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Topological Graphs Based on a new Topology on $\mathbf{Z}^n$ and its Applications

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Abstract.
Up to now there is no homotopy for Marcus-Wyse (for short M-) topological spaces. In relation to the development of a homotopy for the category of Marcus-Wyse (for short M-) topological spaces on $\mathbf{Z}^2$, we need to generalize the M-topology on $\mathbf{Z}^2$ to higher dimensional spaces $X \subset \mathbf{Z}^n$, $n \geq 3$ [18]. Hence the present paper establishes a new topology on $\mathbf{Z}^n, n \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. It is called the *generalized Marcus-Wyse* (for short H-) topology and is denoted by $(\mathbf{Z}^n, \gamma^n)$. Besides, we prove that $(\mathbf{Z}^3, \gamma^3)$ induces only 6- or 18-adjacency relations. Namely, $(\mathbf{Z}^3, \gamma^3)$ does not support a 26-adjacency, which is quite different from the Khalimsky topology for 3D digital spaces. After developing an H-adjacency induced by the connectedness of $(\mathbf{Z}^n, \gamma^n)$, the present paper establishes topological graphs based on the H-topology, which is called an HA-space, so that we can establish a category of HA-spaces. By using the H-adjacency, we propose an H-topological graph homomorphism (for short HA-map) and an HA-isomorphism. Besides, we prove that an HA-map (*resp.* an HA-isomorphism) is broader than an H-continuous map (*resp.* an H-homeomorphism) and is an H-connectedness preserving map. Finally, after investigating some properties of an HA-isomorphism, we propose both an HA-retract and an extension problem of an HA-map for studying HA-spaces.

1. Introduction

An Alexandroff topological structure plays an important role in applied topology [1, 32] so that a locally finite topological structure strongly contributed to the study of digital spaces [1]. We say that a *digital space* is a pair (X, R), where X is a nonempty set and R is a binary symmetric relation on X such that X is R-connected [22]. Here, we say that X is *R-connected* if for any two elements x and y of X there is a finite sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in R$ for $j \in [0, l-1]_{\mathbf{Z}}$. In digital topology, several kinds of tools have been used to study nD digital spaces. One of them is the Khalimsky (for brevity, K-) topology on the Euclidean nD space with integer coordinates, denoted by $(\mathbf{Z}^n, \kappa^n)$ [27]. Furthermore, a graph theoretical approach was often used to study digital spaces [34]. Since M-topology $(\mathbf{Z}^2, \gamma^2)$ [37] was formulated for studying spaces in $\mathbf{Z}^2$ [37], it has been used to study digital spaces from the

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viewpoint of digital geometry. Hereafter, for the set $\mathbf{Z} \subset \mathbf{Z}^2$ the paper assumes its subspace $(\mathbf{Z}, \gamma^2_{\mathbf{Z}}) := (\mathbf{Z}, \gamma)$ induced by $(\mathbf{Z}^2, \gamma^2)$. Besides, we generalize the M-topological structure on $\mathbf{Z}^2$ to nD spaces. To do this work, we consider a product space $(\mathbf{Z}^2, \gamma^2) \times (\mathbf{Z}, \gamma) := (\mathbf{Z}^3, \gamma^3)$ introduced by Kong [29]. Motivated by this approach, we propose a topology on $\mathbf{Z}^n, n \in \mathbf{N}$, denoted by $(\mathbf{Z}^n, \gamma^n)$ (see Definition 3.1), which is an extended topology of both the M-topology and $(\mathbf{Z}^3, \gamma^3)$ in [29]. Under $(\mathbf{Z}^n, \gamma^n)$, we can naturally consider the notions of an H-continuous map and an H-homeomorphism. But we can observe that an H-continuous map is so rigid that it has some limitations of geometric transformations (see Remark 4.1). Thus, after establishing the notion of an H-adjacency induced by the connectedness of $(\mathbf{Z}^n, \gamma^n)$, we obtain H-topological graphs based on the H-topology, which is called HA-spaces. By using these H-topological graphs, we establish a new map, called an HA-map (see Definition 4.8) which is broader than an H-continuous map and finally, it is proved to be an H-connectedness preserving map which can be used in applied topology. Furthermore, to study HA-spaces substantially, we need to establish an HA-isomorphism.

Then we may naturally pose the following queries.

(Q1) On $\mathbf{Z}^n$, what is a relation among an ordinary k-adjacency relation of $\mathbf{Z}^n$, an H-adjacency and a K-adjacency ?

Let $SC^{n,l}_H$ be a simple closed H-curve with l-elements in $(\mathbf{Z}^n, \gamma^n)$, $SC^{n,l}_K$ a simple closed K-curve with l-elements in $(\mathbf{Z}^n, \kappa^n)$ and $SC^{n,l}_k$ a simple closed k-curve with l-elements in $\mathbf{Z}^n$ with a k-adjacency relation.

(Q2) What is a relation among $SC^{n,l}_H$, $SC^{n,l}_K$ and $SC^{n,l}_k$?

Since an HA-map is different from a K-continuous map, a digitally k-continuous map [9, 10] and the other digitally continuous maps [11], we need to establish a new notion of a retract for HA-spaces.

(Q3) What is an HA-retract for HA-spaces ?

In Sections 3–6, we shall address these issues. This approach can contribute to the classification of HA-spaces and nD digital spaces.

The present paper follows traditional approaches in studying digital topology which includes a graph theoretical approach [9–11, 31, 34], an M-topological structure [13, 17, 37] and a K-topological structure [8, 27].

The rest of the paper proceeds as follows: Section 2 provides some basic notions on digital topology. Section 3 develops a new topology on $\mathbf{Z}^n, n \in \mathbf{N}$, that is called the H-topology and investigates its properties. Section 4 refers to some limitations of an H-continuous map and develops the notion of an HA-map (*resp.* an HA-isomorphism) being broader than an H-continuous map (*resp.* an H-homeomorphism). Besides, we prove that an HA-map is an H-connectedness preserving map which can be used in applied topology. Besides, it formulates an adjacency, so called an H-adjacency relation, induced by the H-connectedness. In addition, it develops H-topological graphs based on the H-topology (HA-spaces for short). Section 5 compares among simple closed K-, H- and k-curves. Section 6 proposes an HA-retract and an extension problem of an HA-map. Section 7 concludes the paper with a summary and a further work.

2. Preliminaries

Since almost of all digital topologies are based on Alexandroff topology, let us recall basic notions of the structure. We say that a topological space X is Alexandroff if for each point $x \in X$ there is the minimal open set (or the smallest open set) $V(x)$ containing x [1]. In particular, every locally finite space (where each point has an open neighborhood which is finite) is Alexandroff. It is easy to see that for each point $y \in V(x)$ we have $V(y) \subset V(x)$. This implies that if X is a T_0-space and $x, y \in X$ then $V(x) = V(y)$ if and only if $x = y$. Alexandroff spaces appear by a natural way in studies of topological models of digital spaces. They are quotient spaces of the Euclidean spaces $\mathbf{R}^n$ defined by special decompositions [16, 23]. Some studies of

Alexandroff spaces from the general topological point of view can be found for example in [1].

