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Degrees of Separation Properties in Stratified L-Generalized Convergence Spaces Using Residual Implication

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Abstract. By using the residual implication on a frame L, we develop a theory of separation axioms in the category of stratified L-generalized convergence spaces in the spirit of Lowen, i.e., we define for each space some degrees of fulfilling T_0, T_1, T_2 and regularity axioms from a logical aspect. These degrees of separation axioms generalize the theory of separation axioms in the sense of Jäger.

1. Introduction

Separation is certainly one of the most important properties of topological spaces. Usually, separation axioms are defined by separating different points with open sets, or separating point and open set with open sets. As we all know, axiomatic filter convergence spaces (convergence spaces, in short) have more general sense than topological spaces. Convergence spaces not only possess desirable categorical properties, such as Cartesian-closedness, but also have close relations with topological spaces from a categorical aspect [26]. Moreover, separation axioms in topological spaces can also be characterized by its induced convergence structures. Actually, for convergence spaces, where axiom schemes based on convergence of filters are used, we can use the nice characterization of separation axioms by filter convergence as definitions of separation axioms.

With the development of lattice-valued topology, many researchers extended convergence structures to the lattice-valued case and studied its categorical properties and topological properties, such as Jäger [8–10], Fang [2, 3, 28], Yao [29, 30], Li [14, 15], Pang [20–25]. In the situation of stratified L-topology, Min [19] proposed the concept of fuzzy convergence structures (called fuzzy limit structures in [19]) by using fuzzy prefilters and proved that the resulting category is Cartesian closed and can contain the category of stratified I-topological spaces as a reflective subcategory. Later, Lowen et al. [17, 18] introduced the concept of fuzzy convergence structures by means of fuzzy prime filters. Afterwards, Lee [11–13] defined a kind of fuzzy convergence structures by relaxing the axiomatic conditions of fuzzy convergence structures in the sense of Min and investigated the separation axioms in the resulting fuzzy convergence spaces. In 1997, Jäger gave a new definition of fuzzy convergence structures [6] and discussed separation axioms in the

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corresponding fuzzy convergence spaces [7]. Based on stratified L-filters, Jäger [8] proposed a new kind of fuzzy convergence structures, which is called stratified L-generalized convergence structures (also called lattice-valued convergence structures in [9]). Later, in [10], Jäger introduced T_1, T_2 and regularity axioms in stratified L-generalized convergence spaces and showed T_1 and T_2 axioms in this case are compatible with those in stratified L-topological spaces [5].

This paper goes in a more general direction and is in the spirit of Lowen [16], i.e., we do not only ask if a stratified L-generalized convergence space satisfies T_1, T_2 and regularity axioms or not, but we measure the degree to which it fulfils these properties by using the residual implication on the lattice background. These separation degrees for stratified L-generalized convergence space cover as special case the separation axioms in the sense of Jäger [10] from a logical aspect.

2. Preliminaries

We consider in this paper complete lattice L where finite meets distributive over arbitrary joins, i.e., $a \wedge \bigvee_{j\in J} b_j = \bigvee_{j\in J}(a \wedge b_j)$ holds for all $a,\ b_j\ (j \in J)$. These lattices are called *frames* (or *complete Heyting algebras*). The smallest element and the largest element in L are denoted by $\bot$ and $\top$, respectively. We can then define a *residual implication* by

$$a \to b = \bigvee \{c \in L : a \wedge c \leqslant b\}.$$

We will often use, without explicitly mentioning, the following properties of the residual implication.

Lemma 2.1. ([5]) *Let L be a complete Heyting algebra. The following statements hold:*
(H1) $\top \to a = a$.
(H2) $a \leqslant b$ *if and only if* $a \to b = \top$.
(H3) $(a \to b) \to b \geqslant a$.
(H4) $(a \wedge b) \to (a \wedge c) \geqslant b \to c$.
(H5) $(a \to b) \to (a \to c) \geqslant b \to c$.
(H6) $a \to \bigwedge_{j\in J} a_j = \bigwedge_{j\in J}(a \to a_j)$, *hence* $a \to b \leqslant a \to c$ *whenever* $b \leqslant c$.
(H7) $\bigvee_{j\in J} a_j \to b = \bigwedge_{j\in J}(a_j \to b)$, *hence* $a \to c \geqslant b \to c$ *whenever* $a \leqslant b$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. The smallest element and the largest element in L^X are denoted by $\underline{\bot}$ and $\underline{\top}$, respectively. For each $a \in L$, $\underline{a}$ denotes the constant map $X \longrightarrow L$, $x \longmapsto a$.

Definition 2.2. ([1, 2])The map $\mathcal{S}(-,-) : L^X \times L^X \longrightarrow L$ defined by

$$\forall C,\ D \in L^X,\ \ \mathcal{S}(C, D) = \bigwedge_{x\in X} \Big(C(x) \to D(x)\Big)$$

is called the fuzzy inclusion order of L-subsets.

Definition 2.3. ([4])A map $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies
(F1) $\mathcal{F}(\underline{\bot}) = \bot, \mathcal{F}(\underline{\top}) = \top$;
(F2) $A \leqslant B \Longrightarrow \mathcal{F}(A) \leqslant \mathcal{F}(B)$;
(F3) $\mathcal{F}(A \wedge B) \geqslant \mathcal{F}(A) \wedge \mathcal{F}(B)$;
(Fs) $a \wedge \mathcal{F}(A) \leqslant \mathcal{F}(\underline{a} \wedge A)$.

The family of all stratified L-filters on X will be denoted by $\mathcal{F}_L^s(X)$. For every $x \in X$, $[x] \in \mathcal{F}_L^s(X)$ is defined by $[x](A) = A(x)$ for all $A \in L^X$.

Let $f : X \longrightarrow Y$ be a map and $\mathcal{F}$ be a stratified L-filter on X. Define $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ (see [27]) by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively. Then the map $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $f^{\Rightarrow}(\mathcal{F})(A) = \mathcal{F}(f^{\leftarrow}(A))$ for $A \in L^Y$, is a stratified L-filter on Y, which is called the image of $\mathcal{F}$ under f (see [5]).

On the set $\mathcal{F}_L^s(X)$ of all stratified L-filters on X, an order $\leqslant$ defined by $\mathcal{F} \leqslant \mathcal{G}$ if and only if $\mathcal{F}(A) \leqslant \mathcal{G}(A)$ for all $A \in L^X$, was introduced in [5]. For a nonempty family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of stratified L-filters, the infimum $\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda$ is given by $(\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda)(A) = \bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda(A)$ for all $A \in L^X$. In order to guarantee the least upper bound for a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, Höhle and Šostak presented the following lemma.

