Filomat 31:3 (2017), 709–721
DOI 10.2298/FIL1703709C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ascent, Essential Ascent, Descent and Essential Descent for a Linear Relation in a Linear Space

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Abstract. For a linear relation in a linear space some spectra defined by means of ascent, essential ascent, descent and essential descent are introduced and studied. We prove that the algebraic ascent, essential ascent, descent and essential descent spectrum of a linear relation in a linear space satisfy the polynomial spectral mapping theorem. As an application of the obtained results we show that the topological ascent, essential ascent, descent and essential descent spectrum verify the polynomial spectral mapping theorem.

1. Introduction

Let T be an operator in a linear space. The ascent and the essential ascent of T are defined by

$$a(T) := min\{\mathbb{N} \cup \{0\} : dim \frac{N(T^{n+1})}{N(T^n)} = 0\},$$

$$a_e(T) := min\{n \in \mathbb{N} \cup \{0\} : dim \frac{N(T^{n+1})}{N(T^n)} < \infty\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and the essential ascent of T are defined to be ∞.
The descent and the essential descent of T are defined by

$$d(T) := min\{n \in \mathbb{N} \cup \{0\} : dim \frac{R(T^n)}{R(T^{n+1})} = 0\},$$

$$d_e(T) := min\{n \in \mathbb{N} \cup \{0\} : dim \frac{R(T^n)}{R(T^{n+1})} < \infty\},$$

respectively, whenever these minima exist. If no such numbers exist the descent and the essential descent of T are defined to be ∞.
These notions permit to define the following algebraic spectra

$$\sigma_a(T) := \{\lambda \in \mathbb{K} : a(T - \lambda) = \infty\}, \;\; \sigma_a^e(T) := \{\lambda \in \mathbb{K} : a_e(T - \lambda) = \infty\},$$

2010 *Mathematics Subject Classification*. Primary 47A06; Secondary 47A53, 47A10
Keywords. Ascent; essential ascent; descent; essential descent; polynomial spectral mapping theorem.
Received: 27 December 2014; Accepted: 16 January 2015
Communicated by Dragan S. Djordjević
The work of T. Álvarez was supported by Micinn (Spain) Grant MTM 2013-45643.
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$$\sigma_d(T) := \{\lambda \in \mathbb{K} : d(T-\lambda) = \infty\}, \ \ \sigma_d^e(T) := \{\lambda \in \mathbb{K} : d_e(T-\lambda) = \infty\}.$$

Mbekhta and Müller [11] have shown that if T is a bounded operator in a Banach space, then the above algebraic spectra verify the polynomial version of the spectral mapping theorem, that is, for a complex polynomial P, $\sigma_*(P(T)) = P(\sigma_*(T))$ where $* \in \{a, a^e, d, d^e\}$. In a recent paper [6], Chafai and Mnif prove that if T is a closed linear relation in a Banach space having a finite dimensional multivalued part, then $\sigma_d(P(T)) = P(\sigma_d(T))$ and $\sigma_d^e(P(T)) = P(\sigma_d^e(T))$ for a complex polynomial P. In the present paper we continue the investigation initiated in [6] and thus we establish that the polynomial spectral mapping property is true for the algebraic spectra of a linear relation in a linear space. Our proofs are purely algebraic and they differ considerably from these in [11] which were based in the concept of regularity. We also remark that the techniques employed by Chafai and Mnif [6, Theorem 4.3] are not applicable when the assumption that the multivalued part of the linear relation be finite dimensional is not required.

Combining the algebraic conditions defining the spectra $\sigma_*(T)$, $* \in \{a, a^e, d, d^e\}$ with a topological condition, for a bounded operator T in a Banach space we can consider the following topological spectra

$$\sigma_{ta}(T) := \mathbb{K}\backslash\rho_{ta}(T) \ \text{ where } \ \rho_{ta}(T) := \{\lambda \in \mathbb{K} : a(T-\lambda) < \infty, R((T-\lambda)^{a(T-\lambda)+1}) \ \text{ is closed }\},$$

$$\sigma_{ta}^e(T) := \mathbb{K}\backslash\rho_{ta}^e(T) \ \text{ where } \ \rho_{ta}^e(T) := \{\lambda \in \mathbb{K} : a_e(T-\lambda) < \infty, R((T-\lambda)^{a_e(T-\lambda)+1}) \ \text{ is closed }\},$$

$$\sigma_{td}(T) := \mathbb{K}\backslash\rho_{td}(T) \ \text{ where } \ \rho_{td}(T) := \{\lambda \in \mathbb{K} : d(T-\lambda) < \infty, R((T-\lambda)^{d(T-\lambda)}) \ \text{ is closed }\}$$

and

$$\sigma_{td}^e(T) := \mathbb{K}\backslash\rho_{td}^e(T) \ \text{ where } \ \rho_{td}^e(T) := \{\lambda \in \mathbb{K} : d_e(T-\lambda) < \infty, R((T-\lambda)^{d_e(T-\lambda)}) \ \text{ is closed }\}.$$

In [11], the authors proved that if P is a complex polynomial, then $P(\sigma_*(T)) = \sigma_*(P(T))$ where $* \in \{ta, ta^e, td, td^e\}$. In the second part of this paper, we extend these properties to the case of closed linear relations. The proofs presented here are very different from those in [11]. Our analysis is essentially based on the algebraic spectral mapping results developed in the first part of this paper.

On the other hand, we note that there are many reasons why linear relations are more convenient than operators, one of them is that one can define the inverse of a linear relation. In the last years, several authors have paid attention to the research of the theory of linear relations since it has applications in many problems in Physics and other areas of Applied Mathematical. We cite some of them

• Theory of pseudo-resolvents. Note that any pseudo-resolvent is a resolvent set of a certain linear relation (see for instance [2]).

• Theory of linear bundles. Let S and T be two bounded operators. The map $\mathcal{P}(\lambda) := S + \lambda T$, $\lambda \in \mathbb{C}$, is called a linear bundle. It is known that many problems of Mathematical Physics are reduced to the study of the reversibility conditions of operators $\mathcal{P}(\lambda)$, $\lambda \in \mathbb{C}$ and this study is reduced in many cases to the investigation of spectral properties of the linear relations $T^{-1}S$ and ST^{-1} (see for instance [3] and the references therein).

• Applications of the theory of linear relations in: Game theory and Mathematical Economics, Discontinuous differential equations which occur in the Biological Sciences (for example, population in dynamics and epidemiology), Optimal control and Digital imaging. A systematic bibliography on these applications including references to other and more recent contributions can be found in [10].