Motivated by the Alexandroff topological approach, many mathematical tools have been used in the fields of both digital topology and digital geometry. We need to recall some terminology from a graph theoretical approach and the topologies such as Marcus-Wyse, Khalimsky topology and so forth. It is well known that the study of 2D digital spaces plays an important role in digital geometry related to the fields of mathematical morphology, computer graphics, image analysis, image processing and so forth. Thus the M-topological structure [37], denoted by $(\mathbf{Z}^2, \gamma^2)$ in the paper, was established by using the set U in (2.1) as a base, where for each point $p = (x, y) \in \mathbf{Z}^2$

$$U := \begin{cases} U(p) := \{(x \pm 1, y), (x, y \pm 1)\} \cup \{p\} \text{ if } x + y \text{ is even, and} \\ \{p\} : \text{ else.} \end{cases} \qquad (2.1)$$

As a further term of a point in $\mathbf{Z}^2$, in the paper we call a point $p = (x_1, x_2)$ *completely even* if each x_i is even, $i \in \{1, 2\}$ and further, a point $p = (x_1, x_2)$ *completely odd* if each x_i is odd, $i \in \{1, 2\}$. For a set $X \subset \mathbf{Z}^2$ we can take the subspace induced by $(\mathbf{Z}^2, \gamma^2)$, denoted by $(X, {\gamma^2}_X)$, which has been often studied in the context of digital images [8, 35, 37].

Let us recall some notions on the K-topology. The K-topology κ on $\mathbf{Z}$ is induced by the set $\{[2n-1, 2n+1]_{\mathbf{Z}} \,|\, n \in \mathbf{Z}\}$ as a subbase [27] so that it is a $T_{\frac{1}{2}}$ space (see also [27, 28]), where for two distinct points a and b in $\mathbf{Z}$ let $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \leq n \leq b\}$ be called an integer interval [31]. Furthermore, the product topology on $\mathbf{Z}^n$ induced by $(\mathbf{Z}, \kappa)$ is called the *Khalimsky nD space* which is denoted by $(\mathbf{Z}^n, \kappa^n)$. For a set $X \subset \mathbf{Z}^n$ we can consider a subspace induced by $(\mathbf{Z}^n, \kappa^n)$, denoted by $(X, {\kappa^n}_X)$. Besides, both a K-continuous map and a K-homeomorphism are established from the viewpoint of K-topology.
In $(\mathbf{Z}^n, \kappa^n)$ we say that a simple closed K-curve with l elements in $\mathbf{Z}^n$ is a path $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ that is K-homeomorphic to a quotient space of a Khalimsky line interval $[0, 2m]_{\mathbf{Z}}$ or $[1, 2m+1]_{\mathbf{Z}}$ in terms of the identification of the only two end points x_0 and x_l. We denote it by $SC_K^{n,l}$ [19]. In other words, we can represent $SC_K^{n,l}$ as a simple K-path $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that $x_0 = x_l$ if and only if $|i - j| = 1 (mod\, l)$.

Let us recall some digital topological tools introduced by Rosenfeld [34]. To study a multidimensional space $X \subset \mathbf{Z}^n$ in a graph theoretical approach, we have used the k-adjacency relations of $\mathbf{Z}^n$. As a generalization of the k-adjacency relations of $2D$ and $3D$ digital spaces [31, 34], the k-adjacency relations of $\mathbf{Z}^n$ were established in [9] (see also [12]):
For a natural number m where $1 \leq m \leq n$, two distinct points
$p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n$ are called $k(m, n)$- (briefly, k-) adjacent if

$$\text{at most } t \text{ of their coordinates differs by } \pm 1, \text{ and all others coincide.} \qquad (2.2)$$

Concretely, according to the two numbers $m, n \in \mathbf{N}$, the $k(m, n)$ (or k)-adjacency relations of $\mathbf{Z}^n$ were represented in [9, 12], as follows (for more details, see also [12]).

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \frac{n!}{(n-i)!\, i!}. \qquad (2.3)$$

Rosenfeld [34] called a set $X \subset \mathbf{Z}^n$ with a k-adjacency as a digital image denoted by (X, k). Indeed, to follow a graph theoretical approach of studying nD digital images, both the k-adjacency relations of $\mathbf{Z}^n$ and a digital k-neighborhood have been often used. More precisely, using the k-adjacency of $\mathbf{Z}^n$ in (2.3), we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [34]

$$N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

Furthermore, we often use the notation [31]

$$N_k^*(p) := N_k(p) \setminus \{p\}.$$

For one of the k-adjacency relations of $\mathbf{Z}^n$ in (2.3), a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective finite sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|i-j| = 1$ [31]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x, y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [9], is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $|i-j| = 1(mod\, l)$ [31].

For a digital image (X, k), as a generalization of $N_k(p)$ [9] the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset [10] of X

$$N_k(x_0, \varepsilon) := \{x \in X \,|\, l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.4}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$.
Concretely, for $X \subset \mathbf{Z}^n$ we obtain [10]

$$N_k(x, 1) = N_k(x) \cap X. \tag{2.5}$$

3. A Generalization of the *M*-topology and its Properties

For two M-topological spaces $(X, {\gamma^2}_X) := X$ and $(Y, {\gamma^2}_Y) := Y$, a function $f : X \to Y$ is said to be M-continuous at a point $x \in X$ if f is continuous at the point x from the viewpoint of M-topology. Furthermore, we say that a map $f : X \to Y$ is M-continuous if it is M-continuous at every point $x \in X$. Using M-continuous maps, we establish the *M-topological category*, denoted by MTC, consisting of two sets [17].

(1) The set of objects $(X, {\gamma^2}_X)$,

(2) For every ordered pair of objects $(X, {\gamma^2}_X)$ and $(Y, {\gamma^2}_Y)$, the set of all M-continuous maps $f : (X, {\gamma^2}_X) \to (Y, {\gamma^2}_Y)$ as morphisms.

Besides, in MTC, for two spaces $(X, {\gamma^2}_X)$ and $(Y, {\gamma^2}_Y)$, we say that a map $f : X \to Y$ is an M-homeomorphism [37] if f is an M-continuous bijection and that $f^{-1} : Y \to X$ is M-continuous.

Starting with $(\mathbf{Z}^2, \gamma^2)$, let us now develop a new topology on $\mathbf{Z}^n$ which is not compatible with $(\mathbf{Z}^n, \kappa^n)$. By using $(\mathbf{Z}, \gamma)$ and the M-topology, we now establish high dimensional digital topological structure. More precisely, consider the product topology on $\mathbf{Z}^2 \times \mathbf{Z} := \mathbf{Z}^3$ induced by γ^2 and γ so that we have the topological space $(\mathbf{Z}^3, \gamma^3)$. In general, we define the following:

Definition 3.1. On $\mathbf{Z}^n, n \geq 3$, we define the product topology γ^n induced by the topologies $(\mathbf{Z}^{n-1}, \gamma^{n-1})$ and $(\mathbf{Z}, \gamma)$, *i.e.* $(\mathbf{Z}^{n-1}, \gamma^{n-1}) \times (\mathbf{Z}, \gamma) := (\mathbf{Z}^n, \gamma^n)$.

In this paper $(\mathbf{Z}^n, \gamma^n)$ is called H-topology.

Corollary 3.2. $(\mathbf{Z}^n, \gamma^n)$ *is a proper subspace of* $(\mathbf{Z}^{n+1}, \gamma^{n+1})$ *with the relative topology on* $\mathbf{Z}^n$ *induced by* $(\mathbf{Z}^{n+1}, \gamma^{n+1})$, $n \in \mathbf{N}$.

Remark 3.3. (1) For the case of $n = 3$ of the H-topology, Kong [29] established it.
(2) The case of $n = 4$ is also treated in [30].

Besides, the paper [8] further studied the following:
(•) If a set $X \subset \mathbf{Z}^3$ is 6-connected, then it is topologically connected,
(•) If a set $X \subset \mathbf{Z}^3$ is not 26-connected, then it is not topologically connected.

In all subspaces of $(\mathbf{Z}^n, \gamma^n)$ of Figures 1-3 and 5-6, a black jumbo dot means a point whose all coordinates are odd (for brevity, a completely odd point), the symbols ■ means a point in $\mathbf{Z}^n$ whose all coordinates are even (for short a completely even point) and • means a mixed point. Under the H-topology $(\mathbf{Z}^n, \gamma^n)$, for a given point $p \in \mathbf{Z}^n$ since the smallest open set of the point p, denoted by $O_H(p)$, plays an important role in studying H-topological spaces, let us investigate $O_H(p)$ around the origin point $(0, 0, ..., 0) := 0_n \in \mathbf{Z}^n$.