Lemma 2.4. ([5]) *For a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of stratified L-filters on X, there exists a stratified L-filter $\mathcal{F}$ such that $\mathcal{F}_\lambda \leqslant \mathcal{F}$ ($\forall\lambda \in \Lambda$), if and only if*

$$\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) = \bot \text{ whenever } A_1 \wedge \cdots \wedge A_n = \underline{\bot},$$

for $n \in \mathbb{N}$, $A_1, \cdots, A_n \in L^X$, $\{\lambda_1, \cdots, \lambda_n\} \subseteq \Lambda$. In the case of existence, the supremum $\bigvee_{\lambda\in\Lambda}\mathcal{F}_\lambda$ of a family $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of stratified L-filters is given by

$$\left(\bigvee_{\lambda\in\Lambda}\mathcal{F}_\lambda\right)(A) = \bigvee_{n\in\mathbb{N}}\bigvee\{\mathcal{F}_{\lambda_1}(A_1) \wedge \cdots \wedge \mathcal{F}_{\lambda_n}(A_n) \mid A_1 \wedge \cdots \wedge A_n \leqslant A\}$$

for all $A \in L^X$.

In [8], Jäger also proved that given a map $f : X \longrightarrow Y$ and a stratified L-filter $\mathcal{F}$ on Y, the map $f^{\Leftarrow}(\mathcal{F}) : L^X \longrightarrow L$ defined by

$$\forall A \in L^X,\ f^{\Leftarrow}(\mathcal{F})(A) = \bigvee_{f^{\leftarrow}(B)\leqslant A}\mathcal{F}(B)$$

is a stratified L-filter if and only if $\mathcal{F}(B) = \bot$ whenever $f^{\leftarrow}(B) = \underline{\bot}$ for all $B \in L^Y$. In the case $f^{\Leftarrow}(\mathcal{F}) \in \mathcal{F}_L^s(X)$, it is called the inverse image of $\mathcal{F}$ under f.

In [8], Jäger proposed that the product $\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda$ of a family of stratified L-filters $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, where for each $\lambda \in \Lambda$, X_λ is a nonempty set and $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$, is defined as follows:

$$\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda := \bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda) \in \mathcal{F}_L^s\left(\prod_{\lambda\in\Lambda} X_\lambda\right),$$

where for each $\lambda \in \Lambda$, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ is the projection map.

Lemma 2.5. ([8]) *Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets, $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow X_\mu$ the projection map, $\mathcal{F}_\lambda \in \mathcal{F}_L^s(X_\lambda)$ ($\forall\lambda \in \Lambda$) and $\mathcal{F} \in \mathcal{F}_L^s(\prod_{\lambda\in\Lambda} X_\lambda)$. Then the following statements hold:*

(1) $\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{F}$.

(2) $p_\mu^{\Rightarrow}(\prod_{\lambda\in\Lambda}\mathcal{F}_\lambda) \geqslant \mathcal{F}_\mu$, $\forall\mu \in \Lambda$.

(3) $p_\mu^{\Rightarrow}\left(\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathcal{F})\right) = p_\mu^{\Rightarrow}(\mathcal{F})$, $\forall\mu \in \Lambda$.

Definition 2.6. ([8])A map $\lim : \mathcal{F}_L^s(X) \longrightarrow L^X$ is called a stratified L-generalized convergence structure on X if it satisfies:

(LGC1) $\forall x \in X$, $\lim[x](x) = \top$;

(LGC2) $\forall\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $\mathcal{F} \leqslant \mathcal{G}$ implies $\lim\mathcal{F} \leqslant \lim\mathcal{G}$.

The pair $(X, \lim)$ is called a stratified L-generalized convergence space.

A map $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between stratified L-generalized convergence spaces is called continuous provided that for all $\mathcal{F} \in \mathcal{F}_L^s(X)$, $x \in X$, $\lim_X\mathcal{F}(x) \leqslant \lim_Y f^{\Rightarrow}(\mathcal{F})(f(x))$.

Definition 2.7. ([8]) Let $(f_\lambda : X \longrightarrow (X_\lambda, \lim_\lambda))_{\lambda\in\Lambda}$ be a source. Then

$$Init(\lim_\lambda)\mathcal{F} = \bigwedge_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(\lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F})) \quad (\mathcal{F} \in \mathcal{F}_L^s(X))$$

is the initial stratified L-generalized convergence structure on X. Especially, if $X = \prod_{\lambda\in\Lambda} X_\lambda$ and $p_\mu : X \longrightarrow X_\mu$ are the projections onto X_μ, then we denote π-$(\lim_\lambda)$, defined by

$$\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall (x_\lambda) \in X, \pi\text{-}(\lim_\lambda)\mathcal{F}((x_\lambda)) = \bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F})(x_\lambda),$$

the product structure and call $(X, \pi\text{-}(\lim_\lambda))$ the product space. In the case $(\iota_A : A \longrightarrow (X, \lim))$ we call the initial construction $(A, \lim|_A)$ a subspace of $(X, \lim)$, where $A \subseteq X$ and $\iota_A : A \longrightarrow X$ is the inclusion map.

3. Degrees of T_0 and T_1

In this section, we define the degrees of T_0 and T_1. Then we study their relations and properties.

Definition 3.1. For a stratified L-generalized convergence space $(X, \lim)$, define the degree $T_0(X, \lim)$ to which $(X, \lim)$ is T_0 as follows:

$$T_0(X, \lim) = \bigwedge_{x\neq y} (\lim[y](x) \to \bot) \vee (\lim[x](y) \to \bot).$$

Definition 3.2. For a stratified L-generalized convergence space $(X, \lim)$, define the degree $T_1(X, \lim)$ to which $(X, \lim)$ is T_1 as follows:

$$T_1(X, \lim) = \bigwedge_{x\neq y} (\lim[y](x) \to \bot) \wedge (\lim[x](y) \to \bot).$$

Remark 3.3. For a stratified L-generalized convergence space $(X, \lim)$, Jäger [10] defined T_1 separation axiom as follows:

(T_1) $\forall x, y \in X : \lim[y](x) = \top$ implies $x = y$.

It is easy to see that the definition of T_1 in the sense of Jäger is exactly the case that $T_1(X, \lim) = \top$ in Definition 3.2.

Obviously, the following theorem holds.

Theorem 3.4. *For a stratified L-generalized convergence space* $(X, \lim)$, $T_1(X, \lim) \leqslant T_0(X, \lim)$.

The degrees of T_0 and T_1 are productive and inherited by subspaces:

Theorem 3.5. *If all* $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ *are stratified L-generalized convergence spaces and if the family of maps* $(f_\lambda : X \longrightarrow X_\lambda)$ *separates points (i.e. for* $x \neq y$ *there is* $\lambda \in \Lambda$ *such that* $f_\lambda(x) \neq f_\lambda(y)$*), then*

(1) $\bigwedge_{\lambda\in\Lambda} T_0(X_\lambda, \lim_\lambda) \leqslant T_0(X, Init(\lim_\lambda))$.

(2) $\bigwedge_{\lambda\in\Lambda} T_1(X_\lambda, \lim_\lambda) \leqslant T_1(X, Init(\lim_\lambda))$.

Proof. The verifications of (1) and (2) are similar. We only prove (1).

