,j} \cup \{u, w, v\}$$

$$u *_{\mathcal{X}} (w *_{\mathcal{X}} v) = u *_{\mathcal{X}} (A_{i,j} \cup \{w, v\}) = A_{i,j} \cup \{u, w, v\}$$

- $u, v \in \bar{i}, w \in \bar{j}, i \neq j$

$$(u *_{\mathcal{X}} w) *_{\mathcal{X}} v = (A_{i,j} \cup \{u, w\}) *_{\mathcal{X}} v = A_{i,j} \cup \{u, w, v\}$$

$$u *_{\mathcal{X}} (w *_{\mathcal{X}} v) = u *_{\mathcal{X}} (A_{i,j} \cup \{w, v\}) = A_{i,j} \cup \{u, w, v\}$$

- $w, v \in \bar{i}, u \in \bar{j}, i \neq j$

$$(u *_{\mathcal{X}} w) *_{\mathcal{X}} v = (A_{i,j} \cup \{u, w\}) *_{\mathcal{X}} v = A_{i,j} \cup \{u, w, v\}$$

$$u *_{\mathcal{X}} (w *_{\mathcal{X}} v) = u *_{\mathcal{X}} (\{w, v\}) = A_{i,j} \cup \{u, w, v\}$$

- $u, w, v \in \bar{i}$

$$(u *_{\chi} w) *_{\chi} v = \{u, w, v\} = u *_{\chi} (w *_{\chi} v)$$

So, we conclude that the hypergroupoid $(V, *_{\chi})$ is a semihypergroup. □

Theorem 3.6. *Let $G = (V, E)$ be a colored graph. Then the semihypergroup $(V, *_{\chi})$ is a commutative hypergroup.*

Proof. By prorposition 3.4, we obtain that the semihypergroup $(V, *_{\chi})$ is commutative. Now, it is sufficient to prove that $u *_{\chi} V = V *_{\chi} u = V$ for all $u \in V(G)$. First we show that $u *_{\chi} V = V$ for all $u \in V(G)$.
By definition of hyperoperation $*_{\chi}$, we have $w \in u *_{\chi} w$, for all $u, w \in V(G)$. So $V \subseteq u *_{\chi} V$. Also by definition of hyperoperation in general, we know that $u *_{\chi} V \subseteq V$. Hence, we conclude that $u *_{\chi} V = V$ for all $u \in V(G)$. Since $*_{\chi}$ is commutative, thus $V *_{\chi} u = V$, so that $(V, *_{\chi})$ is a hypergroup. □

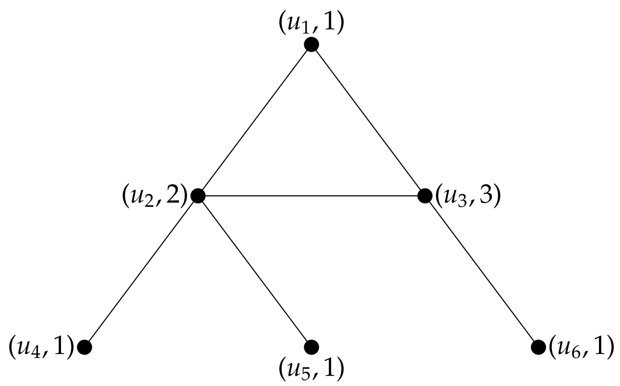
We call this hypergroup a color hypergroup.

Theorem 3.7. *Let $G = (V, E)$ be a colored graph with $\mathcal{X}(G) = k$. Then the hypergroup $(V, *_{\chi})$ is a regular hypergroup.*

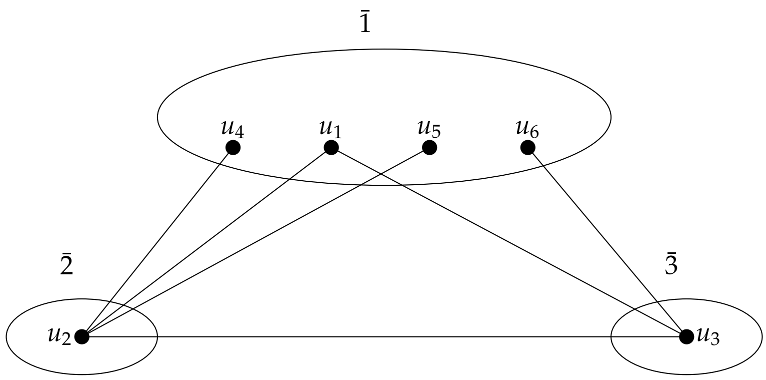
Proof. Assume that e be a vertex of $V(G)$. By definition of hyperoperation $*_{\chi}$, we have $u \in e *_{\chi} u \cap u *_{\chi} e$, for every $u \in V(G)$. So, every vertex of $V(G)$ could be an identity.
Also consider $e \in V(G)$ be an idenity of hypergroup $(V, *_{\chi})$. By definition of hyperoperation $*_{\chi}$, it holds $e \in e *_{\chi} u \cap u *_{\chi} e$ for every $u \in V(G)$. So, the inverse of every vertex of $V(G)$ could be the vertex e. So, we show that the hypergroup $(V, *_{\chi})$ has an identity and every element has at least one inverse. Therefore $(V, *_{\chi})$ is a regular hypergroup. □

From the following example, we obtain that $(V, *_{\chi})$ is not a join space.