In view of the above remarks the attempt to generalize the existing results concerning the spectra of operators to general context of linear relations appears as natural.

The structure of this paper is as follows. To make the paper easily accessible the exposition is more or less self-contained. Some basic notations and results from the theory of linear relations in linear spaces are recalled in Section 2. In this section, some relationships between descent and essential descent, and ascent and essential ascent, respectively, are established. Section 3 is devoted to show that the descent, essential descent, ascent and essential ascent spectrum of a linear relation in a linear space verify the polynomial version of the spectral mapping theorem. These algebraic results are used in Section 4 to show that the topological descent, essential descent, ascent and essential ascent spectrum of a closed linear relation in a Banach space satisfy the polynomial spectral mapping property. We close this paper, by calculating $\sigma_*(L)$ and $\sigma_*(L^{-1})$, $* \in \{ta, ta^e, td, td^e\}$ where L is the left shift operator on l_p, $1 \leq p < \infty$ and L^{-1} denotes the inverse of L (which is a linear relation but it is not an operator).

2. Algebraic Results for Ascent, Essential Ascent, Descent and Essential Descent

This section contains some results concerning the ascent, essential ascent, descent and essential descent of a linear relation in a linear space.
We first recall the notions of the objects which will be studied in this paper. Let E, F and G be linear spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A linear relation A in $E \times F$ is a subspace of the space $E \times F$, the Cartesian product of E and F. The notations $D(A)$ and $R(A)$ denote the domain and the range of A, defined by

$$D(A) := \{x : (x, y) \in A\}, \ \ R(A) := \{y : (x, y) \in A\}.$$

Further, $N(A)$ and $A(0)$ denote the null space and the multivalued part of A, defined by

$$N(A) := \{x : (x, 0) \in A\}, \ \ A(0) := \{y : (0, y) \in A\}.$$

We say that A is injective if $N(A) = \{0\}$, surjective if $R(A) = F$ and A is called bijective if it is injective and surjective.
A linear relation A is (the graph of) an operator if and only if $A(0) = \{0\}$. The inverse A^{-1} of A is given by $\{(y, x) : \ (x, y) \in A\}$, so that $D(A^{-1}) = R(A)$, $R(A^{-1}) = D(A)$, $N(A^{-1}) = A(0)$ and $A^{-1}(0) = N(A)$.
Let M be a subspace of E. The notation $A_{/M}$ will be used for the linear relation $A_{/M} := A \cap (M \times F)$.
For linear relations A and B in $E \times F$ and $\lambda \in \mathbb{K}$, the linear relations $A + B$, and λA are defined by

$$A + B := \{(x, y + z) : (x, y) \in A, (x, z) \in B\},$$

$$\lambda A := \{(x, \lambda y) : (x, y) \in A\}.$$

If A is a linear relation in $E \times E$, or a linear relation in E for short, then $A - \lambda := A - \lambda I$ where I is the identity operator on E, the resolvent set of A is the subset $\rho(A) := \{\lambda \in \mathbb{K} : A - \lambda \text{ is bijective}\}$ and the subset $\sigma(A) := \mathbb{K} \backslash \rho(A)$ is called the spectrum of A.
Let A and B be linear relations in $E \times F$ and $F \times G$ respectively. The product BA is the linear relation in $E \times G$ defined by

$$BA := \{(x, z) : (x, y) \in A, (y, z) \in B \text{ for some } y \in F\}.$$

Let A be a linear relation in E. Then A^n, $n \in \mathbb{Z}$, is defined as usual with $A^0 = I$ and $A^1 = A$. The singular chain manifold $R_c(A)$ of A is defined by

$$R_c(A) := \Big(\bigcup_{n=1}^{+\infty} A^n(0)\Big) \cap \Big(\bigcup_{n=1}^{+\infty} N(A^n)\Big).$$

It is known (see [13, Lemmas 3.4 and 3.5]) that $(N(A^n))_{n\in\mathbb{N}}$ is an increasing sequence and if $N(A^m) = N(A^{m+1})$ for some nonnegative integer m, then $N(A^m) = N(A^n)$ for all $n \geq m$. Similarly, $(R(A^n))_{n\in\mathbb{N}}$ is a decreasing sequence and if $R(A^m) = R(A^{m+1})$ for some $m \in \mathbb{N} \cup \{0\}$, then $R(A^m) = R(A^n)$ for all $n \geq m$. These statements lead to the introduction of the ascent and the descent of a linear relation A in E by

$$a(A) := min\{r \in \mathbb{N} \cup \{0\} : N(A^r) = N(A^{r+1})\},$$

$$d(A) := min\{s \in \mathbb{N} \cup \{0\} : R(A^s) = R(A^{s+1})\},$$

respectively, whenever these minima exist. If no such numbers exist the ascent and descent of A are defined to be ∞.
In [13], the authors introduce and study these notions. They showed that many of the results of Taylor and Kaashoek for operators remain valid in the context of linear relations only under the additional condition that the linear relation A has a trivial singular chain manifold, that is, $R_c(A) = \{0\}$.
The following lemma helps to understand Definition 2.1 below

Lemma 2.1. *[6, Lemmas 2.7 and 2.8]* (*i*) $dimR(A^n)/R(A^{n+1}) \leq dimR(A^{n-1})/R(A^n)$ *for all* $n \in \mathbb{N}$.

(*ii*) *If there exists* $m \in \mathbb{N} \cup \{0\}$ *such that* $dimR(A^m)/R(A^{m+1})$ *is finite, then* $dimR(A^n)/R(A^{n+1})$ *is finite for all* $n \geq m$.

(*iii*) $dimR(A^n)/R(A^{n+1}) < \infty$ *if and only if* $dimR(A^n)/R(A^{n+k}) < \infty$ *for all positive integer k.*
Assume that $R_c(A) = \{0\}$. *Then*

(*iv*) $dimN(A^{n+1})/N(A^n) \leq dimN(A^n)/N(A^{n-1})$ *for all* $n \in \mathbb{N}$.

(*v*) *If there exists* $m \in \mathbb{N} \cup \{0\}$ *such* $dimN(A^{m+1})/N(A^m) < \infty$, *then* $dimN(A^{n+1})/N(A^n) < \infty$ *for all* $n \geq m$.

(*vi*) $dimN(A^{n+1})/N(A^n) < \infty$ *if and only if* $dimN(A^{n+k})/N(A^n) < \infty$ *for all positive integer k.*

The statements in Lemma 2.1 lead to the introduction of the following concepts which are due to Chafai and Mnif [6].