Let $m = \bigwedge_{\lambda\in\Lambda} T_0(X_\lambda, \lim_\lambda)$ and $n = T_0(X, Init(\lim_\lambda))$. By Definitions 2.7 and 3.1, we have

$$m = \bigwedge_{\lambda\in\Lambda} \bigwedge_{x_\lambda\neq y_\lambda} (\lim_\lambda[y_\lambda](x_\lambda) \to \bot) \vee (\lim_\lambda[x_\lambda](y_\lambda) \to \bot)$$

and

$$n = \bigwedge_{x\neq y} \left(\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[f_\lambda(y)](f_\lambda(x)) \to \bot \right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[f_\lambda(x)](f_\lambda(y)) \to \bot \right) \right).$$

In order to show $m \leqslant n$, take any $x \neq y$. Then there exists $\lambda_0 \in \Lambda$ such that $f_{\lambda_0}(x) \neq f_{\lambda_0}(y)$. Hence,

$$\begin{aligned} m &\leqslant (\lim_{\lambda_0}[f_{\lambda_0}(y)](f_{\lambda_0}(x)) \to \bot) \vee (\lim_{\lambda_0}[f_{\lambda_0}(x)](f_{\lambda_0}(y)) \to \bot) \\ &\leqslant \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[f_\lambda(y)](f_\lambda(x)) \to \bot \right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[f_\lambda(x)](f_\lambda(y)) \to \bot \right) \end{aligned}$$

By the arbitrariness of x and y, we obtain $m \leqslant n$, as desired. □

Corollary 3.6. *If* $(A, \lim|_A)$ *is a subspace of* $(X, \lim)$*, then* $T_0(X, \lim) \leqslant T_0(A, \lim|_A)$ *and* $T_1(X, \lim) \leqslant T_1(A, \lim|_A)$.

Theorem 3.7. *If all* $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ *are stratified L-generalized convergence spaces, then*
(1) $\bigwedge_{\lambda\in\Lambda} T_0(X_\lambda, \lim_\lambda) = T_0(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda))$.
(2) $\bigwedge_{\lambda\in\Lambda} T_1(X_\lambda, \lim_\lambda) = T_1(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda))$.

Proof. We only prove (1). The verification of (2) is similar.
By Theorem 3.5, it follows that $\bigwedge_{\lambda\in\Lambda} T_0(X_\lambda, \lim_\lambda) \leqslant T_0(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda))$.
In order to prove the inverse, put $m = \bigwedge_{\lambda\in\Lambda} T_0(X_\lambda, \lim_\lambda)$ and $n = T_0(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda))$. Then we have

$$m = \bigwedge_{\lambda\in\Lambda} \bigwedge_{x_\lambda \neq y_\lambda} (\lim_\lambda[y_\lambda](x_\lambda) \to \bot) \vee (\lim_\lambda[x_\lambda](y_\lambda) \to \bot)$$

and

$$n = \bigwedge_{x\neq y} \left(\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[p_\lambda(y)](p_\lambda(x)) \to \bot \right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[p_\lambda(x)](p_\lambda(y)) \to \bot \right) \right).$$

For each $\lambda_0 \in \Lambda$ and $x_{\lambda_0},\ y_{\lambda_0} \in X_{\lambda_0}$ with $x_{\lambda_0} \neq y_{\lambda_0}$, take $x,\ y \in \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_{\lambda_0}(x) = x_{\lambda_0}$, $p_{\lambda_0}(y) = y_{\lambda_0}$ and $p_\lambda(x) = p_\lambda(y)$ $(\forall \lambda \neq \lambda_0)$. Then for each $\lambda \neq \lambda_0$, it follows that

$$\lim_\lambda[p_\lambda(y)](p_\lambda(x)) = \lim_\lambda[p_\lambda(x)](p_\lambda(y)) = \lim_\lambda[p_\lambda(x)](p_\lambda(x)) = \top.$$

This implies that

$$\begin{aligned} n &\leqslant \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[p_\lambda(y)](p_\lambda(x)) \to \bot \right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda[p_\lambda(x)](p_\lambda(y)) \to \bot \right) \\ &= (\lim_{\lambda_0}[p_{\lambda_0}(y)](p_{\lambda_0}(x)) \to \bot) \vee (\lim_{\lambda_0}[p_{\lambda_0}(x)](p_{\lambda_0}(y)) \to \bot) \\ &= (\lim_{\lambda_0}[y_{\lambda_0}](x_{\lambda_0}) \to \bot) \vee (\lim_{\lambda_0}[x_{\lambda_0}](y_{\lambda_0}) \to \bot). \end{aligned}$$

By the arbitrariness of λ_0 and $x_{\lambda_0}, y_{\lambda_0}$, we obtain $n \leqslant m$, as desired. □

Definition 3.8. A map $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between stratified L-generalized convergence spaces is called a homomorphism provided that f and f^{-1} are bijective and continuous.

Lemma 3.9. *Let* $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *be a homomorphism. Then for each* $\mathcal{F} \in \mathcal{F}_L^s(X)$ *and* $x \in X$, $\lim_X \mathcal{F}(x) = \lim_Y f^{\Rightarrow}(\mathcal{F})(f(x))$.

Proof. Since $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ and $f^{-1} : (Y, \lim_Y) \longrightarrow (X, \lim_X)$ are both continuous, we have

$$\begin{aligned} \lim_X \mathcal{F}(x) &\leqslant \lim_Y f^{\Rightarrow}(\mathcal{F})(f(x)) \\ &\leqslant \lim_X (f^{-1})^{\Rightarrow}(f^{\Rightarrow}(\mathcal{F}))(f^{-1}(f(x))) \\ &= \lim_X (f^{-1} \circ f)^{\Rightarrow}(\mathcal{F})(x) \\ &= \lim_X \mathcal{F}(x), \end{aligned}$$

as desired. □

Theorem 3.10. *If* $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *is a homomorphism, then*
(1) $T_0(X, \lim_X) = T_0(Y, \lim_Y)$.
(2) $T_1(X, \lim_X) = T_1(Y, \lim_Y)$.

Proof. (1) By Lemma 3.9, we have

$$\begin{aligned} T_0(X, \lim_X) &= \bigwedge_{x_1 \neq x_2} (\lim_X[x_2](x_1) \to \bot) \vee (\lim_X[x_1](x_2) \to \bot) \\ &= \bigwedge_{f(x_1) \neq f(x_2)} (\lim_Y[f(x_2)](f(x_1)) \to \bot) \vee (\lim_Y[f(x_1)](f(x_2)) \to \bot) \\ &= \bigwedge_{y_1 \neq y_2} (\lim_Y[y_2](y_1) \to \bot) \vee (\lim_Y[y_1](y_2) \to \bot) \\ &= T_0(Y, \lim_Y). \end{aligned}$$

(2) The verification is similar and we omit it. □

4. Degrees of T_2

In this section, we generalize the T_2 separation axiom in the sense of Jäger [10] to more general case and then investigate its properties.

Definition 4.1. For a stratified L-generalized convergence space $(X, \lim)$, define the degree $T_2(X, \lim)$ to which $(X, \lim)$ is T_2 as follows:

$$T_2(X, \lim) = \bigwedge_{x \neq y} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim \mathcal{F}(x) \to \bot) \wedge (\lim \mathcal{F}(y) \to \bot).$$

Remark 4.2. For a stratified L-generalized convergence space $(X, \lim)$, Jäger [10] defined T_2 separation axiom in the following form:

(T_2) $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x, y \in X : \lim \mathcal{F}(x) = \lim \mathcal{F}(y) = \top$ implies $x = y$.