Example 3.4. Under the H-topological structure of $(\mathbf{Z}^3, \gamma^3)$, for the origin $0_3 \in \mathbf{Z}^3$ we obtain $O_H(0_3) = \{x_i \,|\, i \in [1, 14]_{\mathbf{Z}}\} \cup \{0_3\}$ shown at Figure 1(1). Indeed, for the point $p := (1, 1, 0) \in \mathbf{Z}^3$ we also have $O_H(p)$ which is the same shape of $O_H(0_3)$ (see Definition 3.1). Next, for the point $x_{14} := (0, 0, 1) \in \mathbf{Z}^3$ in Figure 1(2), according to the H-topological structure from Definition 3.1, we obtain $O_H(x_{14}) = \{x_i \,|\, i \in [10, 14]_{\mathbf{Z}}\}$. Similarly, for the point $x_8 := (1, 0, 0)$ we obtain $O_H(x_8) = \{x_i \,|\, i \in \{3, 8, 12\}\}$ and finally, for the point $x_{13} := (0, 1, 1)$ $O_H(x_{13}) = \{x_{13}\}$.

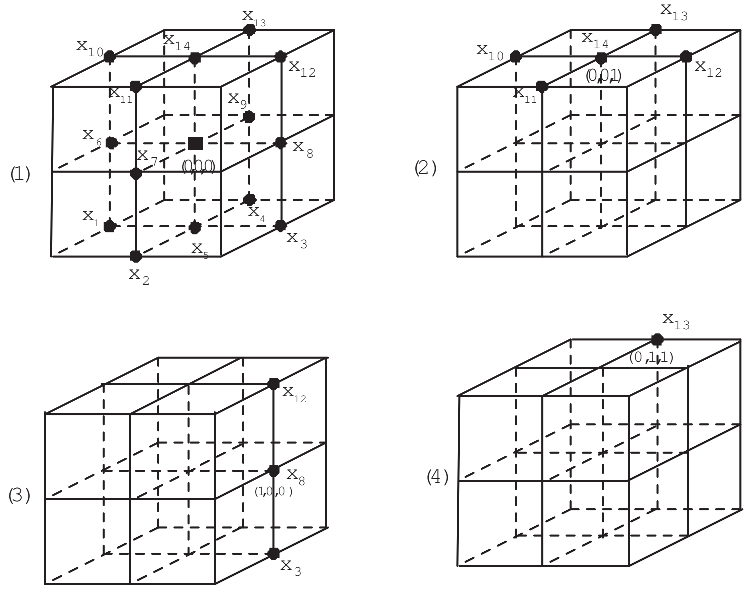

Figure 1: Configuration of $O_H(p)$ under $(\mathbf{Z}^3, \gamma^3)$, where (1) $p = 0_3 \in \mathbf{Z}^3$; (2) p=(0,0,1); (3) p=(1,0, 0); (4) p=(0,1,1).

We can consider naturally an H-continuous map and an H-homeomorphism. For two H-topological spaces $(X, {\gamma^n}_X) := X$ and $(Y, {\gamma^n}_Y) := Y$ a function $f : X \to Y$ is said to be H-continuous at a point $x \in X$ if f is continuous at the point x from the viewpoint of H-topology. In other words, we can represent it as follows:

$$f(O_H(x)) \subset O_H(f(x)). \tag{3.1}$$

Furthermore, a map $f : X \to Y$ is H-continuous if it is H-continuous at every point $x \in X$. For two H-topological spaces $(X, {\gamma^n}_X) := X$ and $(Y, {\gamma^n}_Y) := Y$, a function $f : X \to Y$ is said to be an H-homeomorphism if f is an H-continuous bijection and the inverse f^{-1} is H-continuous.

Example 3.5. Under $(\mathbf{Z}^3, \gamma^3)$, the two spaces in Figure 2 (4) and (6) are H-homeomorphic to each other. Indeed, they have four closed points with the relative topology induced by $(\mathbf{Z}^3, \gamma^3)$.

By Corollary 3.2, we obtain the following:

Proposition 3.6. *$(\mathbf{Z}^m, \gamma^m)$ cannot be H-homeomorphic to $(\mathbf{Z}^n, \gamma^n)$ if $m \neq n$.*

In digital topology simple examples of locally finite $T_{\frac{1}{2}}$-spaces (not T_1) are the Khalimsky line $(\mathbf{Z}, \kappa)$ [27] and the M-topology $(\mathbf{Z}^2, \gamma^2)$. Recall that a set A of a topological space X is called semi-open [24] if there is an open set O such that $O \subset A \subset ClO$, where "Cl" means the closure operator. The semi-closed sets are defined as the complements to the semi-open sets. The separation axioms semi-T_i, where $i = 0, \frac{1}{2}$ etc (see [3, 26]), are obtained from the definitions of the usual separation axioms T_i by the replacing of open sets by semi-open ones. For example, a space X satisfies the separation axiom $T_{\frac{1}{2}}$ [7] if for each point p of X the set $\{p\}$ is either open or closed, *i.e.* for each point p of X at least one of the sets $\{p\}$, $X \setminus \{p\}$ is open. Hence, a space X satisfies the separation axiom semi-$T_{\frac{1}{2}}$ if for each point p of X at least one of the sets $\{p\}$, $X \setminus \{p\}$ is semi-open, *i.e.* for each point p of X the set $\{p\}$ is either semi-open or semi-closed [6]. As a rule (*cf.* [6]) the axiom T_i implies the axiom semi-T_i but the converse does not hold. It is clear that the products $X \times Y$, where X, Y are either $(\mathbf{Z}, \kappa)$ or $(\mathbf{Z}^2, \gamma^2)$, are not $T_{\frac{1}{2}}$ (even not $T_{\frac{1}{4}}$, see [2] for the definition). But they are evidently T_0-spaces. So are their subspaces.

Since digital spaces are so related to both an Alexandroff topological structure and low level separations axioms, the recent paper [5] proved the following:

Proposition 3.7. *[5] A T_0-Alexandroff space is a semi-$T_{\frac{1}{2}}$-space.*

Since each of an M-topological space and a K-topological space is a T_0-Alexandroff space [1], they are proved to be semi-$T_{\frac{1}{2}}$-spaces [5, 13]. Since the H-topology is an Alexandroff space with T_0-separation axiom, we obtain the following:

Corollary 3.8. *The H-topology $(\mathbf{Z}^n, \gamma^n)$ is a semi-$T_{\frac{1}{2}}$-space.*

Definition 3.9. In $(\mathbf{Z}^n, \gamma^n)$, two distinct points x, y in $\mathbf{Z}^n$ are H-adjacent if $y \in O_H(x)$ or $x \in O_H(y)$.

By using this notion, let us now establish the following terminology which can be used to study H-topological spaces.

Definition 3.10. Let $(X, \gamma^n{}_X) := X$ be an H-topological space. Then we define the following:
(1) Two distinct points $x, y \in X$ are called *H-path connected* if and only if there is a finite sequence (or a path) $(x_0, x_1, ..., x_m)$ on X with $\{x_0 = x, x_1, ..., x_m = y\}$ such that $\{x_i, x_{i+1}\}$ is H-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \geq 1$. Besides, the number m is called the *length* of this H-path. Furthermore, an H-path is called a closed H-curve if $x_0 = x_m$.