Definition 2.1. *Let A be a linear relation in a linear space E. The essential descent of A is defined by*

$$d_e(A) := min\{s \in \mathbb{N} \cup \{0\} : dimR(A^s)/R(A^{s+1}) < \infty\},$$

where the minimum over the empty set is taken to be infinite. If $R_c(A) = \{0\}$, *then the essential ascent of A is given by*

$$a_e(A) := min\{r \in \mathbb{N} \cup \{0\} : dimN(A^{r+1})/N(A^r) < \infty\},$$

where the minimum over the empty set is taken to be infinite.

It is proved in [13, Theorem 5.7] that if $R_c(A) = \{0\}$, $a(A) < \infty$ and $d(A) < \infty$, then $a(A) \leq d(A)$ with equality if A is everywhere defined. The main purpose of this section is to prove the same properties for the essential ascent and essential descent. For this end, we need some auxiliary results. Firstly, we recall the following elementary lemma

Lemma 2.2. *Let M, N and W be subspaces of E. Then*

(*i*) $\dfrac{M}{M \cap N} \simeq \dfrac{M+N}{N}$.

(*ii*) *If* $M \subset W$, *then* $(M + N) \cap W = M + (N \cap W)$.

Lemma 2.3. *Let A be a linear relation in a linear space E such that* $d_e(A) := s < \infty$. *Then*

$$N(A) \cap R(A^s) = N(A) \cap R(A^{s+n}) \text{ for all } n \in \mathbb{N}.$$

Proof. Consider the linear relation $\widehat{A}$ in $\dfrac{R(A^s)}{R(A^{s+1})} \times \dfrac{R(A^{s+1})}{R(A^{s+2})}$ defined by

$$\widehat{A} := \{(\overline{x}, \overline{y}) \in \frac{R(A^s)}{R(A^{s+1})} \times \frac{R(A^{s+1})}{R(A^{s+2})} : (x, y) \in A\}.$$

We shall show that

$$\widehat{A} \text{ is a bijective operator and } dim\frac{R(A^s)}{R(A^{s+1})} = dim\frac{R(A^n)}{R(A^{n+1})} \text{ for all } n \geq s. \tag{2.1}$$

Indeed, since $A(0) \subset R(A^n)$ for all nonnegative integer n, one deduces trivially from the definition of $\widehat{A}$ that $\widehat{A}$ is a surjective operator. So that $dim \frac{R(A^s)}{R(A^{s+1})} \leq dim \frac{R(A^{s+1})}{R(A^{s+2})}$ and thus we infer from Lemma 2.1(i) that $dim \frac{R(A^s)}{R(A^{s+1})} = dim \frac{R(A^{s+1})}{R(A^{s+2})}$. Now, a repeated reasoning proves that

$$dim \frac{R(A^s)}{R(A^{s+1})} = dim \frac{R(A^n)}{R(A^{n+1})} \text{ for all } n \geq s.$$

This equality combined with the fact that $dim \frac{R(A^s)}{R(A^{s+1})} < \infty$ ensures that $\widehat{A}$ is injective. Hence (2.1) holds.

$$dim N(\widehat{A}) = dim \frac{N(A) \cap R(A^s)}{N(A) \cap R(A^{s+1})}. \tag{2.2}$$

The use of [8, Proposition I.3.1] together with Lemma 2.2 gives

$$\begin{aligned}
dim N(\widehat{A}) &= dim \frac{A^{-1}(R(A^{s+2})) \cap [R(A^s) \cap D(A)]}{R(A^{s+1}) \cap D(A)} \\
&= dim \frac{[N(A) + (R(A^{s+1}) \cap D(A))] \cap [R(A^s) \cap D(A)]}{R(A^{s+1}) \cap D(A)} \\
&= dim \frac{[N(A) \cap R(A^s) \cap D(A)] + [R(A^{s+1}) \cap D(A)]}{R(A^{s+1}) \cap D(A)} \\
&= dim \frac{[N(A) \cap R(A^s)] + [R(A^{s+1}) \cap D(A)]}{R(A^{s+1}) \cap D(A)} \quad (\text{since } N(A) \subset D(A)) \\
&= dim \frac{N(A) \cap R(A^s)}{[(N(A) \cap R(A^s)] \cap [R(A^{s+1}) \cap D(A))]} \quad (by\ Lemma\ 2.2) \\
&= dim \frac{N(A) \cap R(A^s)}{N(A) \cap R(A^{s+1})} \quad (since\ N(A) \subset D(A)\ and\ R(A^{s+1}) \subset R(A^s)).
\end{aligned}$$

Hence (2.2) holds.
A combination of (2.1) and (2.2) now implies that $N(A) \cap R(A^s) = N(A) \cap R(A^{s+1})$ and a repeated reasoning then gives $N(A) \cap R(A^s) = N(A) \cap R(A^{s+n})$ for all $n \in \mathbb{N}$, as desired. □

Lemma 2.4. *[13, Lemmas 4.1 and 4.4] Let A be a linear relation in a linear space E and let $n, m \in \mathbb{N} \cup \{0\}$. Then*

(*i*) $\frac{D(A^n)}{D(A^n) \cap (N(A^n) + R(A^m))} \simeq \frac{R(A^n)}{R(A^{n+m})}$.

(*ii*) *If $R_c(T) = \{0\}$, then* $\frac{N(A^{n+m})}{N(A^m)} \simeq N(A^n) \cap R(A^m)$.

Theorem 2.1. *Assume that $R_c(A) = \{0\}$. If $a_e(A)$ and $d_e(A)$ are both finite, then $a_e(A) \leq d_e(A)$ and there is equality if $D(A) = E$.*

Proof. Let $a_e(A) := r < \infty$ and $d_e(A) := s < \infty$. Assume that $r > s$, then $r = s + q$ for some $q \in \mathbb{N}$ and thus it follows from Lemma 2.3 that $N(A) \cap R(A^s) = N(A) \cap R(A^r)$. So that one has from Lemma 2.4 (ii) that $dim \frac{N(A^{s+1})}{N(A^s)} = dim \frac{N(A^{r+1})}{N(A^r)} < \infty$. Therefore $r \leq s$, that is $a_e(A) \leq d_e(A)$. Suppose now that A is everywhere defined and let us consider various cases for s.

Case 1: $s = 0$. Since $r \leq s$, we have that $r = 0$.