Observe that if $T_2(X, \lim) = \top$, where $T_2(X, \lim)$ is defined as in Definition 4.1, then it is exactly the definition of T_2 in the sense of Jäger.

Theorem 4.3. *For a stratified L-generalized convergence space* $(X, \lim)$, $T_2(X, \lim) \leqslant T_1(X, \lim)$.

Proof. By Definition 4.1, we have

$$\begin{aligned} T_2(X, \lim) &= \bigwedge_{x \neq y} \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim \mathcal{F}(x) \to \bot) \wedge (\lim \mathcal{F}(y) \to \bot) \\ &\leqslant \bigwedge_{x \neq y} \Big(\big((\lim[y](x) \to \bot) \wedge (\lim[y](y) \to \bot) \big) \\ &\qquad \wedge \big((\lim[x](x) \to \bot) \wedge (\lim[x](y) \to \bot) \big) \Big) \\ &= \bigwedge_{x \neq y} (\lim[y](x) \to \bot) \wedge (\lim[x](y) \to \bot) \\ &= T_1(X, \lim), \end{aligned}$$

as desired. □

Corollary 4.4. *If* $(X, \lim)$ *is a stratified L-generalized convergence space, then*

$$T_2(X, \lim) \leqslant T_1(X, \lim) \leqslant T_0(X, \lim).$$

The degree of T_2 are productive and inherited by subspaces:

Theorem 4.5. *If all* $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ *are stratified L-generalized convergence spaces and if the family of maps* $(f_\lambda : X \longrightarrow X_\lambda)$ *separates points (i.e. for* $x \neq y$ *there is* $\lambda \in \Lambda$ *such that* $f_\lambda(x) \neq f_\lambda(y)$*), then*

$$\bigwedge_{\lambda\in\Lambda} T_2(X_\lambda, \lim_\lambda) \leqslant T_2(X, Init(\lim_\lambda)).$$

Proof. For convenience, put $m = \bigwedge_{\lambda\in\Lambda} T_2(X_\lambda, \lim_\lambda)$ and $n = T_2(X, Init(\lim_\lambda))$. Then

$$m = \bigwedge_{\lambda\in\Lambda} \bigwedge_{x_\lambda\neq y_\lambda} \bigwedge_{\mathcal{F}_\lambda\in\mathcal{F}_L^s(X_\lambda)} (\lim_\lambda\mathcal{F}_\lambda(x_\lambda) \to \bot) \vee (\lim_\lambda\mathcal{F}_\lambda(y_\lambda) \to \bot)$$

and

$$n = \bigwedge_{x\neq y} \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(X)} \left(\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(x)) \to \bot\right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(y)) \to \bot\right)\right).$$

Take any $x \neq y$. Then there exists $\lambda_0 \in \Lambda$ such that $f_{\lambda_0}(x) \neq f_{\lambda_0}(y)$. Hence

$$\begin{aligned} m &\leqslant \bigwedge_{\mathcal{F}_{\lambda_0}\in\mathcal{F}_L^s(X_{\lambda_0})} (\lim_{\lambda_0}\mathcal{F}_{\lambda_0}(f_{\lambda_0}(x)) \to \bot) \vee (\lim_{\lambda_0}\mathcal{F}_{\lambda_0}(f_{\lambda_0}(y)) \to \bot) \\ &\leqslant \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(X)} \left(\lim_{\lambda_0} f_{\lambda_0}^{\Rightarrow}(\mathcal{F})(f_{\lambda_0}(x)) \to \bot\right) \vee \left(\lim_{\lambda_0} f_{\lambda_0}^{\Rightarrow}(\mathcal{F})(f_{\lambda_0}(y)) \to \bot\right) \\ &\leqslant \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(X)} \left(\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(x)) \to \bot\right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(y)) \to \bot\right)\right). \end{aligned}$$

By the arbitrariness of x and y, we obtain $m \leqslant n$, as desired. □

Corollary 4.6. *If* $(A, \lim|_A)$ *is a subspace of* $(X, \lim)$*, then* $T_2(X, \lim) \leqslant T_2(A, \lim|_A)$*.*

Theorem 4.7. *If all* $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ *are stratified L-generalized convergence spaces, then*

$$\bigwedge_{\lambda\in\Lambda} T_2(X_\lambda, \lim_\lambda) = T_2\left(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda)\right).$$

Proof. By Theorem 4.5, it follows that

$$\bigwedge_{\lambda\in\Lambda} T_2(X_\lambda, \lim_\lambda) \leqslant T_2\left(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda)\right).$$

Conversely, put $m = \bigwedge_{\lambda\in\Lambda} T_2(X_\lambda, \lim_\lambda)$ and $n = T_2\left(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda)\right)$. Then

$$m = \bigwedge_{\lambda\in\Lambda} \bigwedge_{x_\lambda\neq y_\lambda} \bigwedge_{\mathcal{F}_\lambda\in\mathcal{F}_L^s(X_\lambda)} (\lim_\lambda\mathcal{F}_\lambda(x_\lambda) \to \bot) \vee (\lim_\lambda\mathcal{F}_\lambda(y_\lambda) \to \bot)$$

and

$$n = \bigwedge_{x\neq y} \bigwedge_{\mathcal{F}\in\mathcal{F}_L^s(X)} \left(\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F})(p_\lambda(x)) \to \bot\right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F})(p_\lambda(y)) \to \bot\right)\right).$$

For each $\lambda_0 \in \Lambda$, $x_{\lambda_0} \neq y_{\lambda_0} \in X_{\lambda_0}$ and $\mathcal{F}_{\lambda_0} \in \mathcal{F}_L^s(X_{\lambda_0})$, take $x,\ y \in \prod_{\lambda\in\Lambda} X_\lambda$ such that $p_{\lambda_0}(x) = x_{\lambda_0}$, $p_{\lambda_0}(y) = y_{\lambda_0}$ and $p_\lambda(x) = p_\lambda(y)$ for all $\lambda \neq \lambda_0$. Also, let $\mathcal{F}_\lambda = \mathcal{F}_{\lambda_0}$ for $\lambda = \lambda_0$ and $\mathcal{F}_\lambda = [p_\lambda(x)] = [p_\lambda(y)]$ for $\lambda \neq \lambda_0$. Define

$\mathcal{F}^* = \prod_{\lambda\in\Lambda}\mathcal{F}_\lambda$. Then

$$\begin{aligned} n &\leqslant \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F}^*)(p_\lambda(x)) \to \bot\right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F}^*)(p_\lambda(y)) \to \bot\right) \\ &\leqslant \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda \mathcal{F}_\lambda(p_\lambda(x)) \to \bot\right) \vee \left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda p_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)(p_\lambda(y)) \to \bot\right) \\ &= (\lim_{\lambda_0}\mathcal{F}_{\lambda_0}(x_{\lambda_0}) \to \bot) \vee (\lim_{\lambda_0}\mathcal{F}_{\lambda_0}(y_{\lambda_0}) \to \bot). \end{aligned}$$

By the arbitrariness of λ_0, x_{λ_0}, y_{λ_0} and $\mathcal{F}_{\lambda_0}$, we obtain $n \leqslant m$, as desired. □

Theorem 4.8. *If $f : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between stratified L-generalized convergence spaces is a homomorphism, then $T_2(X, \lim_X) = T_2(Y, \lim_Y)$.*

Proof. By Lemma 3.9, it is obvious and we omit it. □

5. Degrees of Regularity

In this section, we endow each stratified L-generalized convergence space with some degrees of fulfilling regularity and then investigate its relations with T_1 and T_2.