Example 3.8. *Consider the following colored graph G, with $\mathcal{X}(G) = 3$. For every (u_j, i), $j \in \{1, \cdots, 6\}$ and $i \in \{1, 2, 3\}$, u_j is the vertex of the graph G and i is the color of this vertex.*



To show the color classes, we draw the graph according to the color of each vertex as follows:



Now by definition of hyperoperation " $*_{\mathcal{X}}$ ", we have the following table:

$*_{\mathcal{X}}$	u_1	u_2	u_3	u_4	u_5	u_6
u_1	$\{u_1\}$	$\{u_1,u_2,u_4,u_5\}$	$\{u_1,u_3,u_6\}$	$\{u_1,u_4\}$	$\{u_1,u_5\}$	$\{u_1,u_6\}$
u_2	$\{u_1,u_2,u_4,u_5\}$	$\{u_2\}$	$\{u_2,u_3\}$	$\{u_1,u_2,u_4,u_5\}$	$\{u_1,u_2,u_4,u_5\}$	$\{u_1,u_2,u_4,u_5,u_6\}$
u_3	$\{u_1,u_3,u_6\}$	$\{u_2,u_3\}$	$\{u_3\}$	$\{u_1,u_3,u_4,u_6\}$	$\{u_1,u_3,u_5,u_6\}$	$\{u_1,u_3,u_6\}$
u_4	$\{u_1,u_4\}$	$\{u_1,u_2,u_4,u_5\}$	$\{u_1,u_3,u_4,u_6\}$	$\{u_4\}$	$\{u_4,u_5\}$	$\{u_4,u_6\}$
u_5	$\{u_1,u_5\}$	$\{u_1,u_2,u_4,u_5\}$	$\{u_1,u_3,u_5,u_6\}$	$\{u_4,u_5\}$	$\{u_5\}$	$\{u_5,u_6\}$
u_6	$\{u_1,u_6\}$	$\{u_1,u_2,u_4,u_5,u_6\}$	$\{u_1,u_3,u_6\}$	$\{u_4,u_6\}$	$\{u_5,u_6\}$	$\{u_6\}$

According to the above table, we get that

$$\begin{array}{l} u_1 \in u_2 *_{\mathcal{X}} u_4 \\ \\ u_5 \in u_2 *_{\mathcal{X}} u_6 \end{array} \Longrightarrow \quad u_2 \in u_1/u_4 \cap u_5/u_6$$

Also, it is easy to see that $u_1 *_{\mathcal{X}} u_6 \cap u_4 *_{\mathcal{X}} u_5 = \{u_1, u_6\} \cap \{u_4, u_5\} = \emptyset$. So $(V(G), *_{\mathcal{X}})$ is not a join space.

4. Color Join Space and Graph Theory

In this section, we want to construct a join space which is extracted from a colored graph. To achieve this aim, let $G = (V, E)$ be a graph with $\mathcal{X}(G) = k$. First, we define the hyperoperation $*_{\mathcal{X}'} : V \times V \longrightarrow \mathcal{P}^*(V)$ as follows:
For every $u, w \in V(G)$ and $i, j \in \{1, \cdots, k\}$

$$u *_{\mathcal{X}'} w = \begin{cases} A_{i,j} \cup \{u, w\} & \text{if } u \in \bar{i}, w \in \bar{j} \text{ and } i \neq j \\ \bar{i} & if u, w \in \bar{i} \end{cases}$$

Proposition 4.1. *Let $G = (V, E)$ be a colored graph. Then the hypergroupoid $(V, *_{\mathcal{X}'})$ is commutative.*

Proof. By definitions of the hyperoperation $*_{\mathcal{X}'}$ and $A_{i,j}$, it is easy to see that the hypergroupoid $(V, *_{\mathcal{X}'})$ is commutative. □

Theorem 4.2. *Let $G = (V, E)$ be a colored graph with $\mathcal{X}(G) = k$. Then the hypergroupoid $(V, *_{\mathcal{X}'})$ is a semihypergroup.*