Case 2: $s = 1$. Assume first that $r = 0$. Then $dimN(A) = dimN(A^{0+1})/N(A^0) < \infty$ and $dimR(A^1)/R(A^2) < \infty$. The use of these facts combined with Lemmas 2.2 and 2.4 shows that $dim\dfrac{N(A)+R(A)}{R(A)} < \infty$ and

$$dim\frac{\frac{E}{R(A)}}{\frac{N(A)+R(A)}{R(A)}} = dim\frac{E}{N(A)+R(A)} = dim\frac{R(A)}{R(A^2)} < \infty$$

which implies that $dim\dfrac{R(A^0)}{R(A)} = dim\dfrac{E}{R(A)} < \infty$ and this fact contradicts the assumption $d_e(A) = 1$. Hence $r = s = 1$.

Case 3: $s > 1$. In this case, $dim\dfrac{R(A^s)}{R(A^{s+1})} < \infty$ and $dim\dfrac{R(A^{s-1})}{R(A^s)} = \infty$. Define the linear relation $\widehat{A}$ in $\dfrac{R(A^{s-1})}{R(A^s)} \times \dfrac{R(A^s)}{R(A^{s+1})}$ in the same way that in Lemma 2.3. Then we obtain that $\widehat{A}$ is a surjective operator with

$$dimN(\widehat{A}) = dim\frac{N(A)\cap R(A^{s-1})}{N(A)\cap R(A^s)}.$$

Consequently, $dimN(\widehat{A}) = \infty$; in particular $dimN(A) \cap R(A^{s-1}) = \infty$, so that, $dim\frac{N(A^s)}{N(A^{s-1})} = \infty$ by Lemma 2.4. A combination of this last equality and Lemma 2.1 ensures that $a_e(A) \geq s$ and since $a_e(A) \leq d_e(A)$, we conclude that $r = s$ as desired.

□

Remark 2.1. *In [7, Theorem 2.2], Chandra and Kumar proved the above Theorem 2.1 when A is an operator. Our proof is very different (and more short) that the proof of Theorem 2.2 in [7].*

Remark 2.2. *It is possible to find operators A such that*

(*i*) $a_e(A) < \infty$ *and* $d_e(A) = \infty$.

(*ii*) $a_e(A) = \infty$ *and* $d_e(A) < \infty$.

(*iii*) $a_e(A) = d_e(A) = \infty$.

See [7, Examples 3.2, 3.3, and 3.4].

3. Polynomial Spectral Mapping Theorem for the Algebraic Spectra $\sigma_a(.)$, $\sigma_a^e(.)$, $\sigma_d(.)$ and $\sigma_d^e(.)$

Definition 3.1. *Let A be a linear relation in a linear space E. The descent resolvent set and the essential descent resolvent set of A are respectively defined by*

$$\rho_d(A) := \{\lambda \in \mathbb{K} : d(T-\lambda) < \infty\},$$
$$\rho_d^e(A) := \{\lambda \in \mathbb{K} : d_e(T-\lambda) < \infty\}.$$

The complementary sets $\sigma_d(A) := \mathbb{K}\backslash\rho_d(A)$ *and* $\sigma_d^e(A) := \mathbb{K}\backslash\rho_d^e(A)$ *are called the algebraic descent spectrum and the algebraic essential descent spectrum of A, respectively.*

Assume that A has a trivial singular chain manifold. Then the ascent resolvent set and the essential ascent resolvent set of A are respectively defined by

$$\rho_a(A) := \{\lambda \in \mathbb{K} : a(T-\lambda) < \infty\},$$
$$\rho_a^e(A) := \{\lambda \in \mathbb{K} : a_e(T-\lambda) < \infty\}.$$

The sets $\sigma_a(A) := \mathbb{K}\backslash\rho_a(A)$ *and* $\sigma_a^e(A) := \mathbb{K}\backslash\rho_a^e(A)$ *are called the algebraic ascent spectrum and the algebraic essential ascent spectrum of A, respectively.*

In this section our interest concentrates to show that the above algebraic spectra satisfy the polynomial spectral mapping property. For this end, we need some auxiliary results.

Lemma 3.1. *[6, Lemmas 3.2 and 4.5] Let A be a linear relation in a linear space E. Then $d(A^m)$ is finite for some $m \in \mathbb{N}$ if and only if $d(A^n)$ is finite for all $n \in \mathbb{N}$. The same property is true for the essential descent, ascent and essential ascent.*

Lemma 3.2. *[8, Proposition VI.5.1],[13, Lemma 7.2] and [12, Lemma 6.1] Let A be a linear relation in a linear space E, $\lambda, \mu \in \mathbb{K}$ and let $m, n \in \mathbb{N} \cup \{0\}$. We have*

(*i*) $(A-\lambda)^n(A-\mu)^m = (A-\mu)^m(A-\lambda)^n$.

(*ii*) *If* $\lambda \neq \mu$, *then* $N(A-\lambda)^n \subset R((A-\mu)^m)$.

(*ii*) *If* $\rho(A) \neq \emptyset$, *then* $E = D(A^n) + R(A^m)$ *and* $\{0\} = A^m(0) \cap N(A^n)$. *Particularly,* $R_c(A) = \{0\}$ *if* $\rho(A) \neq \emptyset$.

We recall the notion of polynomial in a linear relation which is due to Sandovic [12].

Definition 3.2. *Let A be a linear relation in a linear space E. Let p and m_i, $1 \leq i \leq p$ be some positive integers and $\lambda_i \in \mathbb{K}, 1 \leq i \leq p$ be some distinct constants. Then the polynomial P in A is the linear relation*

$$P(A) := \prod_{i=1}^{p}(A-\lambda_i)^{m_i}.$$

The following lemma will be very useful, it describes the behaviour of the domain, the range, the null space and the multivalued part of $P(A)$.

Lemma 3.3. *[12, Theorems 3.2, 3.3, 3.4 and 3.6] Let A be a linear relation in a linear space E and let $P(A)$ be as in Definition 3.2. Then*

(*i*) $D(P(A)) = D(A^{\sum_{i=1}^{p} m_i})$.

(*ii*) $R(P(A)) = \bigcap_{i=1}^{p} R((A-\lambda_i)^{m_i})$.

(*iii*) $N(P(A)) = \sum_{i=1}^{p} N((A-\lambda_i)^{m_i})$.

(*iv*) $P(A)(0) = A^{\sum_{i=1}^{p} m_i}(0)$.

As an immediate consequence of Lemmas 2.2, 3.2 and 3.3 we get

Lemma 3.4. *Let A be a linear relation in a linear space E and let $P(A)$ be as in Definition 3.2. Then*

(*i*) $N(P(A)) \cap R(P(A)^n) = \sum_{i=1}^{p}(N((A-\lambda_i)^{m_i})) \cap R((A-\lambda_i)^{nm_i})$.