Definition 5.1. ([10])Let J be a set, $\mathcal{G} \in \mathcal{F}_L^s(J)$ and for all $j \in J$, $\mathcal{F}_j \in \mathcal{F}_L^s(X)$, we define for $A \in L^X$,

$$\mathcal{F}_{(\cdot)}(A) : \begin{cases} J & \longrightarrow & L \\ j & \longmapsto & \mathcal{F}_j(A). \end{cases}$$

i.e., $\mathcal{F}_{(\cdot)}(A) \in L^J$. Then the map $\mathcal{G}(\mathcal{F}_{(\cdot)})$ defined by

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(A) = \mathcal{G}(\mathcal{F}_{(\cdot)}(A)) \quad (A \in L^X)$$

is a stratified L-filter on X. It is called the stratified L-diagonal filter of $(\mathcal{G}, (\mathcal{F}_j)_{j\in J})$.

Definition 5.2. For a stratified L-generalized convergence space $(X, \lim)$, define the degree $Reg(X, \lim)$ to which $(X, \lim)$ is regular as follows:

$$Reg(X, \lim) = \bigwedge_{J} \bigwedge_{\psi} \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(J)} \bigwedge_{j\in J} \bigwedge_{\mathcal{F}_j\in\mathcal{F}_L^s(X)} \left(\bigwedge_{j\in J} \lim\mathcal{F}_j(\psi(j)) \to \mathcal{S}(\lim\mathcal{G}(\mathcal{F}_{(\cdot)}), \lim\psi^{\Rightarrow}(\mathcal{G}))\right),$$

where J is any set and $\psi : J \longrightarrow X$ is any map.

Remark 5.3. For a stratified L-generalized convergence space $(X, \lim)$, if $Reg(X, \lim) = \top$, then we can interpret it as follows:

$\forall J, \forall\psi : J \longrightarrow X, \forall\mathcal{G} \in \mathcal{F}_L^s(J), \forall\mathcal{F}_j \in \mathcal{F}_L^s(X)\ (j \in J)$,

$$\bigwedge_{j\in J} \lim\mathcal{F}_j(\psi(j)) \leqslant \mathcal{S}(\lim\mathcal{G}(\mathcal{F}_{(\cdot)}), \lim\psi^{\Rightarrow}(\mathcal{G})) = \bigwedge_{x\in X} \lim\mathcal{G}(\mathcal{F}_{(\cdot)})(x) \to \lim\psi^{\Rightarrow}(\mathcal{G}))(x),$$

i.e., $\forall x \in X$,

$$\bigwedge_{j\in J} \lim\mathcal{F}_j(\psi(j)) \wedge \lim\mathcal{G}(\mathcal{F}_{(\cdot)})(x) \leqslant \lim\psi^{\Rightarrow}(\mathcal{G}))(x).$$

It is exactly the definition of regularity in the sense of Jäger [10].

Lemma 5.4. ([10]) *Let J be a set, $\mathcal{G} \in \mathcal{F}_L^s(X)$ and $f : X \longrightarrow Y$ be a map. Then*

$$\mathcal{G}(f^{\Rightarrow}(\mathcal{F}_{(\cdot)})) = f^{\Rightarrow}(\mathcal{G}(\mathcal{F}_{(\cdot)})).$$

Theorem 5.5. *If all $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ are stratified L-generalized convergence spaces, then*

$$\bigwedge_{\lambda \in \Lambda} Reg(X_\lambda, \lim_\lambda) \leqslant Reg(X, Init(\lim_\lambda)).$$

Proof. For convenience, put $m = \bigwedge_{\lambda\in\Lambda} Reg(X_\lambda, \lim_\lambda)$ and $n = Reg(X, Init(\lim_\lambda))$. By Definition 5.2, we have

$$m = \bigwedge_{\lambda\in\Lambda} \bigwedge_{J_\lambda} \bigwedge_{\psi_\lambda} \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(J_\lambda)} \bigwedge_{j\in J_\lambda} \bigwedge_{\mathcal{F}_j^\lambda\in\mathcal{F}_L^s(X_\lambda)} \left(\bigwedge_{j\in J_\lambda} \lim_\lambda \mathcal{F}_j^\lambda(\psi_\lambda(j)) \to \mathcal{S}(\lim_\lambda \mathcal{G}_\lambda(\mathcal{F}_{(\cdot)}^\lambda), \lim_\lambda \psi_\lambda^{\Rightarrow}(\mathcal{G}_\lambda)) \right)$$

and

$$n = \bigwedge_{J} \bigwedge_{\psi} \bigwedge_{\mathcal{G}\in\mathcal{F}_L^s(J)} \bigwedge_{j\in J} \bigwedge_{\mathcal{F}_j\in\mathcal{F}_L^s(X)} \Big(\bigwedge_{j\in J} Init(\lim_\lambda)\mathcal{F}_j(\psi(j)) \to \mathcal{S}(Init(\lim_\lambda)\mathcal{G}(\mathcal{F}_{(\cdot)}), Init(\lim_\lambda)\psi^{\Rightarrow}(\mathcal{G})) \Big)$$

Take each J, $\psi : J \longrightarrow X$, $\mathcal{G} \in \mathcal{F}_L^s(J)$, $\mathcal{F}_j \in \mathcal{F}_L^s(X)$ $(\forall j \in J)$. For each $\lambda \in \Lambda$, put $J_\lambda = J$, $\psi_\lambda = f_\lambda \circ \psi$, $\mathcal{G}_\lambda = \mathcal{G}$, $\mathcal{F}_j^\lambda = f_\lambda^{\Rightarrow}(\mathcal{F}_j)$. Then by Lemma 5.4, we have