(2) A *simple H-path* in X is an H-path $(x_i)_{i\in[0,m]_{\mathbf{Z}}}$ such that the set $\{x_i, x_j\}$ are H-adjacent if and only if $|i - j| = 1$.
Furthermore, we say that a simple closed H-curve with l elements $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$, denoted by $SC_H^{n,l}, l \geq 4$, is a simple H-path with $x_0 = x_l$ if and only if $|i - j| = 1(mod\, l)$.

Example 3.11. Let us consider the spaces in Figure 2 with the relative topologies on the given sets. Then we obtain the following:

(1) Under $(\mathbf{Z}^3, \gamma^3)$, each of the spaces in Figure 2(4) and (6) is a kind of $SC_H^{3,8}$.

(2) Under $(\mathbf{Z}^3, \gamma^3)$, the space in Figure 2(5) cannot be an $SC_H^{3,8}$ because of the completely even points in the space such as w_4 and w_8. More precisely, we obtain the smallest open set of the point w_4, *i.e.* $O_H(w_4) = \{w_i \,|\, i \in [2, 6]_{\mathbf{Z}}\}$.

(3) Under $(\mathbf{Z}^3, \gamma^3)$, while the space in Figure 2(7) is an $SC_H^{3,6}$, the space $Y := (y_i)_{i\in[0,5]_{\mathbf{Z}}}$ in Figure 2(8) cannot be an $SC_H^{3,6}$ because each of the points $y_0 := (0, -1, 0)$ and $y_3 := (0, 1, 1)$ in the space (Y, γ_Y^3) is not H-connected with the other points, which implies that this space is disconnected.

(4) Under $(\mathbf{Z}^3, \gamma^3)$, the space in Figure 2(9) cannot be an $SC_H^{3,4}$ because the space is disconnected from the viewpoint of the H-topology because every singleton $\{t_i\}_{i\in[1,4]_{\mathbf{Z}}}$ is a smallest open set, which means the space in Figure 2(9), denoted by (T, γ_T^3), is a discrete topological space derived from H-topology.

Reminding Example 3.11, in order to investigate some difference between the H-topology and the K-topology, let us examine the spaces in Figure 2 from the viewpoint of K-topology. Let KTC be the category of K-topological spaces.

Example 3.12. (1) Under $(\mathbf{Z}^3, \kappa^3)$, each of the spaces in Figure 2(4) and (6) is a kind of $SC_K^{3,8}$.

(2) Under $(\mathbf{Z}^3, \kappa^3)$, the space in Figure 2(5) cannot be an $SC_K^{3,8}$ (see the points w_4, w_8).

(3) Under $(\mathbf{Z}^3, \kappa^3)$, the space in Figure 2(7) cannot be an $SC_K^{3,6}$ (see the points y_0, y_2).

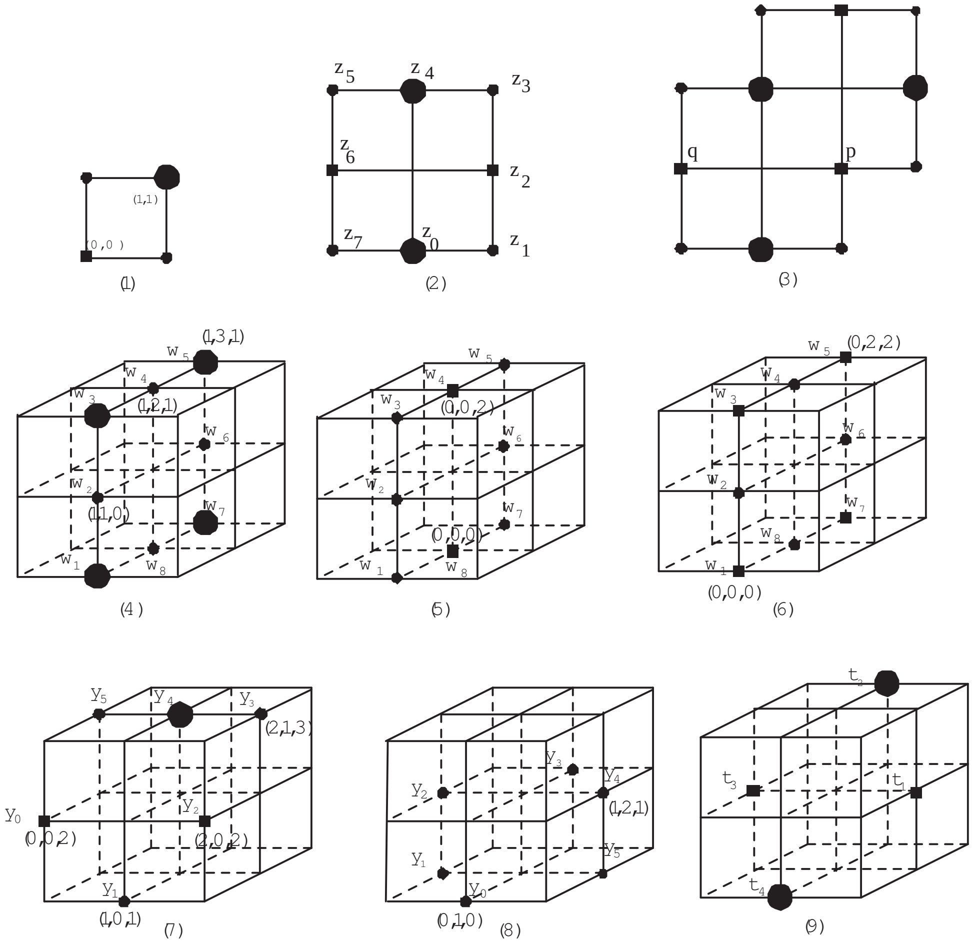

Figure 2: Configuration of several types of an $SC_H^{n,l}$, $n \in \{2, 3\}$ (see (1)-(3), (4), (6) and (7)): The spaces in (5),(8) and (9) cannot be $SC_H^{3,l}$ for the given l elements.

Remark 3.13. In view of Examples 3.11 and 3.12, compared with the K-topology, it turns out that the H-topology has its own features (see also Figure 2(7)).

Let us now establish the category of H-topological spaces which is an extension of MTC, denoted by HTC, as follows:

(1) The set of objects $(X, \gamma^n{}_X)$,

(2) For every ordered pair of objects $(X, \gamma^n{}_X)$ and $(Y, \gamma^n{}_Y)$, the set of all H-continuous maps $f : (X, \gamma^n{}_X) \to (Y, \gamma^n{}_Y)$ as morphisms.

Let us now compare between the H-topology and the K-topology.

Theorem 3.14. *The H-topology* $(\mathbf{Z}^n, \gamma^n)$ *cannot be compatible with the K-topology on* $\mathbf{Z}^n$ *up to H-homeomorphism or K-homeomorphism, where* $n \neq 1$.

Before proving this assertion, we need to recall the following: $(\mathbf{Z}, \gamma)$ is equivalent to the Khalimsky line $(\mathbf{Z}, \kappa)$ and further, $(\mathbf{Z}^2, \gamma^2)$ cannot be compatible with the digital plane with the K-topology.