(*ii*) $N(P(A^n)) + R(P)) \subset \bigcap_{i=1}^{p}[N((A-\lambda_i)^{nm_i}) + R((A-\lambda_i)^{m_i})]$.

Proposition 3.1. *Let A be a linear relation in a linear space E such that $\rho(A) \neq \emptyset$ and let $P(A)$ be as in Definition 3.2. Then*

(*i*) $d(P(A)) < \infty$ *if and only if* $d(A-\lambda_i) < \infty$ *for all* $i \in \{1, 2, ..., p\}$.

(*ii*) $d_e(P(A)) < \infty$ *if and only if* $d_e(A-\lambda_i) < \infty$ *for all* $i \in \{1, 2, ..., p\}$.

(*iii*) *$a(P(A)) < \infty$ if and only if $a(A-\lambda_i) < \infty$ for all $i \in \{1,2,...,p\}$.*

(*iv*) *$a_e(P(A)) < \infty$ if and only if $a_e(A-\lambda_i) < \infty$ for all $i \in \{1,2,...,p\}$.*

Proof. Arguing exactly as in the proof of Theorem VI.5.4 in [8] we obtain that $\rho((A-\lambda_i)^{m_i}) \neq \emptyset$ for each $i \in \{1,2,...,p\}$ and that $\rho(P(A)) \neq \emptyset$.
Write $U := (A-\lambda_i)^{m_i}$ and $V := (A-\lambda_j)^{m_j}$.

(*i*) Assume that $d(P(A)) := d < \infty$. Then

$$\begin{aligned}\{0\} &= \frac{R(P(A)^d)}{R(P(A)^{d+1})} \\ &\simeq \frac{D(P(A)^d)}{D(P(A)^d)\cap[R(P(A))+N(P(A)^d)]} \quad \text{(Lemma 2.4(i))} \\ &\simeq \frac{D(P(A)^d)+R(P(A))}{N(P(A)^d)+R(P(A))} \quad \text{(Lemma 2.2(i))} \\ &= \frac{E}{N(P(A)^d)+R(P(A))} \quad \text{(Lemma 3.2(iii))}.\end{aligned}$$

Hence $E = N(P(A)^d) + R(P(A))$. So that $E = N((A-\lambda_i)^{m_i d}) + R(A-\lambda_i)^{m_i})$ by Lemma 3.4(ii) and thus, again Lemmas 2.2(i), 2.4(i) and 3.2(iii), show that $d((A-\lambda_i)^{m_i})$ is finite. Now, the use of Lemma 3.1 makes us to conclude that $d(A-\lambda_i)$ is finite.
Conversely, assume that $d(A-\lambda_i)$ is finite for every i, $1 \leq i \leq p$. Then $d((A-\lambda_i)^{m_i}) < \infty$ by virtue of Lemma 3.1. Let $q := max\{d((A-\lambda_i)^{m_i}), d((A-\lambda_j)^{m_j})\}$. Then

$$\begin{aligned}R((UV)^q) &= U^q(R(V^q)) = U^q(R(V^{q+1})) \\ &= V^{q+1}(R(U^q)) = V^{q+1}(R(U^{q+1})) \\ &= R((UV)^{q+1}).\end{aligned}$$

Hence $d(UV) := d((A-\lambda_i)^{m_i}(A-\lambda_j)^{m_j}) < \infty$ and repeating the same reasoning we obtain that $d(P(A)) < \infty$.

(*ii*) Suppose that $d_e(P(A)) < \infty$. Proceeding exactly as for the proof of the assertion (i) we obtain that

$$dim \frac{E}{N(P(A)^{d_e(P(A))}) + R(P(A))} < \infty$$

and applying Lemma 3.4(ii) we get that

$$dim \frac{E}{R((A-\lambda_i)^{m_i}) + N((A-\lambda_i)^{m_i d_e(P(A))})} < \infty$$

which implies that $d_e((A-\lambda_i)^{m_i})$ is finite. So that $d_e(A-\lambda_i) < \infty$ by virtue of Lemma 3.1. Conversely, assume that for every $i \in \{1,2,...,p\}$, $d_e(A-\lambda_i) < \infty$. By Lemma 3.1, $max\{d_e((A-\lambda_i)^{m_i}), d_e((A-\lambda_j)^{m_j})\} := s < \infty$. We first claim that

$$\frac{R((UV)^s)}{R(U^sV^{s+1})} \text{ and } \frac{R(U^sV^{s+1})}{R((UV)^{s+1})} \text{ are both finite dimensional spaces} \tag{3.1}$$

As a direct consequence of Lemmas 3.2(iii) and 3.3(i) we obtain that $E = D(U^s) + R(V^{s+1})$. Using this equality combined with Lemmas 2.2 and 2.4(i), we deduce that

$$\begin{aligned} dim\frac{R(V^s)}{R(V^{s+1})} &= dim\frac{R(V^s)\cap(D(U^s)+R(V^{s+1}))}{R(V^{s+1})} \\ &= dim\frac{R(V^{s+1})+(D(U^s)\cap R(V^s))}{R(V^{s+1})} \\ &= dim\frac{D(U^s)\cap R(V^s)}{D(U^s)\cap R(V^{s+1})}. \end{aligned}$$

Hence $dim\frac{D(U^s)\cap R(V^s)}{D(U^s)\cap R(V^{s+1})} < \infty$. On the other hand, we define $W := (U_{/R(V^s)})^s$ and $M := R(V^{s+1})$. Then

$$\begin{aligned} dim\frac{R((UV)^s)}{R(U^sV^{s+1})} &= dim\frac{R(W)}{W(M)} \leq dim\frac{D(W)}{D(W)\cap M} \quad \text{([8, Proposition I.6.1])} \\ &= dim\frac{D(U^s)\cap R(V^s)}{D(U^s)\cap R(V^{s+1})}. \end{aligned}$$

Consequently, $dim\frac{((UV)^s)}{R(U^sV^{s+1})} < \infty$. Similarly, if we consider $(V_{/R(U^s)})^{s+1}$ and $R(U^{s+1})$ instead of W and M respectively we obtain that $dim\frac{R(U^sV^{s+1})}{R((UV)^{s+1})} < \infty$. Hence (3.1) holds.
One finds, by (3.1), that

$$dim\frac{R((UV)^s)}{R((UV)^{s+1})} = dim\frac{R((UV)^s)}{R(U^sV^{s+1})} + dim\frac{R(U^sV^{s+1})}{R((UV)^{s+1})} < \infty.$$

Hence $d_e((A-\lambda_i)^{m_i}(A-\lambda_j)^{m_j}) < \infty$ and a repeated reasoning then gives $d_e(P(A)) < \infty$.
(*iii*) and (*iv*) follow immediately from Lemmas 2.4(ii), 3.2(iii) and 3.4(i).