$$\begin{aligned}
m &\leqslant \bigwedge_{\lambda\in\Lambda} \left(\bigwedge_{j\in J} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F}_j)(f_\lambda(\psi(j))) \to \mathcal{S}(\lim_\lambda \mathcal{G}(f_\lambda^{\Rightarrow}(\mathcal{F}_{(\cdot)})), \lim_\lambda f_\lambda^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{G})) \right) \\
&\leqslant \bigwedge_{j\in J} \bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{F}_j)(f_\lambda(\psi(j))) \to \bigwedge_{\lambda\in\Lambda} \mathcal{S}(\lim_\lambda \mathcal{G}(f_\lambda^{\Rightarrow}(\mathcal{F}_{(\cdot)})), \lim_\lambda f_\lambda^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{G})) \\
&= \bigwedge_{j\in J} Init(\lim_\lambda)\mathcal{F}_j(\psi(j)) \to \bigwedge_{\lambda\in\Lambda} \mathcal{S}(\lim_\lambda \mathcal{G}(f_\lambda^{\Rightarrow}(\mathcal{F}_{(\cdot)})), \lim_\lambda f_\lambda^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{G})) \\
&\leqslant \bigwedge_{j\in J} Init(\lim_\lambda)\mathcal{F}_j(\psi(j)) \to \mathcal{S}\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda \mathcal{G}(f_\lambda^{\Rightarrow}(\mathcal{F}_{(\cdot)})), \bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{G})) \right) \\
&= \bigwedge_{j\in J} Init(\lim_\lambda)\mathcal{F}_j(\psi(j)) \to \mathcal{S}\left(\bigwedge_{\lambda\in\Lambda} \lim_\lambda f_\lambda^{\Rightarrow}(\mathcal{G}(\mathcal{F}_{(\cdot)})), Init(\lim_\lambda)(\psi^{\Rightarrow}(\mathcal{G})) \right) \\
&= \bigwedge_{j\in J} Init(\lim_\lambda)\mathcal{F}_j(\psi(j)) \to \mathcal{S}\Big(Init(\lim_\lambda)(\mathcal{G}(\mathcal{F}_{(\cdot)})), Init(\lim_\lambda)(\psi^{\Rightarrow}(\mathcal{G})) \Big).
\end{aligned}$$

By the arbitrariness of J, ψ, $\mathcal{G}$ and $\mathcal{F}_j$, we obtain $m \leqslant n$, as desired. □

Corollary 5.6. *If $(A, \lim|_A)$ is a subspace of $(X, \lim)$, then*

$$Reg(X, \lim) \leqslant Reg(A, \lim|_A).$$

Corollary 5.7. *If all $(X_\lambda, \lim_\lambda)$ $(\lambda \in \Lambda)$ are stratified L-generalized convergence spaces, then*

$$\bigwedge_{\lambda\in\Lambda} Reg(X_\lambda, \lim_\lambda) \leqslant Reg\left(\prod_{\lambda\in\Lambda} X_\lambda, \pi\text{-}(\lim_\lambda) \right).$$

In classical convergence theory, regularity can also be characterized by closures of filters. In the lattice-valued context, Jäger [10] generalized this concept and obtained so called α-closures of stratified L-filters. Moreover, he introduced the regularity axiom with this concept. However, the lattice must be required to be a complete Boolean algebra. The reason that we need this requirement is the following result.

Lemma 5.8. ([10]) *Let L be a complete Boolean algebra. For $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\alpha \in L$ and $A \in L^X$, let*

$$\overline{\mathcal{F}}^{\alpha}(A) = \bigvee \{\mathcal{F}(B) : B \in L^X \text{ such that for all } \mathcal{G} \in \mathcal{F}_L^s(X) \text{with } \lim \mathcal{G}(x) \geqslant \alpha, \text{ we have } \mathcal{G}(B) \leqslant B(x)\}.$$

Then $\overline{\mathcal{F}}^{\alpha} \in \mathcal{F}_L^s(X)$.

In this case, we call $\overline{\mathcal{F}}^{\alpha}$ he α-closure of $\mathcal{F}$. In the sequel, we will require that L be a complete Boolean algebra.

The next theorem shows that $Reg(X, \lim)$ can be characterized by $\overline{\mathcal{F}}^{\alpha}$.

Theorem 5.9. *Let $(X, \lim)$ be a stratified L-generalized convergence space. Then*

$$Reg(X, \lim) = \bigwedge_{\alpha,\beta \in L} \bigwedge_{x \in X} \bigwedge_{\lim \mathcal{F}(x) \geqslant \beta} \Big((\alpha \wedge \beta) \to \lim \overline{\mathcal{F}}^{\alpha}(x)\Big).$$

Proof. For convenience, put $n = \bigwedge_{\alpha,\beta \in L} \bigwedge_{x \in X} \bigwedge_{\lim \mathcal{F}(x) \geqslant \beta} \Big((\alpha \wedge \beta) \to \lim \overline{\mathcal{F}}^{\alpha}(x)\Big)$ and $m = Reg(X, \lim)$.

Firstly, we show $m \leqslant n$. Take any $\alpha, \beta \in L$, $x \in X$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\lim \mathcal{F}(x) \geqslant \beta$. Then we define

$$J = \{(\mathcal{G}, y) : \mathcal{G} \in \mathcal{F}_L^s(X),\ \lim \mathcal{G}(y) \geqslant \alpha\}.$$

For $j = (\mathcal{G}, y) \in J$, we define $\mathcal{F}_{((\mathcal{G},y))} = \mathcal{G}$ and $\psi : J \longrightarrow X$ by $\psi((\mathcal{G}, y)) = y$. Then $\lim \mathcal{F}_{((\mathcal{G},y))}(\psi((\mathcal{G}, y))) = \lim \mathcal{G}(y) \geqslant \alpha$. We define a stratified L-filter $\kappa \in \mathcal{F}_L^s(J)$ by

$$\kappa(a) = \bigvee_{\mathcal{G}(A) \leqslant a((\mathcal{G},y)), \forall (\mathcal{G},y) \in J} \mathcal{F}(a) \quad (a \in L^J).$$

Then $\mathcal{F} \leqslant \kappa(\mathcal{F}_{(\cdot)})$ and $\psi^{\Rightarrow}(\kappa) = \overline{\mathcal{F}}^{\alpha}$, which can be found in Lemma 7.2 [10]. Hence, $\lim \kappa(\mathcal{F}_{(\cdot)})(x) \geqslant \lim \mathcal{F}(x) \geqslant \beta$. Then it follows that

$$\begin{aligned} m &\leqslant \bigwedge_{(\mathcal{G},y) \in J} \lim \mathcal{G}(y) \to \mathcal{S}(\lim \kappa(\mathcal{F}_{(\cdot)}), \lim \psi^{\Rightarrow}(\kappa)) \\ &\leqslant \bigwedge_{\alpha \leqslant \lim \mathcal{G}(y)} \lim \mathcal{G}(y) \to (\lim \kappa(\mathcal{F}_{(\cdot)})(x) \to \lim \psi^{\Rightarrow}(\kappa)(x)) \\ &\leqslant \alpha \to (\beta \to \lim \overline{\mathcal{F}}^{\alpha}(x)) \\ &= (\alpha \wedge \beta) \to \lim \overline{\mathcal{F}}^{\alpha}(x). \end{aligned}$$

By the arbitrariness of α, β, x and $\mathcal{F}$, we obtain $m \leqslant n$.