Proof. It is enough to check the associativity of " $*_{\mathcal{X}'}$ ", i.e. $(u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v = u *_{\mathcal{X}'} (w *_{\mathcal{X}'} v)$ For all $u, w, v \in V(G)$. To achieve this aim we check the following situations for every $u, w, v \in V(G)$ and $i, j, f \in \{1, \cdots, k\}$:

- $u \in \bar{i}, w \in \bar{j}, v \in \bar{f}, i \neq j \neq f \neq i$

$$(u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v = (A_{i,j} \cup \{u, w\}) *_{\mathcal{X}'} v = A_{i,j} \cup A_{i,f} \cup A_{j,f} \cup \{u, w, v\}$$

$$u *_{\mathcal{X}'} (w *_{\mathcal{X}'} v) = u *_{\mathcal{X}'} (A_{j,f} \cup \{w, v\}) = A_{i,j} \cup A_{i,f} \cup A_{j,f} \cup \{u, w, v\}$$

- $u, w \in \bar{i}, v \in \bar{j}, i \neq j$

$$(u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v = \bar{i} *_{\mathcal{X}'} v = \bar{i} \cup A_{i,j} \cup \{v\}$$

$$u *_{\mathcal{X}'} (w *_{\mathcal{X}'} v) = u *_{\mathcal{X}'} (A_{i,j} \cup \{w, v\}) = \bar{i} \cup A_{i,j} \cup \{v\}$$

- $u, v \in \bar{i}, w \in \bar{j}, i \neq j$

$$(u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v = (A_{i,j} \cup \{u, w\}) *_{\mathcal{X}'} v = \bar{i} \cup A_{i,j} \cup \{w\}$$

$$u *_{\mathcal{X}'} (w *_{\mathcal{X}'} v) = u *_{\mathcal{X}'} (A_{i,j} \cup \{w, v\}) = \bar{i} \cup A_{i,j} \cup \{w\}$$

- $w, v \in \bar{i}, u \in \bar{j}, i \neq j$

$$(u *_{\chi'} w) *_{\chi'} v = (A_{i,j} \cup \{u, w\}) *_{\chi'} v = \bar{i} \cup A_{i,j} \cup \{u\}$$

$$u *_{\chi'} (w *_{\chi'} v) = u *_{\chi'} \bar{i} = \bar{i} \cup A_{i,j} \cup \{u\}$$

- $u, w, v \in \bar{i}$

$$(u *_{\chi'} w) *_{\chi'} v = \bar{i} = u *_{\chi'} (w *_{\chi'} v)$$

So, we obtain that the hypergroupoid $(V, *_{\chi'})$ is a semihypergroup. □

The proofs of the next Theorems are similar to those of Theorems 3.6 and 3.7 respectively.

Theorem 4.3. *Let $G = (V, E)$ be a colored graph. Then the semihypergroup $(V, *_{\chi'})$ is a commutative hypergroup.*

Theorem 4.4. *Let $G = (V, E)$ be a colored graph. Then the hypergroup $(V, *_{\chi'})$ is a regular hypergroup.*

The most important information on the hypergroup $(V, *_{\chi'})$ is obtained from the following result:

Theorem 4.5. *Let $G = (V, E)$ be a colored graph with $\chi(G) = k$. Then $(V, *_{\chi'})$ is a join space.*

Proof. By theorem 4.3, we found that $(V, *_{\chi'})$ is a commutative hypergroup. According to the definition of join space, we must prove

$$u_1/u_2 \cap u_3/u_4 \neq \emptyset \Longrightarrow u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3 \neq \emptyset,$$

for every $u_1, u_2, u_3, u_4 \in V(G)$.
Since $u_1/u_2 \cap u_3/u_4 \neq \emptyset$, we assume that $t \in u_1/u_2 \cap u_3/u_4$ and $t \in V(G)$. So, we get that

$$u_1 \in t *_{\chi'} u_2 \text{ and } u_3 \in t *_{\chi'} u_4$$

In this proof, we show the color class of u_i by $\overline{u}_i$ for $i \in \{1, \cdots, 4\}$ and the color class of t by $\bar{t}$. First we consider the following situations:

- $u_1 = u_2$
  By definition of hyperoperation $*_{\chi'}$, we have $u, w \in u *_{\chi'} w$ for all $u, w \in V(G)$. So, we get that

$$u_1 = u_2 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_1 = u_3$
  In this case, the proof is similar to the previous situation. So, we have

$$u_1 = u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_1 = u_4$
  We know that $u_1 \in t *_{\chi'} u_2$, so by definition of hyperoperation $*_{\chi'}$ we have

$$\text{either } u_1 \in \bar{t} \text{ or } u_1 \in \overline{u}_2$$

■ Assume that $u_1 \in \overline{u_2}$
Since $u_1 = u_4$ we have $u_1 \in u_1 *_{\chi'} u_4 = \overline{u_1}$. Also, we know that $u_1 \in \overline{u_2}$ so $\overline{u_1} = \overline{u_2}$ and $u_2 \in \overline{u_1}$. Thus, we get that