□

Now, we are in the position to give the main result of this section.

Theorem 3.1. *Let A be a linear relation in a linear space E with $\rho(A) \neq \emptyset$ and let P be a complex polynomial. Then*

(*i*) $P(\sigma_d(A)) = \sigma_d(P(A))$.

(*ii*) $P(\sigma_d^e(A)) = \sigma_d^e(P(A))$.

(*iii*) $P(\sigma_a(A)) = \sigma_a(P(A))$.

(*iv*) $P(\sigma_a^e(A)) = \sigma_a^e(P(A))$.

Proof. Fix $\mu \in \mathbb{C}$ and let $P(\mu) = \prod_{i=1}^{p}(\mu-\lambda_i)^{m_i}$ where p and m_i, $1 \leq i \leq p$ are positive integers and $\lambda_i \in \mathbb{K}, 1 \leq i \leq p$, are distinct constants. Then $P(A) - \lambda = \prod_{i=1}^{p}(A-\lambda_i)^{m_i}$.

(*i*) Let $\lambda \in P(\sigma_d(A))$, so that $\lambda = P(\beta)$ for some $\beta \in \sigma_d(A)$. Then $\beta = \lambda_j$ for some j. It follows from Proposition 3.1(i) that $d(P(A)-\lambda) = \infty$ which implies that $\lambda \in \sigma_d(P(A))$. Conversely, assume that $\lambda \in \sigma_d(P(A))$. Then $d(A-\lambda_i) = \infty$ for some $i \in \{1,2,...,p\}$ by virtue of Proposition 3.1(i) and since $P(\lambda_i) = \lambda$ we have that $\lambda \in P(\sigma_d(A))$. Therefore (i) holds.
The proof of the statements (ii), (iii) and (iv) my be sketched in the same way as the proof of (i).

□

For bounded operators in Banach spaces the above Theorem 3.1 was proved by Mbekhta and Müller [11] by using the notion of regularity.
For an everywhere defined closed linear relation T in a Banach space with $dim T(0) < \infty$, the statements (i) and (ii) have been proved in [6, Theorem 3.4]. Hence, Theorem 3.1 provides an improvement of Theorem 3.4 in [6] to the general case of a linear relation in a linear space.

4. Polynomial Spectral Mapping Theorem for the Topological Spectra $\sigma_{ta}(.)$, $\sigma^e_{ta}(.)$, $\sigma_{td}(.)$ and $\sigma^e_{td}(.)$

It is more interesting from the point of view of the operator theory to combine the algebraic conditions defining the resolvent sets $\rho_d(.)$, $\rho^e_d(.)$, $\rho_a(.)$ and $\rho^e_a(.)$ with a topological condition. The following observations explain the exponents in Definition 4.1 below.
Let T be a closed linear relation in a complex Banach space X (that is, T is a closed subspace of $X \times X$). It is easy to see that if $dim \frac{R(T^d)}{R(T^{d+1})} < \infty$. for some $d \in \mathbb{N} \cup \{0\}$ and $R(T^n)$ is closed for some $n \geq d$, then $R(T^i)$ is closed for all $i \geq d$. Further, in [6, Lemma 4.3], the authors prove that if $D(T) = X, \rho(T) \neq \emptyset, a_e(T) < \infty$ and $R(T^n)$ is closed for some $n > a_e(T)$, then $R(T^n)$ is closed for all $n \geq a_e(T)$.
In the rest of this section X will be a complex Banach space and T will always denote an everywhere defined closed linear relation in X with $\rho(T) \neq \emptyset$.

Definition 4.1. *The topological resolvent sets, $\rho_*(T)$ for $* \in \{ta, ta^e, td, td^e\}$ are defined as follows*

$$\rho_{ta}(T) := \{\lambda \in \mathbb{C} : a(T-\lambda) < \infty, R((T-\lambda)^{a(T-\lambda)+1}) \text{ is closed}\},$$

$$\rho^e_{ta}(T) := \{\lambda \in \mathbb{C} : a_e(T-\lambda) < \infty, R((T-\lambda)^{a_e(T-\lambda)+1}) \text{ is closed}\},$$

$$\rho_{td}(T) := \{\lambda \in \mathbb{C} : d(T-\lambda) < \infty, R((T-\lambda)^{d(T-\lambda)}) \text{ is closed}\},$$

$$\rho^e_{td}(T) := \{\lambda \in \mathbb{C} : d_e(T-\lambda) < \infty, R((T-\lambda)^{d_e(T-\lambda)}) \text{ is closed}\}.$$

The subsets $\sigma_{ta}(T) := \mathbb{C}\backslash\rho_{ta}(T); \sigma^e_{ta}(T) := \mathbb{C}\backslash\rho^e_{ta}(T)$; $\sigma_{td}(T) := \mathbb{C}\backslash\rho_{td}(T)$ and $\sigma^e_{ta}(T) := \mathbb{C}\backslash\rho^e_{td}(T)$ are called the topological descent spectrum, the topological essential descent spectrum, the topological ascent spectrum and the topological essential ascent spectrum of T, respectively.

Recently, Chafai and Mnif [6, Theorem 4.7] showed that if P is a complex polynomial then $P(\sigma_{ta}(T)) = \sigma_{ta}(P(T))$ and $P(\sigma^e_{ta}(T)) = \sigma^e_{ta}(P(T))$. However, the notions of $\sigma_{td}(T)$ and $\sigma^e_{td}(T)$ as well as the validity of the polynomial spectral mapping property for both topological spectra seem still unknown. The objective of this section is to show that if P is a complex polynomial then $P(\sigma^e_{td}(T)) = \sigma^e_{td}(P(T))$.
The analysis is essentially based on the algebraic results developed in the previous section combined with some topological properties.

Lemma 4.1. *[6, Lemma 4.2] Assume that M is a closed subspace of X such that $T(0) \subset M$. Then $T^{-1}(M)$ is closed.*

Lemma 4.2. *Let $P(T) = \prod_{i=1}^{p} (T-\lambda_i)^{m_i}$ with p, m_i and λ_i, $1 \leq i \leq p$ as in Definition 3.2. Then*

(*i*) *$P(T)$ is an everywhere defined closed linear relation in X having a nonempty resolvent set.*

(*ii*) *For any $n \in \mathbb{N}$, $R(P(T)^n)$ is closed if and only if, for each $i \in \{1, 2, ..., p\}$, $R((T-\lambda_i)^{nm_i})$ is closed.*

Proof. (*i*) See [9, Lemma 3.1] and [8, Theorem VI.5.4].