Proof. In case of $n = 3$, let us investigate the smallest open sets of the points around the point $p := 0_3 \in \mathbf{Z}^3$. More precisely, without loss of generality it suffice to examine the following eight points around the origin

$(0,0,0) \in \mathbf{Z}^3$. Put

$$\begin{cases} q_0 := (p_0, 0), q_1 := (p_0, 1), \text{ where } p_0 := (0,0) \in \mathbf{Z}^2, \\ q_2 := (p_1, 0), q_3 := (p_1, 1), \text{ where } p_1 := (0,1) \in \mathbf{Z}^2, \\ q_4 := (p_2, 0), q_5 := (p_2, 1), \text{ where } p_2 := (1,0) \in \mathbf{Z}^2, \\ q_6 := (p_3, 0), q_7 := (p_3, 1), \text{ where } p_3 := (1,1) \in \mathbf{Z}^2. \end{cases} \tag{3.2}$$

Then we obtain the following smallest open sets of the given points $q_i, i \in [0,7]_{\mathbf{Z}}$ (see Figure 1).

$$\begin{cases} O_H(q_0) = N_4(p_0) \times [-1,1]_{\mathbf{Z}}, O_H(q_1) = N_4(p_0) \times \{1\}, \\ O_H(q_2) = \{p_1\} \times [-1,1]_{\mathbf{Z}}, O_H(q_3) = \{q_3\}, \\ O_H(q_4) = \{p_2\} \times [-1,1]_{\mathbf{Z}}, O_H(q_5) = \{q_5\}, \text{ and} \\ O_H(q_6) = N_4(p_3) \times [-1,1]_{\mathbf{Z}}, O_H(q_7) = N_4(p_3) \times \{1\}. \end{cases} \tag{3.3}$$

Meanwhile, for the given points $q_i, i \in [0,7]_{\mathbf{Z}}$ under the K-topology $(\mathbf{Z}^3, \kappa^3)$ we have their smallest open sets $SN(q_i)$ as follows:

$$\begin{cases} SN(q_0) = N_{26}(q_0), SN(q_1) = N_8(p_0) \times \{1\}, \\ SN(q_2) = \cup_{t \in [-1,1]_{\mathbf{Z}}} \{t\} \times [-1,1]_{\mathbf{Z}} \\ SN(q_3) = \{(\pm 1, 1, 1), q_3\}, \\ SN(q_4) = \{1\} \times N_8(0,0), \\ SN(q_5) = \{(1, \pm 1, 1), q_5\}, \\ SN(q_6) = \{(1,1,\pm 1), q_6\}, \text{ and} \\ SN(q_7) = \{q_7\}. \end{cases} \tag{3.4}$$

In view of (3.3) and (3.4), we obtain

$$\begin{cases} O_H(q_i) \subsetneq SN(q_i), i \in [0,5]_{\mathbf{Z}} \text{ and} \\ SN(q_j) \subsetneq O_H(q_j), j \in \{6,7\}. \end{cases} \tag{3.5}$$

and further, their cardinalities are the following (see (3.3) and (3.4)):

$$\begin{cases} \sharp O_H(q_0) = 15, \sharp O_H(q_1) = 5, ..., \sharp O_H(q_7) = 5 \text{ and} \\ \sharp SN(q_0) = 27, SN(q_1) = 8, ..., \sharp SN(q_7) = 1. \end{cases} \tag{3.6}$$

Owing to the properties of (3.5) and (3.6), we prove that $(\mathbf{Z}^3, \gamma^3)$ is not compatible with $(\mathbf{Z}^3, \kappa^3)$ up to H-homeomorphism or K-homeomorphism because these two homeomorphisms are established in terms of smallest open sets according to their corresponding topologies. By using the method similar to the proof of the case $n = 3$, in view of the H-topological structure from Definition 3.1, we prove that $(\mathbf{Z}^n, \gamma^n), n \geq 4$ is not compatible with $(\mathbf{Z}^n, \kappa^n)$ either. □

In view of Definition 3.1 (see Figure 1 and Figure 2(9)), we obtain the following property which is quite different from the K-topology.

Corollary 3.15. *Under (X, γ^3_X), depending on the situation of X, the HA-space X induces a* 6- *or an* 18-*adjacency relation. Namely, (X, γ^3_X) does not support a* 26-*adjacency relation.*

4. A Development of an *HA*-Map which is an *H*-Connectedness Preserving Map

In view of the H-continuity of a map between two H-topological spaces (see (3.1)), an H-continuous map is so rigid that it has some limitations of geometric transformations.

Remark 4.1. Consider the space (Y, γ^3_Y) in Figure 2(7), where $Y := \{y_i | i \in [0,5]_{\mathbf{Z}}\}$. Consider self maps f_1 and f_2 of (Y, γ^3_Y) in such ways:
(1) $f_1 : (Y, \gamma^3_Y) \to (Y, \gamma^3_Y)$ given by $f_1(y_i) = y_{i+1(mod\,6)}$ as one click geometric transformation. Then it is clear that f_1 is not an H-continuous map because the smallest open set of y_i is the set $\{y_{i-1(mod\,6)}, y_i, y_{i+1(mod\,6)}\}$ if $i \in \{0,2,4\}$ and the smallest open set of y_j is the set $\{y_j\}$ if $j \in \{1,3,5\}$.

(2) $f_2 : (Y, \gamma^3_Y) \to (Y, \gamma^3_Y)$ given by $f_1(y_i) = y_{i+2(mod\,6)}$ as a two-clicks geometric transformation. Then f_2 is an H-continuous map.

Owing to this property of an H-continuous map (in particular, see Remark 4.1(1)), we need to formulate another map preserving H-connectedness. Hence let us now establish a map which is broader than an H-continuous map. To do this work, first of all, we need to consider the H-topological adjacency of $(\mathbf{Z}^n, \gamma^n)$.

Under the H-topology $(\mathbf{Z}^n, \gamma^n)$, let us define the notions of an adjacency relation for two points in $\mathbf{Z}^n$ and an H-adjacency (for short HA-) neighborhood of a point $x \in \mathbf{Z}^n$. Furthermore, by using an HA-map, we establish the notion of an HA-isomorphism. These notions will play important roles in studying digital nD spaces.

Definition 4.2. Given a point $p \in \mathbf{Z}^n$, we define the (smallest) HA-neighborhood of p, denoted by $HA(n,p)$ (for short $HA(p)$ if there is no danger of ambiguity), as follows:

$$HA(n,p) = \{q \mid p \text{ is } H\text{-adjacent to } q\}. \tag{4.1}$$

In view of (4.1), we observe that $p \notin HA(n,p)$. Namely, we see that the H-adjacency holds only the *symmetric relation* without the reflexive relation. Since $(\mathbf{Z}^n, \gamma^n)$ is an Alexandroff topological space, we obtain the following:

Remark 4.3. Under $(\mathbf{Z}^n, \gamma^n)$, for two distinct points there is an equivalence between their H-adjacency and H-connectedness.

Under $(\mathbf{Z}^n, \gamma), n \geq 3$, owing to the property of H-connectedness, for any point $p \in \mathbf{Z}^n$ we clearly observe the following (see Figure 1)

Theorem 4.4. *Assume that p(resp. p') is a completely even point in $\mathbf{Z}^n$(resp. $\mathbf{Z}^{n-1}$), e.g. $p := 0_n \in \mathbf{Z}^n$ as the origin point of $\mathbf{Z}^n$ (resp. $p' := 0_{n-1} \in \mathbf{Z}^{n-1}$). Then we obtain $\sharp HA(n,p) = 3\sharp HA(n-1,p') + 2, n \geq 3$, where $\sharp HA(n,p)$ (resp. $\sharp HA(n-1,p')$) means the cardinality of the set $HA(n,p)$(resp. $HA(n-1,p')$) in $\mathbf{Z}^n$ (resp. $\mathbf{Z}^{n-1}$).*

Proof. For the given point $p := (\overbrace{0, \cdots, 0}^{n\text{-tuples}}) \in \mathbf{Z}^n$, in view of the product topological construction of the H-topology $(\mathbf{Z}^n, \gamma^n), n \geq 3$, we see that

$$\sharp HA(n,p) = 3(\sharp HA(n-1,p') + 1) - 1 = 3\sharp HA(n-1,p') + 2, \tag{4.2}$$

which proves the assertion. □

Example 4.5. In view of (4.2) and Definition 3.1, we obtain that for the origin point $p := (\overbrace{0, \cdots, 0}^{n\text{-tuples}}) \in \mathbf{Z}^n, n \geq 3$

$$\sharp HA(3,p) = 14, \sharp HA(4,p) = 44 \text{ and so forth.}$$

Definition 4.6. For an H-topological space $(X, {\gamma^n}_X)$ consider two H-topological spaces $(A, {\gamma^n}_A) := A$ and $(B, {\gamma^n}_B) := B$ such that A and B are nonempty subsets of X. Then we say that two subspaces A and B of X are H-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are H-adjacent to each other.