$$u_2 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- Assume that $u_1 \in \bar{t}$
  We know that $u_3 \in t *_{\chi'} u_4$, hence it holds

$$\text{either } u_3 \in \bar{t} \text{ or } u_3 \in \overline{u_4}.$$

If $u_3 \in \bar{t}$, then by considering $u_1 \in \bar{t}$ and $u_1 *_{\chi'} u_4 = \overline{u_1}$ we have $\overline{u_1} = \bar{t} = \overline{u_3}$ and

$$u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

If $u_3 \in \overline{u_4}$, then by considering $u_1 = u_4$ we have $\overline{u_1} = \overline{u_3} = \overline{u_4}$ and

$$u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_2 = u_4$
  By definition of hyperoperation $*_{\chi'}$, we have $u, w \in u *_{\chi'} w$ for all $u, w \in V(G)$. So, we have

$$u_2 = u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_3 = u_4$
  The proof is similar to the previous situation. Thus, we have

$$u_3 = u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_2 = u_3$
  We know that $u_1 \in t *_{\chi'} u_2$, so by definition of hyperoperation $*_{\chi'}$ we have

$$\text{either } u_1 \in \bar{t} \text{ or } u_1 \in \overline{u_2}$$

- Assume that $u_1 \in \overline{u_2}$
  Since $u_2 = u_3$ we have $u_2 *_{\chi'} u_3 = \overline{u_2} = \overline{u_3}$. Also, we know that $u_1 \in \overline{u_2}$ and $u_1 \in u_1 *_{\chi'} u_4$. Thus, it holds

$$u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- Assume that $u_1 \in \bar{t}$
  We know that $u_3 \in t *_{\chi'} u_4$, hence, we have

$$\text{either } u_3 \in \bar{t} \text{ or } u_3 \in \overline{u_4}.$$

If $u_3 \in \bar{t}$, then by considering $u_1 \in \bar{t}$ and $u_1 \in u_1 *_{\chi'} u_4$ we have $\overline{u_1} = \bar{t} = \overline{u_3} = \overline{u_2}$ and

$$u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

If $u_3 \in \overline{u_4}$, then by considering $u_2 = u_3$ we have $\overline{u_2} = \overline{u_3} = \overline{u_4}$ and

$$u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

Now we consider the cases in which all vertices (u_1, u_2, u_3 and u_4) are different. Since $u_1 \in t *_{\chi'} u_2$ and $u_3 \in t *_{\chi'} u_4$ we find that

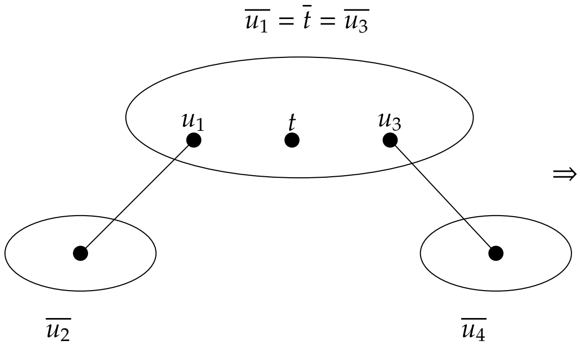
$$\text{either } u_1 \in \bar{t} \text{ or } u_1 \in \overline{u}_2,$$

$$\text{either } u_3 \in \bar{t} \text{ or } u_3 \in \overline{u}_4$$

Therefore, the proof will be completed if we check the following four situations:

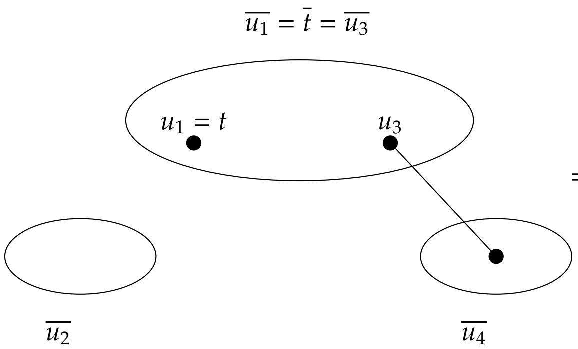
1) $u_1 \in \bar{t}$ and $u_3 \in \bar{t}$
   Note that since $u_1, u_3 \in \bar{t}$ we have $\overline{u_3} = \bar{t} = \overline{u_1}$. To check this situation, we should consider the following subcases:

- $u_1 \neq t \neq u_3$ and $\overline{u_2} \neq \bar{t} \neq \overline{u_4} \neq \overline{u_2}$
  Since $u_1 \in t *_{\chi'} u_2, u_1 \neq t$ and $u_1 \in \bar{t}$ there exists an edge between classes $\bar{t}$ and $\overline{u_2}$ such that one of its end point is u_1 and the other one is in class $\overline{u_2}$. Similarly, since $u_3 \in t *_{\chi'} u_4, u_3 \neq t$ and $u_3 \in \bar{t}$, we can say that there exists also an edge between classes $\bar{t}$ and $\overline{u_4}$ such that one of its end point is u_3 and the other one is in class $\overline{u_4}$. Thus, by definition of hyperoperation $*_{\chi'}$ we have



$\Rightarrow u_3, u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$

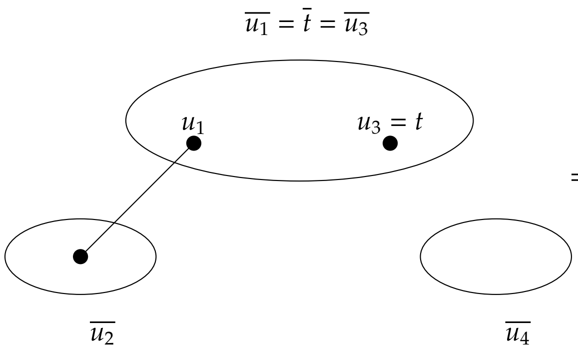
- $u_1 = t \neq u_3$ and $\overline{u_2} \neq \bar{t} \neq \overline{u_4} \neq \overline{u_2}$
  By similar discussion, we have the following shape:



$\Rightarrow u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$

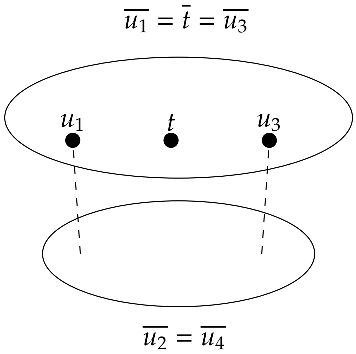
Note that we are not sure there exists an edge between vertex u_1 and class $\overline{u}_2$ or not.

- $u_1 \neq t = u_3$ and $\overline{u}_2 \neq \bar{t} \neq \overline{u_4} \neq \overline{u_2}$
  Similarly, in this case, we have the following shape:



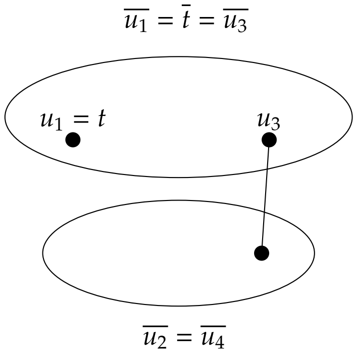
$\Rightarrow u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$

- $u_1 \neq t \neq u_3$ and $\overline{u_2} = \overline{u_4} \neq \bar{t}$
  Since $u_1 \in t *_{\chi'} u_2, u_1 \neq t$ and $u_1 \in \bar{t}$ there exists an edge between classes $\bar{t}$ and $\overline{u_2}$ such that one of its end point is u_1 and the other one is in class $\overline{u_2}$. Similarly, since $u_3 \in t *_{\chi'} u_4, u_3 \neq t$ and $u_3 \in \bar{t}$, we can say that there exists also an edge between classes $\bar{t}$ and $\overline{u_4}$ such that one of its end point is u_3 and the other one is in class $\overline{u_4}$.



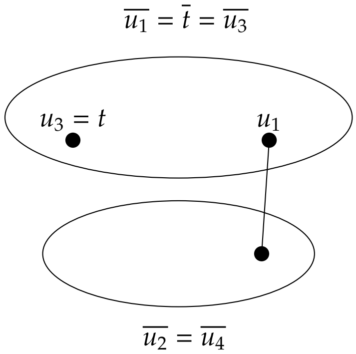
$$\Rightarrow u_1, u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- ■ $u_1 = t \neq u_3$ and $\overline{u}_2 = \overline{u_4} \neq \bar{t}$
  By similar discussion, we have the following shape:



$$\Rightarrow u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- ■ $u_1 \neq t = u_3$ and $\overline{u_2} = \overline{u_4} \neq \bar{t}$
  Similar to the previous case, we have



$$\Rightarrow u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- ■ $\bar{t} = \overline{u_3} = \overline{u_1} = \overline{u_2} \neq \overline{u_4}$
  Since $u_1 \in u_1 *_{\chi'} u_4$ and $u_2 *_{\chi'} u_3 = \overline{u_1}$, we get that