(*ii*) Write $P(T) = US$ where $U := (T-\lambda_i)^{m_i}$ and $S = \prod_{j=1, j\neq i}^{p} (T-\lambda_i)^{m_i}$. By Lemmas 3.2 and 3.3 we have that $U^nS^n = S^nU^n$, $S^n(0) \subset R(P(T)^n)$ and $N(S^n) \subset R(U^n)$. Assume that $R(P(T)^n)$ is closed. Then $R(U^n) = R(U^n) + N(S^n) = S^{-n}S^n(R(U^n))$ [8, Proposition I.3.1] $= S^{-n}(R(P(T)^n))$ which is closed by virtue

of (i) and Lemma 4.1. Hence $R((T-\lambda_i)^{nm_i})$ is closed.
Conversely, suppose that $R((T-\lambda_i)^{nm_i})$ is closed for every $i, 1 \leq i \leq p$. According to Lemma 3.3, $R(P(T)^n)$ is closed.

□

Theorem 4.1. *Let P be a complex polynomial. Then*

(*i*) $P(\sigma_{td}(A)) = \sigma_{td}(P(A))$.

(*ii*) $P(\sigma^e_{td}(A)) = \sigma^e_{td}(P(A))$.

(*iii*) $P(\sigma_{ta}(A)) = \sigma_{ta}(P(A))$.

(*iv*) $P(\sigma^e_{ta}(A)) = \sigma^e_{ta}(P(A))$.

Proof. For $\lambda \in \mathbb{C}$ let $P(T) - \lambda = \prod_{i=1}^{p}(T-\lambda_i)^{m_i}$. where p and m_i, $1 \leq i \leq p$, are positive integers and $\lambda_i \in \mathbb{K}$, $1 \leq i \leq p$ are distinct constants. We only prove (i), the proofs of the statements (ii), (iii) and (iv) are similar.

(*i*) Let $\lambda \in P(\sigma_{td}(T))$. Then $\lambda = P(\beta)$ for some $\beta \in \sigma_{td}(T)$, so that $\beta = \lambda_j$ for some λ_j. Let us consider two possibilities for λ_j:

Case 1: $d(T-\lambda_j) = \infty$. In such case we infer, from Proposition 3.1(i) that $d(P(T)-\lambda) = \infty$ and hence $\lambda \in \sigma_d(P(T)) \subset \sigma_{td}(P(T))$.

Case 2: $d(T-\lambda_j) = q_j < \infty$ and $R(T-\lambda_j)^{q_j}$ is not closed. If $d(P(T)-\lambda) = \infty$, then $\lambda \in \sigma_d(P(T)) \subset \sigma_{td}(P(T))$, so that we can suppose that $d(P(T)-\lambda) = q < \infty$. Then $R((P(T)-\lambda)^{m_j(q+q_j)})$ is not closed and one has from Lemma 4.2(ii) that $R((P(T)-\lambda)^{(q+q_j)})$ is not closed and since $d(P(T)-\lambda) = q < \infty$, we conclude that $R((P(T)-\lambda)^q)$ is not closed. Hence $\lambda \in \sigma_{td}(P(T))$. Consequently $P(\sigma_{td}(T)) \subset \sigma_{td}(P(T))$.

Conversely, assume that $\lambda \in \sigma_{td}(P(T))$. Various cases for λ will be considered :

Case 1: $d(P(T)-\lambda) = \infty$. In such case, Proposition 3.1(i) ensures that $d(T-\lambda_j) = \infty$ for some $j \in \{1, 2, ..., p\}$ and since $P(\lambda_j) = \lambda$ we get that $\lambda \in P(\sigma_{td}(T))$.

Case 2: $d(P(T)-\lambda) = q < \infty$ and $R(P(T)-\lambda)^q$ is not closed. Again Proposition 3.1(i) says that $d(T-\lambda_j)$ is finite. Let $d := max\{q, d(T-\lambda_1), d(T-\lambda_2), ..., d(T-\lambda_p)\}$. Then, it follows from Lemma 4.2(ii) that there is $j \in \{1, 2, ..., p\}$ for which $R(T-\lambda_j)^{dm_j}$ is not closed, so that $R((T-\lambda_j)^{d(T-\lambda_j)})$ is not closed which shows that $\lambda_j \in \sigma_{td}(T)$ and since $P(\lambda_j) = \lambda$ we conclude that $\lambda \in P(\sigma_{td}(T))$. Consequentely, $\sigma_{td}(P(A)) \subset P(\sigma^e_{ta}(A))$.

□

For bounded operators the above Theorem 4.1 was proved by Mbekhta and Müller [11] by using the notion of regularity.

5. Example

Let $X = l_p$, $1 \leq p < \infty$ be the Banach space of all complex sequences $x = (x_1, x_2, ...)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. We define the right shift operator R and the left shift operator L in X by

$$R : (x_1, x_2, x_3, ...) \in X \mapsto (0, x_1, x_2, x_3, ...) \in X \text{ and } L : (x_1, x_2, x_3, ...) \in X \mapsto (x_2, x_3, ...) \in X.$$

Clearly L is a bounded operator in X with $N(L) = span\{e_1\}$ where $e_1 := (1, 0, 0, ...)$, and $R(L) = X$. So that L^{-1} is a closed linear relation in X with $D(L^{-1}) = X, N(L^{-1}) = \{0\}$ $R(L^{-1}) = X$ and $L^{-1}(0) = span\{e_1\} \neq \{0\}$. This section is devoted to calculate $\sigma_*(L)$ and $\sigma_*(L^{-1})$ where $* \in \{td, td^e, ta, ta^e\}$. For this end, the following entirely algebraic results will play a crucial role.

Lemma 5.1. *[13, Theorem 5.6 and Corollary 6.8] Let A be a linear relation in a linear space E. Then*

(*i*) *If* $R_c(T) = \{0\}$ *and* $a(A) < \infty$, *then* $dim N(A) \leq dim \dfrac{E}{R(A)}$.

(*ii*) *If A is everywhere defined and* $d(A) < \infty$, *then* $dim \dfrac{E}{R(A)} \leq dim N(A)$.