In order to compare the H-adjacency with the K-adjacency, let us now recall some basic notions and terminology on K-topology. In $(\mathbf{Z}^n, \kappa^n)$, we say that two distinct points x and y are K-adjacent if $y \in SN(x)$ or $x \in SN(y)$ [28], where $SN(x)$ is the smallest open neighborhood of the point x. For a point $p \in \mathbf{Z}^n$, since a permutation of coordinates and a translation by an even vector is a homeomorphism of $\mathbf{Z}^n$ onto itself [33], we can consider the point p as $p = (p_i)_{i \in [1,n]_{\mathbf{Z}}}$ where there are consecutively k even coordinates from 1 to α coordinates and the other $n - \alpha$ coordinates are odd such as $p := (\overbrace{0, \cdots, 0}^{\alpha}, \overbrace{1, \cdots, 1}^{\beta}) :=< \alpha, \beta >$, where there are α zeros and $n - \alpha$ ones. Let $p :=< \alpha, \beta >$ be a point in $\mathbf{Z}^n$. For a point $p \in \mathbf{Z}^n$ under $(\mathbf{Z}^n, \kappa^n)$, the Khalimsky adjacency neighborhood is denoted by $KA(p)$ and further, it turns out that [33] (see also [19])

$$\sharp KA(p) = (3^\alpha - 1) + (3^\beta - 1) = 3^\alpha + 3^\beta - 2. \tag{4.3}$$

Owing to the properties of (2.3), (4.2) and (4.3), we obtain the following:

Theorem 4.7. *For $\mathbf{Z}^n, n \geq 3$, the K-, the H-adjacency and the digital k-adjacency of (2.3) are not compatible with each other.*

Proof. By Theorem 4.4 and Example 4.5, and owing to the property (4.3) and the k-adjacency in (2.2), the proof is completed. More precisely, assume $p := 0_3$, $q := (0, 0, 1)$ and $r = (1, 0, 0)$ in $\mathbf{Z}^3$

$$\left\{\begin{array}{l} \sharp KA(p) = 26, \sharp HA(p) = 14, k \in \{6, 18, 26\}, \\ \sharp KA(q) = 10, \sharp HA(q) = 6, k \in \{6, 18, 26\} \text{ and} \\ \sharp KA(r) = 10, \sharp HA(r) = 4, k \in \{6, 18, 26\}. \end{array}\right\} \tag{4.4}$$

After comparing the cardinalities of adjacency neighborhoods of p and q in (4.4), we complete the proof. □

Let us now establish an H-adjacency neighborhood of a given point $x \in (X, \gamma^n)$ as follows:

$$HN(x) := HA(x) \cup \{x\} \tag{4.5}$$

Since any point $x \in (X, \gamma^n)$ always has $HN(x)$, we may consider the following map.

Definition 4.8. For two HA-spaces $(X, \gamma_X^{n_0}) := X$ and $(Y, \gamma_Y^{n_1}) := Y$, we say that a function $f : X \to Y$ is an HA-map at a point $x \in X$ if

$$f(HN(x)) \subset HN(f(x)).$$

Furthermore, we say that a map $f : X \to Y$ is an HA-map if the map f is an HA-map at every point $x \in X$.

In view of Definitions 3.1 and 4.8, Remarks 4.1 and 4.3, we obtain the following:

Remark 4.9. (1) While an H-continuous map implies an HA-map, the converse does not hold.
(2) An HA-map is an H-connectedness preserving map.

Using HA-maps, we establish an *HA-category*, denoted by HAC, consisting of two sets.

(1) For any set $X \subset \mathbf{Z}^n$, the set of HA-spaces as objects of HAC,

(2) For every ordered pair of objects $(X, \gamma_X^{n_0}) := X$ and $(Y, \gamma_Y^{n_1}) := Y$, the set of all HA-maps $f : X \to Y$ as morphisms of HAC.

Since the inverse of an HA-map (resp. H-continuous map) need not be an HA-map (resp. H-continuous map), we need to establish the following notion.

Definition 4.10. For two spaces $(X, \gamma_X^{n_0}) := X$ and $(Y, \gamma_Y^{n_1}) := Y$, a map $h : X \to Y$ is called an HA-isomorphism if h is a bijective HA-map (for short HA-bijection) and further, $h^{-1} : Y \to X$ is an HA-map.

In Definition 4.10, we denote by $X \approx_{HA} Y$ an HA-isomorphism from X to Y.

5. A Comparison Among Simple Closed Curves such as $SC_K^{n,l}$, $SC_k^{n,l}$ and $SC_H^{n,l}$

In digital topology, since the study of simple closed curves associated with the given topologies plays an important role in classifying digital spaces, this section investigates some properties of $SC_H^{n,l}$ and compares $SC_H^{n,l} \in HTC$ with both $SC_K^{n,l} \in KTC$ and $SC_k^{n,l}$. To do this work, we need to investigate the following:

Theorem 5.1. *SC_H^{n,l_1} is H-homeomorphic to SC_H^{n,l_2} if and only if $l_1 = l_2$.*

Proof. Before proceeding the proof, we recall some properties of $SC_H^{n,l} := (x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$. For $i \in [0, l-1]_{\mathbf{Z}}$, the subspace $\{x_i, x_{i+1(mod\, l)}\}$ induced by $(\mathbf{Z}^n, \gamma^n)$ is H-connected and further, each point $x_i \in SC_H^{n,l}, i \in [0, l-1]_{\mathbf{Z}}$, has the smallest open set, as follows:

$$\left\{\begin{array}{l} O_H(x_i) = \{x_{i-1(mod\, l)}, x_i, x_{i+1(mod\, l)}\} \text{ or} \\ O_H(x_i) = \{x_i\} \end{array}\right\} \tag{5.1}$$

in terms of the property of $SC_H^{n,l}$ (see Definition 3.10(2) and the structure $O_H(x_i)$ shown in Figure 1). Furthermore, in (5.1), in case $O_H(x_i) = \{x_{i-1(mod\, l)}, x_i, x_{i+1(mod\, l)}\}$, we obtain $O_H(x_{i+1(mod\, l)}) = \{x_{i+1(mod\, l)}\}$, and in case $O_H(x_i) = \{x_i\}$, we have

$$O_H(x_{i+1(mod\, l)}) = \{x_{i(mod\, l)}, x_{i+1(mod\, l)}, x_{i+2(mod\, l)}\}.$$

Let us now consider the two spaces $SC_H^{n,l_1} := (x_i)_{i\in[0,l_1-1]_{\mathbf{Z}}}$ and $SC_H^{n,l_2} := (y_j)_{j\in[0,l_2-1]_{\mathbf{Z}}}$. If $l_1 = l_2$, then owing to the property (5.1), we have a map $h : SC_H^{n,l_1} \to SC_H^{n,l_2}$ satisfying that the restriction map to $O_H(x_i)$, denoted by $h|_{O_H(x_i)}$, is (locally) H-homeomorphic to $O_H(y_j)$ for each $i \in [0, l_1 - 1]_{\mathbf{Z}}$ because the numbers $l_i, i \in \{1, 2\}$ are even. Thus the map h is an H-homeomorphism.
Conversely, if SC_H^{n,l_1} is H-homeomorphic to SC_H^{n,l_2}, then it is clearly $l_1 = l_2$. □

Let us study the set $SC_{2n}^{n,l}$ in the H-topological approach.