$$u_1 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

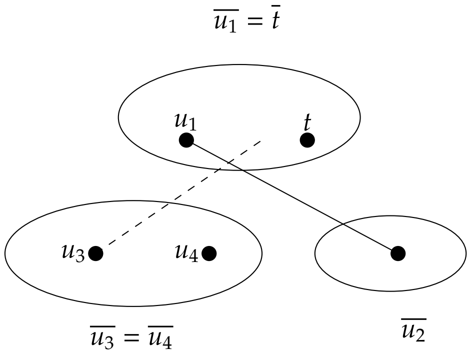
- ■ $\bar{t} = \overline{u_3} = \overline{u_1} = \overline{u_4} \neq \overline{u_2}$
  Since $u_3 \in u_2 *_{\chi'} u_3$ and $u_1 *_{\chi'} u_4 = \overline{u_3}$, we have

$$u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- ■ $\bar{t} = \overline{u_1} = \overline{u_3} = \overline{u_2} = \overline{u_4}$
  It is easy to see that $u_1, u_2, u_3, u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$.
  Checking of the first situation is complete.

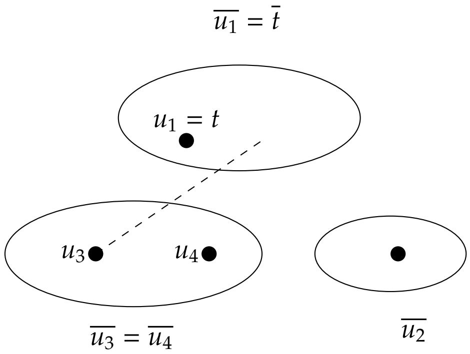
2) $u_1 \in \bar{t}$ and $u_3 \in \overline{u_4}$
Note that since $u_1 \in \bar{t}$ and $u_3 \in \overline{u_4}$ we have $\bar{t} = \overline{u_1}$ and $\overline{u_3} = \overline{u_4}$. To check this situation, we should consider the following subcases:

- $u_1 \neq t$ and $\bar{t} \neq \overline{u_2} \neq \overline{u_4} \neq \bar{t}$
Since $u_1 \in t *_{\chi'} u_2, u_1 \neq t$ and $u_1 \in \bar{t}$ there exists an edge between classes $\bar{t}$ and $\overline{u_2}$ such that one of its end point is u_1 and the other one is in class $\overline{u_2}$. Similarly, since $u_3 \in t *_{\chi'} u_4, u_3 \neq u_4$ and $u_3 \notin \bar{t}$, we find that there exists also an edge between classes $\bar{t}$ and $\overline{u_3}$ such that one of its end point is u_3 and the other one is in class $\bar{t}$. So, we have the following shape:



$$\Rightarrow u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

- $u_1 = t$ and $\bar{t} \neq \overline{u_2} \neq \overline{u_4} \neq \bar{t}$
We can discuss similar to the previous case and we get the following shape:

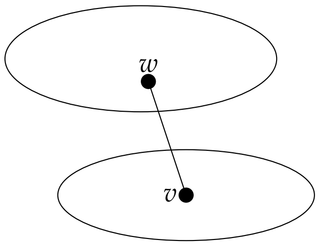


$$\Rightarrow u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

Note that we are not sure there exists an edge between vertex u_1 and class $\overline{u_2}$ or not.

- $\bar{t} \neq \overline{u_4} = \overline{u_2}$
By definition of hyperoperation $*_{\chi'}$, it holds $u_4 \in u_1 *_{\chi'} u_4$. Also, we know that $\overline{u_4} = \overline{u_2} = \overline{u_3} = u_2 *_{\chi'} u_3$. Hence, we get that
$$u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$
- $\bar{t} = \overline{u_4} \neq \overline{u_2}$
Since $\overline{u_1} = \bar{t} = \overline{u_3} = \overline{u_4}$ we find that $\overline{u_3} = \overline{u_1} = u_1 *_{\chi'} u_4$. So $u_3 \in u_1 *_{\chi'} u_4$. Also, we know that $u_3 \in u_2 *_{\chi'} u_3$. Therefore, we have
$$u_3 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$
- $\bar{t} = \overline{u_2} \neq \overline{u_4}$
According to the remark 2.7, since $\overline{u_1} \neq \overline{u_4}$ there exists an edge between class $\overline{u_1}$ and class $\overline{u_4}$. Now assume that two end points of this edge be vertex w and vertex v. So, by definition of hyperoperation $*_{\chi'}$, it is easy to see that $w, v \in u_1 *_{\chi'} u_4$. In addition, we know that $\overline{u_1} = \overline{u_2}$ and $\overline{u_3} = \overline{u_4}$. Hence, $w, v \in u_2 *_{\chi'} u_3$ and it holds
$$w, v \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$
If we want to show the above discussion in a shape, we have the following shape:

$\overline{u_1} = \bar{t} = \overline{u_2}$



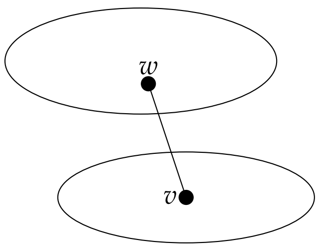
$\overline{u_3} = \overline{u_4}$

- ■ $\bar{t} = \overline{u_2} = \overline{u_4}$
  It is easy to see that $u_1, u_2, u_3, u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$.
  Checking of the second situation is finished.