Secondly, we show $n \leqslant m$. Take any set J, $\psi : J \longrightarrow X$, $\mathcal{G} \in \mathcal{F}_L^s(J)$, $\mathcal{F}_j \in \mathcal{F}_L^s(X)$ for all $j \in J$. Then let $\alpha = \bigwedge_{j \in J} \lim \mathcal{F}_j(\psi(j))$ and $\beta_x = \lim \mathcal{G}(\mathcal{F}_{(\cdot)})(x)$ for each $x \in X$. We first show

$$\overline{\mathcal{G}(\mathcal{F}_{(\cdot)})}^{\alpha} \leqslant \psi^{\Rightarrow}(\mathcal{G}).$$

If now $B \in L^X$ such that for all $\mathcal{H} \in \mathcal{F}_L^s(X)$ with $\lim \mathcal{H}(y) \geqslant \alpha$, we have $\mathcal{H}(B) \leqslant A(y)$, then for all $\mathcal{F}_j \in \mathcal{F}_L^s(X)$, it follows from $\alpha \leqslant \lim \mathcal{F}_j(\psi(j))$ that $\mathcal{F}_j(B) \leqslant A(\psi(j)) \leqslant \psi^{\leftarrow}(A)(j)$. Therefore,

$$\mathcal{G}(\mathcal{F}_{(\cdot)})(B) = \mathcal{G}(\mathcal{F}_{(\cdot)}(B)) \leqslant \mathcal{G}(\psi^{\leftarrow}(A)) = \psi^{\Rightarrow}(\mathcal{G})(A).$$

From this we conclude $\overline{\mathcal{G}(\mathcal{F}_{(\cdot)})}^{\alpha}(A) \leqslant \psi^{\Rightarrow}(\mathcal{G})(A)$. Hence we have

$$\begin{aligned} n &\leqslant \bigwedge_{x\in X}(\alpha\wedge\beta_x)\to \lim\overline{\mathcal{G}(\mathcal{F}_{(\cdot)})}^{\alpha}(x)\\ &= \bigwedge_{x\in X}\alpha\to(\beta_x\to\lim\overline{\mathcal{G}(\mathcal{F}_{(\cdot)})}^{\alpha}(x))\\ &\leqslant \bigwedge_{x\in X}\alpha\to(\beta_x\to\lim\psi^{\Rightarrow}(\mathcal{G})(x))\\ &= \alpha\to\bigwedge_{x\in X}(\beta_x\to\lim\psi^{\Rightarrow}(\mathcal{G})(x))\\ &= \alpha\to\bigwedge_{x\in X}(\lim\mathcal{G}(\mathcal{F}_{(\cdot)})(x)\to\lim\psi^{\Rightarrow}(\mathcal{G})(x))\\ &= \alpha\to\mathcal{S}(\lim\mathcal{G}(\mathcal{F}_{(\cdot)}),\lim\psi^{\Rightarrow}(\mathcal{G}))\\ &= \bigwedge_{j\in J}\lim\mathcal{F}_j(\psi(j))\to\mathcal{S}(\lim\mathcal{G}(\mathcal{F}_{(\cdot)}),\lim\psi^{\Rightarrow}(\mathcal{G})). \end{aligned}$$

By the arbitrariness of J, ψ, $\mathcal{G}$ and $\mathcal{F}_j$, we have $n \leqslant m$. As a consequence, we obtain $m = n$, as desired. □

Theorem 5.10. *Let* $(X, \lim)$ *be a stratified L-generalized convergence space. Then*

$$Reg(X,\lim)\wedge T_1(X,\lim)\leqslant T_2(X,\lim).$$

Proof. Take any $\alpha \in J(L)$ with $\alpha \nleqslant T_2(X,\lim)$. Then there exist $x \neq y$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\alpha \nleqslant \lim\mathcal{F}(x) \to \bot$ and $\alpha \nleqslant \lim\mathcal{F}(y) \to \bot$. Since L is a complete Boolean algebra, we have $\alpha \leqslant \lim\mathcal{F}(x)$ and $\alpha \leqslant \lim\mathcal{F}(y)$. Put

$$J_1 = \{\mathcal{G}\in\mathcal{F}_L^s(X) : \alpha\leqslant\lim\mathcal{F}(y)\}.$$

We define $\mathcal{F}^1_{\mathcal{G}} = \mathcal{G}$, $\psi_1 : J_1 \longrightarrow X : \mathcal{G} \longmapsto y$. Then it follows from $\alpha \leqslant \lim\mathcal{F}(y)$ that $\mathcal{F} \in J_1$. For the point filter $[\mathcal{F}] \in \mathcal{F}_L^s(J_1)$, we have

$$\psi_1^{\Rightarrow}([\mathcal{F}])(A) = [\mathcal{F}](\psi_1^{\Leftarrow}(A)) = \psi_1^{\Leftarrow}(A)(\mathcal{F}) = A(\psi_1^{\Rightarrow}(\mathcal{F})) = A(y) = [y](A)$$

for all $A \in L^X$, i.e., $\psi_1^{\Rightarrow}([\mathcal{F}]) = [y]$. Further,

$$[\mathcal{F}](\mathcal{F}^1_{(\cdot)})(A) = [\mathcal{F}](\mathcal{F}^1_{(\cdot)}(A)) = \mathcal{F}_{\mathcal{F}}(A) = \mathcal{F}(A).$$

This shows $[\mathcal{F}](\mathcal{F}^1_{(\cdot)}) = \mathcal{F}$.

Similarly, put

$$J_2 = \{\mathcal{H}\in\mathcal{F}_L^s(X) : \alpha\leqslant\lim\mathcal{H}(x)\}.$$

We define $\mathcal{F}^2_{\mathcal{H}} = \mathcal{H}$, $\psi_2 : J_2 \longrightarrow X : \mathcal{H} \longmapsto x$. Then we obtain $[\mathcal{F}] \in \mathcal{F}_L^s(J_2)$, $\psi_2^{\Rightarrow}([\mathcal{F}]) = [x]$ and $[\mathcal{F}](\mathcal{F}^2_{(\cdot)}) = \mathcal{F}$.

By the definition of $Reg(X,\lim)$, we have

$$\begin{aligned} &Reg(X,\lim)\\ \leqslant\ & \bigwedge_{\mathcal{G}\in J_1}\lim\mathcal{F}^1_{\mathcal{G}}(\psi_1(\mathcal{G}))\to\mathcal{S}(\lim[\mathcal{F}](\mathcal{F}^1_{(\cdot)}),\lim\psi_1^{\Rightarrow}([\mathcal{F}]))\\ &\wedge\bigwedge_{\mathcal{H}\in J_2}\lim\mathcal{F}^2_{\mathcal{H}}(\psi_2(\mathcal{H}))\to\mathcal{S}(\lim[\mathcal{F}](\mathcal{F}^2_{(\cdot)}),\lim\psi_2^{\Rightarrow}([\mathcal{F}]))\\ =\ & \bigwedge_{\mathcal{G}\in J_1}\lim\mathcal{G}(y)\to\mathcal{S}(\lim\mathcal{F},\lim[y])\wedge\bigwedge_{\mathcal{H}\in J_2}\lim\mathcal{H}(x)\to\mathcal{S}(\lim\mathcal{F},\lim[x])\\ \leqslant\ & (\alpha\to(\lim\mathcal{F}(x)\to\lim[y](x))\wedge(\alpha\to(\lim\mathcal{F}(y)\to\lim[x](y))\\ =\ & (\alpha\wedge\lim\mathcal{F}(x))\to\lim[y](x))\wedge(\alpha\wedge\lim\mathcal{F}(y))\to\lim[x](y)). \end{aligned}$$