Theorem 5.2. *For the set $SC_{2n}^{n,l} := X$, the subspace (X, γ_X^n) becomes an $SC_H^{n,l}$ depending on the situation as follows.*
(1) For every $SC_4^{2,l} := X$ the subspace (X, γ_X^2) always becomes an $SC_H^{2,l}$.

(2) For every $SC_{2n}^{n,l} := X, n \geq 3$, the subspace (X, γ_X^n) need be neither an $SC_H^{n,l}$ nor an $SC_K^{n,l}$. The validity depends on the situation.

Proof. (1) As shown in (5.1), each point $x_i \in (X, \gamma_X^2)$ has the following property: $O_H(x_i) = \{x_{i-1(mod\, l)}, x_i, x_{i+1(mod\, l)}\}$ induced by the set $N_4(x_i, 1)$ or $O_H(x_i) = \{x_i\}$ and further, if $\sharp O_H(x_i) = 3$, then $\sharp O_H(x_{i+1(mod\, l)}) = 1$. Owing to the property of $SC_H^{n,l}$ (see Definition 3.10(2) and the structure $O_H(x_i)$ shown in Figure 1), the proof is completed because the number l is even.

(2) Consider the two spaces in Figure 3(1) and (2). While they are $SC_6^{3,8}$, they are neither $SC_H^{3,8}$ nor $SC_K^{3,8}$. More precisely, take the completely even points in Figure 3(1), *e.g.* x_2, x_6 and the completely odd points in Figure 3(2), *e.g.* x_2, x_6. Owing to these points, they cannot be $SC_H^{3,8}$. By using the method similar to the above proof, we prove that the spaces in Figure 3(1) and (2) cannot be an $SC_K^{3,8}$ either. However, the space in Figure 3(3) is both an $SC_H^{3,8}$ and an $SC_K^{3,8}$. □

Remark 5.3. For the set $SC_{18}^{3,l} := X$, the subspace (X, γ_X^n) (resp.(X, κ_X^n)) need not be an $SC_H^{3,l}$ (resp. $SC_K^{3,l}$). The validity depends on the situation.

Proof. Owing to the smallest open set structures from the H- and the K-topology, the assertion is proved. For instance, consider the space in Figure 3 (5). While it is an $SC_H^{3,6}$, it cannot be an $SC_K^{3,6}$ owing to the completely even points y_0 and y_2 in the space. Motivated by the subset $X_1 := \{(1, 1, 2), (2, 2, 2), (3, 3, 2)\}$ of Figure 3(4), we can easily take $SC_{18}^{3,l} := X$ such that (X, κ_X^3) is neither $SC_K^{3,l}$ nor $SC_H^{3,l}$. □

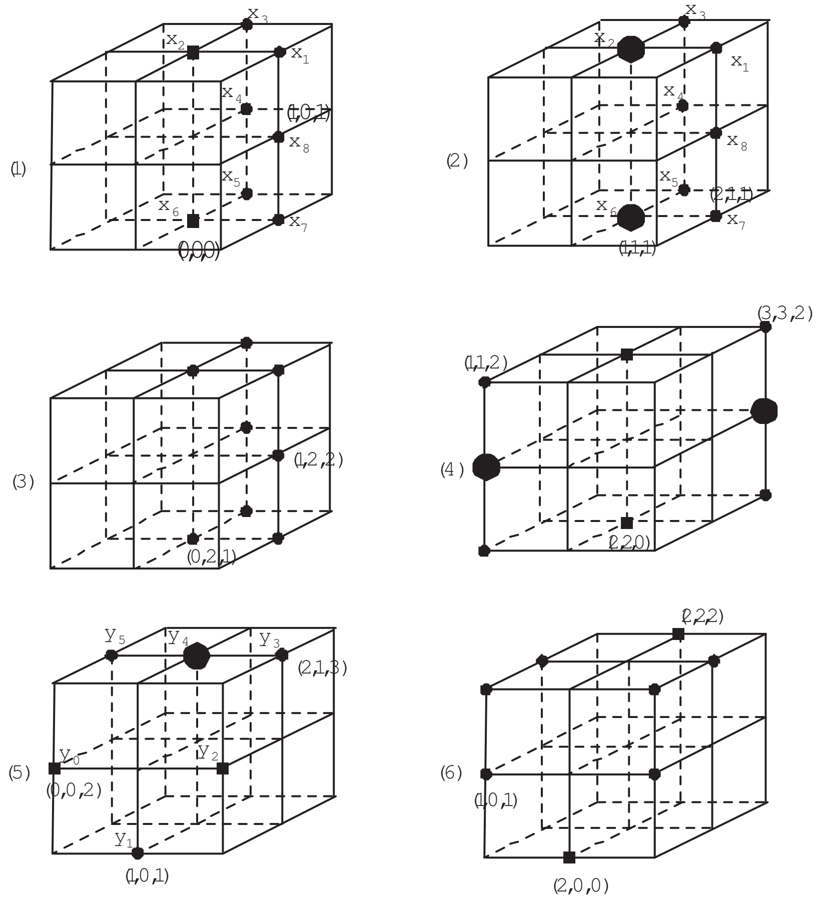

Figure 3: Similarity and difference among simple closed curves such as $SC_K^{n,l}$, $SC_k^{n,l}$ and $SC_H^{n,l}$.

By Corollary 3.15, we obtain the following:

Corollary 5.4. *Consider the set* $SC_k^{3,l} = (x_i)_{i\in[0,l-1]_{\mathbf{Z}}} := X$, *where the adjacency* $k = 26$. *Then the subspace* (X, γ_X^3) *need not be an* $SC_H^{3,l}$.

Proof. Owing to the structure of the smallest open set of the point $x \in SC_{26}^{3,l} := X$, the assertion is valid. For instance, consider $SC_{26}^{3,4} := X$ in Figure 2(9). Then it is clear that under H-topology each point of X has the singleton as its smallest open set. Namely, the space (X, γ_X^3) is a discrete space from the viewpoint of the H-topology. Meanwhile, consider the set $Y := \{w_2, w_4, w_6, w_8\}$ in Figure 2(5). Then the set Y is a kind of $SC_k^{3,4}, k = in\{18, 26\}$ and further, (Y, γ_Y^3) can be $SC_H^{3,4}$. □

In Corollary 5.4, in case $SC_k^{3,4} := X$ such that $k = 26$ and $k \neq 18$, (X, γ_X^3) is not an $SC_H^{3,4}$.

Let us now investigate similarity and difference among simple closed H-and K-curves and digital k-connectivity, $k \in \{6, 18\}$.

Space No. in Figure 3 \ Geometric feature	Simple closed H-curve	Simple closed K-curve	Digital k-onnectivity
1	No	No	6-connectivity
2	No	No	6-connectivity
3	Yes	Yes	6-connectivity
4	No	No	18-connectivity
5	Yes	No	18-connectivity
6	No	Yes	18-connectivity

Figure 4: For the spaces in Figure 3, an observation of similarity and difference among simple closed *H*-and *K*-curves and digital *k*-connectivity, $k \in \{6, 18\}$.

Remark 5.5. In view of Theorems 3.14, 5.2(1) and Remark 5.3, we observe that there are some similarity and difference among simple closed *H*-and *K*-curves and digital *k*-connectivity, $k \in \{6, 18\}$ (see Figure 4).

6. An *HA*-Retract and an Extension of an *HA*-Map in the Category *HAC*

In view of Theorem 4.7, we see that an *HA*-map is different from several continuous maps in [11]. Since an extension problem plays an important role in *HAC*, this section introduces the notion of an *HA*-retract and studies an extension problem of an *HA*-map. Motivated by the several kinds of retracts in [14], we now introduce a new retract for *HA*-spaces and study its properties related to an extension problem of an *HA*-map. Owing to Theorem 3.14, Corollary 3.15, the following *HA*-retract is different from the *K*-, *KA*-, *KD*-*k*-retract in [14].