3) $u_3 \in \bar{t}$ and $u_1 \in \overline{u_2}$
   Checking of this situation is completely similar to the proof of situation 2.
4) $u_1 \in \overline{u_2}$ and $u_3 \in \overline{u_4}$
   Note that since $u_1 \in \overline{u_2}$ and $u_3 \in \overline{u_4}$ we have $\overline{u_1} = \overline{u_2}$ and $\overline{u_3} = \overline{u_4}$. To check this situation, we should consider following subcases:

- ■ $\overline{u_2} \neq \overline{u_4}$
  According to the remark 2.7, since $\overline{u_2} \neq \overline{u_4}$ there exists an edge between class $\overline{u_2}$ and class $\overline{u_4}$. Assume that two end points of this edge be vertex w and vertex v. So, it is easy to see that $w, v \in u_1 *_{\chi'} u_4$. In addition, we know that $\overline{u_1} = \overline{u_2}$ and $\overline{u_3} = \overline{u_4}$. Hence, $w, v \in u_2 *_{\chi'} u_3$ and we have

$\overline{u_1} = \overline{u_2}$



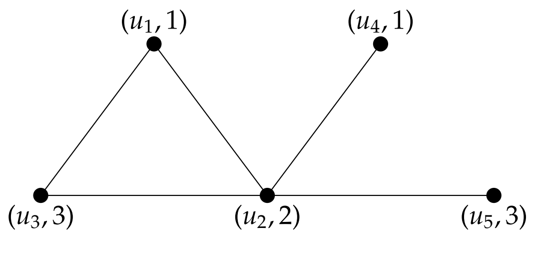
$$\Rightarrow w, v \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$$

$\overline{u_3} = \overline{u_4}$

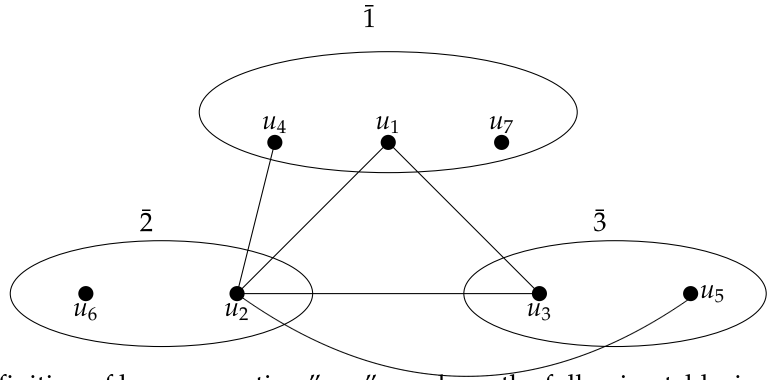
- ■ $\overline{u_2} = \overline{u_4}$
  This case is checked before and $u_1, u_2, u_3, u_4 \in u_1 *_{\chi'} u_4 \cap u_2 *_{\chi'} u_3$.
  Finally, we conclude that $(V, *_{\chi'})$ is a join space.

□

Example 4.6. *Consider the following colored graph G, with $\mathcal{X}(G) = 3$. For every (u_j, i), $j \in \{1, \cdots, 7\}$ and $i \in \{1, 2, 3\}$, u_j is the vertex of the graph G and i is the color of this vertex.*



To show the color classes, we draw the graph according to the color of each vertex as follows:



According to the definition of hyperoperation $'' *_{\mathcal{X}'} ''$, we have the following table, in which it shows a join space structure:

$*_{\mathcal{X}'}$	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	$\{u_1, u_4, u_7\}$	$\{u_1, u_2, u_4\}$	$\{u_1, u_3\}$	$\{u_1, u_4, u_7\}$	$\{u_1, u_3, u_5\}$	$\{u_1, u_2, u_4, u_6\}$	$\{u_1, u_4, u_7\}$
u_2	$\{u_1, u_2, u_4\}$	$\{u_2, u_6\}$	$\{u_2, u_3, u_5\}$	$\{u_1, u_2, u_4\}$	$\{u_2, u_3, u_5\}$	$\{u_2, u_6\}$	$\{u_1, u_2, u_4, u_7\}$
u_3	$\{u_1, u_3\}$	$\{u_2, u_3, u_5\}$	$\{u_3, u_5\}$	$\{u_1, u_3, u_4\}$	$\{u_3, u_5\}$	$\{u_2, u_3, u_5, u_6\}$	$\{u_1, u_3, u_7\}$
u_4	$\{u_1, u_4, u_7\}$	$\{u_1, u_2, u_4\}$	$\{u_1, u_3, u_4\}$	$\{u_1, u_4, u_7\}$	$\{u_1, u_3, u_4, u_5\}$	$\{u_1, u_2, u_4, u_6\}$	$\{u_1, u_4, u_7\}$
u_5	$\{u_1, u_3, u_5\}$	$\{u_2, u_3, u_5\}$	$\{u_3, u_5\}$	$\{u_1, u_3, u_4, u_5\}$	$\{u_3, u_5\}$	$\{u_2, u_3, u_5, u_6\}$	$\{u_1, u_3, u_5, u_7\}$
u_6	$\{u_1, u_2, u_4, u_6\}$	$\{u_2, u_6\}$	$\{u_2, u_3, u_5, u_6\}$	$\{u_1, u_2, u_4, u_6\}$	$\{u_2, u_3, u_5, u_6\}$	$\{u_2, u_6\}$	$\{u_1, u_2, u_4, u_6, u_7\}$
u_7	$\{u_1, u_4, u_7\}$	$\{u_1, u_2, u_4, u_7\}$	$\{u_1, u_3, u_7\}$	$\{u_1, u_4, u_7\}$	$\{u_1, u_3, u_5, u_7\}$	$\{u_1, u_2, u_4, u_6, u_7\}$	$\{u_1, u_4, u_7\}$

The relation of join space $(V, *_{\mathcal{X}'})$ with Graph Theory is illustrated by the following proposition:

Proposition 4.7. *Let $G = (V, E)$ be a colored graph with $\mathcal{X}(G) = k$ and $H \subseteq V(G)$, then $(H, *_{\mathcal{X}'})$ is a subhypergroup of the hypergroup $(V, *_{\mathcal{X}'})$ if and only if H is the union of some color classes of graph G.*

Proof. ($\Longrightarrow$) Let H be a subhypergroup of the hypergroup $(V, *_{\mathcal{X}'}), u \in H$ and u is colored by color i. Also consider that $v \in \bar{i}$. By definition of subhypergroup and hyperoperation $*_{\mathcal{X}'}$ we have:

$$u *_{\mathcal{X}'} H = H \implies u *_{\mathcal{X}'} u = \bar{i} \subseteq H \Longrightarrow v \in H$$

So H should be obtained as the union of some color classes.
($\Longleftarrow$) Assume conversely that H is the union of some color classes. Consume that $H = \overline{i_1} \cup \cdots \cup \overline{i_j}$, for all $m \in \{1, \cdots, j\}$ we have $\overline{i_m} \in \{\bar{1}, \cdots, \bar{k}\}$. Let $u, w, v \in H$ such that $u \in \overline{i_m}, w \in \overline{i_n}, v \in \overline{i_l}$ and $m, n, l \in \{1, \cdots, j\}$. By definition of hyperoperation $*_{\mathcal{X}'}$ we have

$$u *_{\mathcal{X}'} w \subseteq \overline{i_m} \cup \overline{i_n} \subseteq H \implies (u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v \subseteq \overline{i_m} \cup \overline{i_n} \cup \overline{i_l} \subseteq H$$

Since $(V, *_{\mathcal{X}'})$ is a hypergroup, for every $u, w, v \in H$ we get that

$$(u *_{\mathcal{X}'} w) *_{\mathcal{X}'} v = u *_{\mathcal{X}'} (w *_{\mathcal{X}'} v) \subseteq \overline{i_m} \cup \overline{i_n} \cup \overline{i_l} \subseteq H\cdot$$

Therefore $(H, *_{\mathcal{X}'})$ is a semihypergroup.
Now we are going to show that $u *_{\mathcal{X}'} H = H$ for every $u \in H$. By definition of hyperoperation $*_{\mathcal{X}'}$, for every $u, w \in V(G)$ we have

$$u, w \in u *_{\mathcal{X}'} w$$

So, it is obvious that $H \subseteq u *_{\mathcal{X}'} H$, for every $u \in H$. To complete the proof, we must show that for all $u \in H$ it holds $u *_{\mathcal{X}'} H \subseteq H$. Let $u, w \in H$ such that $u \in \overline{i_m}, w \in \overline{i_n}$ and $m, n \in \{1, \cdots, j\}$. By definition of hyperoperation $*_{\mathcal{X}'}$ we get that

$$u *_{\mathcal{X}'} w \subseteq \overline{i_m} \cup \overline{i_n} \subseteq H \Longrightarrow u *_{\mathcal{X}'} H \subseteq H.$$

Thus, $u *_{\mathcal{X}'} H = H$ for every $u \in H$ and H is a subhypergroup. $\square$

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