With $T_1(X,\lim) = \bigwedge_{z_1\neq z_2}(\lim[z_2](z_1) \to \bot) \wedge (\lim[z_1](z_2) \to \bot)$, we further obtain

$$\begin{aligned}
& Reg(X,\lim) \wedge T_1(X,\lim) \\
\leqslant\ & Reg(X,\lim) \wedge \Big((\lim[y](x) \to \bot) \wedge (\lim[x](y) \to \bot)\Big) \\
=\ & (Reg(X,\lim) \wedge (\lim[y](x) \to \bot)) \wedge (Reg(X,\lim) \wedge (\lim[x](y) \to \bot)) \\
\leqslant\ & (\alpha \wedge \lim\mathcal{F}(x)) \to \lim[y](x)) \wedge (\lim[y](x) \to \bot) \wedge (\alpha \wedge \lim\mathcal{F}(y)) \to \lim[x](y) \wedge (\lim[x](y) \to \bot) \\
\leqslant\ & (\alpha \wedge \lim\mathcal{F}(x)) \to \bot) \wedge (\alpha \wedge \lim\mathcal{F}(y)) \to \bot) \\
=\ & \alpha \to \bot.
\end{aligned}$$

Hence, $\alpha \leqslant (Reg(X,\lim) \wedge T_1(X,\lim)) \to \bot$. Since L is a complete Boolean algebra, it follows that

$$\alpha \nleqslant Reg(X,\lim) \wedge T_1(X,\lim).$$

By the arbitrariness of α, we obtain $Reg(X,\lim) \wedge T_1(X,\lim) \leqslant T_2(X,\lim)$. □

6. Conclusions

In this paper, we endowed each stratified L-generalized convergence space with some degrees of fulfilling T_1, T_2 and regularity axioms. Based on the definitions by degrees, we presented the lattice-valued forms of several important conclusions. This theory generalizes the separation theory in the sense of Jäger [10]. However, in Theorems 5.9 and 5.10, we required that L should be a complete Boolean algebra. In the future, we will consider whether it can be generalized to more general lattice.

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References

[1] R. Bělohlávek, Fuzzy Relation Systems, Foundation and Principles, Kluwer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.
[2] J.M. Fang, Stratified L-ordered convergence structures, Fuzzy Sets Syst. 161 (2010) 2130–2149.
[3] J.M. Fang, Relationships between L-ordered convergence structures and strong L-topologies, Fuzzy Sets Syst. 161 (2010) 2923–2944.
[4] U. Höhle, Locales and L-topologies, in: Categorical Methods in Algebra and Topology–Collection of Papers in Honour of Horst Herrlich, Mathematik-Arbeitspapiere, 48 (1997) 233–250.
[5] U. Höhle, A.P. Šostak, Axiomatic foudations of fixed-basis fuzzy topology, in: U. Höhle, S.E. Rodabaugh (Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Handbook Series, vol.3, Kluwer Academic Publishers, Boston, Dordrecht, London, 1999, 123–173.
[6] G. Jäger, Compactness in fuzzy convergence spaces, Fuzzy Sets Syst. 90 (1997) 341–348.
[7] G. Jäger, Separation axioms for fuzzy convergence spaces, J. Fuzzy Math. 5 (1997) 915–924.
[8] G. Jäger, A category of L-fuzzy convergence spaces, Quaest. Math. 24 (2001) 501–517.
[9] G. Jäger, Subcategories of lattice-valued convergence spaces, Fuzzy Sets Syst. 156 (2005) 1–24.
[10] G. Jäger, Lattice-valued convergence spaces and regularity, Fuzzy Sets Syst. 159 (2008) 2488–2502.
[11] B.Y. Lee, J.H. Park, B.H. Park, Fuzzy convergence structures, Fuzzy Sets Syst. 56 (1993) 309–315.
[12] B.Y. Lee, S.H. Sohn, J.H. Park, Separation axioms of fuzzy convergence spaces, Fuzzy Sets Syst. 75 (1995) 111–115.
[13] B.Y. Lee, S.H. Sohn, Fuzzy regular convergence structures, Fuzzy Sets Syst. 101 (1999) 505–508.
[14] L.Q. Li, Q. Jin, On adjunctions between **Lim**, **S***L*-**Top**, and **S***L*-**Lim**, Fuzzy Sets Syst. 182 (2011) 66–78.
[15] L.Q. Li, Q. Jin, On stratified L-convergence spaces: Pretopological axioms and diagonal axioms, Fuzzy Sets Syst. 204 (2012) 40–52.
[16] R. Lowen, Convergence in fuzzy topological spaces, Gen. Topl. Appl. 10 (1979) 147–160.
[17] E. Lowen, R. Lowen, P. Wuyts, The categorical topology approach to fuzzy topology and fuzzy convergence, Fuzzy Sets Syst. 40 (1991) 347–373.
[18] E. Lowen, R. Lowen, A topological universe extension of FTS, in: Applications of category theory to fuzzy sets, (S.E. Rodabaugh, E.P. Klement, U. Höhle, eds.), Kluwer, Dordrecht, 1992.
[19] K.C. Min, Fuzzy limit spaces, Fuzzy Sets Syst. 32 (1989) 343–357.

[20] B. Pang, J.M. Fang, *L*-fuzzy Q-convergence structures, Fuzzy Sets Syst. 182 (2011) 53–65.
[21] B. Pang, On (L, M)-fuzzy convergence spaces, Fuzzy Sets Syst. 238 (2014) 46–70.
[22] B. Pang, Enriched (L, M)-fuzzy convergence spaces, J. Intell. Fuzzy Syst. 27 (2014) 93–103.
[23] B. Pang, F.-G. Shi, Degrees of compactness in (L, M)-fuzzy convergence spaces and its applications, Fuzzy Sets Syst. 251 (2014) 1–22.
[24] B. Pang, Y. Zhao, Several types of enriched (L, M)-fuzzy convergence spaces, Fuzzy Sets Syst. 321 (2017) 55–72.
[25] B. Pang, Y. Zhao, Stratified (L, M)-fuzzy Q-convergence spaces, Iran. J. Fuzzy Syst. 13 (2016) 95–111.
[26] G. Preuss, Foundations of Topology–An Approach to Convenient Topology, Kluwer Academic Publisher, Dordrecht, Boston, London, 2002.
[27] S.E. Rodabaugh, Categorical foundations of variable-basis fuzzy topology, in [5], pp. 273–388.
[28] W.C. Wu, J.M. Fang, *L*-ordered fuzzifying convergence spaces, Iran. J. Fuzzy Syst. 9 (2012) 147–161.
[29] W. Yao, On many-valued stratified *L*-fuzzy convergence spaces, Fuzzy Sets Syst. 159 (2008) 2503–2519.
[30] W. Yao, On *L*-fuzzifying convergence spaces, Iran. J. Fuzzy Syst. 6 (2009) 63–80.
[31] W. Rudin, Real and Complex Analysis, (3rd edition), McGraw-Hill, New York, 1986.