Definition 6.1. In *HAC*, we say that an *HA*-map $r : (X', \gamma^n_{X'}) \to (X, \gamma^n_X)$ is an *HA*-retraction if
(1) $X \subset X'$, and
(2) $r(x) = x$ for all $x \in X$.
Then we say that (X, γ^n_X) is an *HA*-retract of $(X', \gamma^n_{X'})$. Furthermore, we say that a point $x \in X' \setminus X$ is *HA*-retractable.

In view of Theorem 4.7 and Corollary 3.15 and 5.4, an *HA*-retract is different from *K*-, *M*-, *MA*-, *A*-, *k*-, *KD*-*k*-retracts in [14, 15, 20, 21].

Example 6.2. Consider the space $(X', \gamma^3_{X'})$ in Figure 5, where $X' = \{x_i | i \in [1, 6]_{\mathbb{Z}}\}$. Assume that $X = X' \setminus \{x_3, x_5\}$ in Figure 5. Consider the map $r : (X', \gamma^3_{X'}) \to (X, \gamma^3_X)$ given by $r(\{x_3, x_5\}) = \{x_4\}$ and $r(x_i) = x_i, i \in \{1, 2, 4, 6\}$. Then r is an *HA*-retraction.

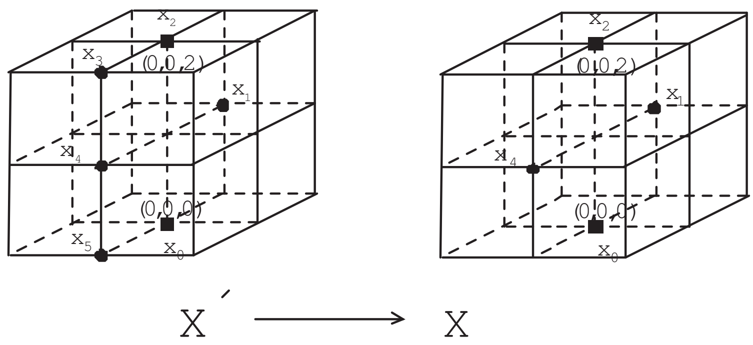

Figure 5: Configuration of an HA-retract from $(X', \gamma^3_{X'})$ onto (X, γ^3_X).

Motivated by the digital isomorphic property of the several types of retracts in [14], we obtain the following property of an HA-isomorphism.

Theorem 6.3. *In HAC, let (X, γ^n_X) be an HA-retract of $(X', \gamma^n_{X'})$ and let $h : (X', \gamma^n_{X'}) \to (Y, \gamma^{n_1}_Y)$ be an HA-isomorphism. Then $h((X, \gamma^n_X))$ is an HA-retract of $(Y, \gamma^{n_1}_Y)$.*

Proof. Let $r : (X', \gamma^n_{X'}) \to (X, \gamma^n_X)$ be an HA-retraction. Then $h \circ r \circ h^{-1} : (Y, \gamma^{n_1}_Y) \to h((X, \gamma^n_X))$ is an HA-retraction because the composition of HA-maps is also an HA-map. □

In topology [4], we recall the following: Let (X', T') be a topological space and (X, T) a subspace of (X', T'). Then (X, T) is a retract of (X', T') if and only if every continuous map $f : (X, T) \to (Y, T_1)$ has a continuous map $F : X' \to Y$ such that $F|_X = f$ for any (Y, T_1) [4]. Even though an HA-map need not be an H-continuous map, motivated by this approach, we need to study an extension problem of an HA-map.

Theorem 6.4. *In HAC, (X, γ^n_X) is an HA-retract of $(X', \gamma^n_{X'})$ if and only if every HA-map $f : (X, \gamma^n_X) \to (Y, \gamma^{n_1}_Y)$ has an HA-map $F : (X', \gamma^n_{X'}) \to (Y, \gamma^{n_1}_Y)$ such that $F|_X = f$ for any $(Y, \gamma^{n_1}_Y)$.*

Proof. Let $r : (X', \gamma^n_{X'}) \to (X, \gamma^n_X)$ be an HA-retraction and $f : (X, \gamma^n_X) \to (Y, \gamma^{n_1}_Y)$ an HA-map. Then the composition $F := f \circ r : (X', \gamma^n_{X'}) \to (Y, \gamma^{n_1}_Y)$ is an HA-map which is an extension of f.

Conversely, suppose that every HA-map $f : (X, \gamma^n_X) \to (Y, \gamma^{n_1}_Y)$ has an extension $F : (X', \gamma^n_{X'}) \to (Y, \gamma^{n_1}_Y)$ for every $(Y, \gamma^{n_1}_Y)$ as an HA-map. Then the identity map $1_{(X, \gamma^n_X)}$ has an extension $r : (X', \gamma^n_{X'}) \to (X, \gamma^n_X)$ as an HA-map. Thus (X, γ^n_X) is an HA-retract of $(X', \gamma^n_{X'})$. □

Let us consider an example guaranteeing Theorem 6.4, as follows:

Example 6.5. Let us consider the map $f : (X, \gamma^2_X) \to (Y, \gamma_Y)$ given by $f(\{x_0, x_1, x_4\}) = \{1\}, f(x_2) = 2, f(x_3) = 3$ (Figure 6). Then it is clear that f is an HA-map. Then there is no extension map $f : (X', \gamma^2_{X'}) \to (Y, \gamma_Y)$, where $X' = X \cup \{(1, 1, 3)\}$ because there is no HA-retraction from $(X', \gamma^2_{X'})$ to (X, γ^2_X).

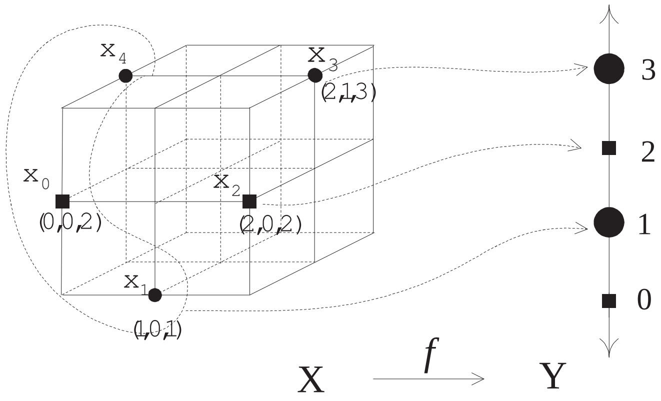

Figure 6: Non-existence of an HA-extension of the given map f to X'.

7. Summary and Further Works

Our goal was to develop a new topology, called as H-topology, for studying nD digital spaces (or nD digital images). Thus the paper has developed the new H-topology which can be used to study digital nD images. Besides, we have developed several notions such as an H-adjacency, an HA-map, an HA-isomorphism and so forth. We have also proved that an HA-map is different from a K-continuous map, a digitally k-continuous map and a K-adjacency map in [19] (see the property (3.6)). Comparing these notions on H-topology with those on K-topology, they have their own merits and utilities depending on the situations (see Theorems 3.14, 4.7 and 5.2, Corollary 3.15, and Remarks 5.3 and 5.5). We observed that an HA-map is not compatible with the several continuous maps in [11]. In HAC we have studied an extension problem of an HA-map and its properties, which is different from the earlier retracts in [14].
As further works, owing to $(\mathbf{Z}^3, \gamma^3)$, for an M-topological space (X, γ_X^2), we can develop the notion of a homotopy on (X, γ_X^2) in an algebraic topological approach [36] which can be used for compressing digital images and doing an HA-homotopic thinning.

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