Lemma 5.2. *[1, Proposition 4.4] Let A be a linear relation in a linear space E such that* $D(A) = E$. *Then, for* $\lambda \in \mathbb{K}\backslash\{0\}$, *we have*

(*i*) $N(A - \lambda)^n = N(A^{-1} - \lambda^{-1})^n$, *for all* $n \in \mathbb{N} \cup \{0\}$.

(*ii*) $R(A - \lambda)^n = R(A^{-1} - \lambda^{-1})^n$, *for all* $n \in \mathbb{N} \cup \{0\}$.

As a direct consequence we get

Lemma 5.3. *Let T be an everywhere defined closed linear relation in a complex Banach space X such that* $0 \in \rho(T)$. *Then, for* $\lambda \in \mathbb{K}\backslash\{0\}$, *we have that*

$$\lambda \in \sigma_*(T) \text{ if and only if } \frac{1}{\lambda} \in \sigma_*(T^{-1})$$

where $* \in \{a, d, {}^e a, {}^e d\}$.

Write $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\mathbb{S} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Lemma 5.4. *[4, Theorem 2.1] and [5, Corollary 2.6 and Theorem 2.7] Let T be a bounded operator in a Banach space X.*

(*i*) *Assume that* $d_e(T) := q < \infty$. *Then there exists* $\delta > 0$ *such that for* $0 < |\lambda| < \delta$, *we have the following assertions :*

(*a*) $R(T - \lambda)$ *is closed and* $N((T - \lambda)^n) \subset R(T - \lambda)$ *for all* $n \in \mathbb{N}$.

(*b*) $dim N((T - \lambda)^n) = n \, dim \dfrac{N(T^{q+1})}{N(T^q)}$ *for all* $n \in \mathbb{N}$.

(*c*) $dim \dfrac{X}{R((T - \lambda)^n)} = n \, dim \dfrac{R(T^q)}{R(T^{q+1})}$ *for all* $n \in \mathbb{N}$.

(*ii*) $\sigma_{ta}(T)$ *is a compact subset of* $\sigma(T)$.

(*iii*) $\sigma_{ta}(T) = \emptyset$ *if and only if the boundary of* $\sigma(T)$ *is contained in* $\rho^e_{ta}(T)$.

Proposition 5.1. $\sigma_{td}(L) = \sigma^e_{td}(L) = \sigma^e_{ta}(L) = \mathbb{S}$ *and* $\sigma_{ta}(L) = \mathbb{D}$.

It is known that $\sigma(R) = \sigma(L) = \mathbb{D}$. We first note that

$$L - \lambda = \lambda L(\lambda^{-1} - R), \text{ whenever } \lambda \in \mathbb{K}\backslash\{0\}. \tag{5.1}$$

Indeed, $L - \lambda := L - \lambda I = \lambda\lambda^{-1}L - \lambda LR = \lambda L(\lambda^{-1} - R)$ for every $\lambda \neq 0$. Hence (5.1) holds.
$\sigma_{td}(L) = \sigma^e_{td}(L) = \mathbb{S}$: Since $R(L) = X$ we have that $0 \in \rho_{td}(L) \subset \rho^e_{td}(L)$.
Let $0 < |\lambda| < 1$. Then it follows from (5.1) and the fact that $\sigma(R) = \mathbb{D}$ that $R(L - \lambda) = R(L) = X$. Therefore $B(0, 1) := \{\lambda \in \mathbb{K} : |\lambda| < 1\} \subset \rho_{td}(L) \subset \rho^e_{td}(L)$ so that $\sigma_{td}(L) \cup \sigma^e_{td}(L) \subset \mathbb{S}$.
Let $|\lambda| = 1$. Clearly $L - \lambda$ is injective and thus we infer from Lemma 5.1(ii) that if $d(L - \lambda)$ is finite then $dim \dfrac{X}{R(L - \lambda)} = 0$ which implies that $\lambda \in \rho(L)$ whenever $|\lambda| = 1$, a contradiction. Hence $\mathbb{S} \subset \sigma_d(L) \subset \sigma_{td}(L)$.
Consequently, $\sigma_{td}(L) = \mathbb{S}$.
On the other hand, let λ be in $\mathbb{S}$, the boundary of $\sigma(L)$ and assume that $d_e(L - \lambda) := q < \infty$. It follows

from Lemma 5.4 (i) that there exists a neighbourhood M of λ such that $dimN(L-\mu) = dim\dfrac{N((L-\lambda)^{q+1})}{N((L-\lambda)^q)}$ and $dim\dfrac{X}{R(L-\lambda)} = dim\dfrac{R((L-\lambda)^q)}{R((L-\lambda)^{q+1})}$ for all $\mu \in M$. Further, $M\backslash\sigma(L)$ is nonempty because λ belongs to the boundary of $\sigma(L)$. Therefore

$$dim\frac{N((L-\lambda)^{q+1})}{N((L-\lambda)^q)} = dim\frac{R((L-\lambda)^q)}{R((L-\lambda)^{q+1})} = 0.$$

Thus $a(L-\lambda) < \infty$ and $d(L-\lambda) < \infty$ and, since $a(L-\lambda) = 0$ with $D(L) = X$, we have that $d(L-\lambda) = 0$. So that $\lambda \in \rho(L)$ which contradicts the fact $\sigma(L) = \mathbb{D}$. Hence $\mathbb{S} \subset \sigma^e_d(L) \subset \sigma^e_{td}(L)$. Therefore $\sigma_{td}(L) = \sigma^e_{td}(L) = \mathbb{S}$.

$\sigma_{ta}(L) = \mathbb{D}$: Note that $N(L) = span\{e_1\}$, $R(L) = X$ and we infer from (5.1) that $N(L-\lambda) = \dfrac{N(L-\lambda)}{N(\lambda^{-1}-R)} \simeq R(\lambda^{-1}-R) \cap N(\lambda L) = N(L)$ if $0 < |\lambda| < 1$. So that $B(0,1) \subset \sigma_{ta}(L)$ by Lemma 5.1(i). According to Lemma 5.4(ii) we conclude that $\sigma_{ta}(L) = \mathbb{D}$, as desired.

$\sigma^e_{ta}(L) = \mathbb{S}$: Since $N(L) = span\{e_1\}$ we have that $0 \in \rho^e_{ta}(L) = \mathbb{S}$ and if $0 < |\lambda| < 1$ then one has by (5.1) that $a_e(L-\lambda) = 0$ and $R(L-\lambda) = X$ which implies that $B(0,1) \subset \rho^e_{ta}(L)$, so that $\sigma^e_{ta}(L) \subset \mathbb{S}$. On the other hand, since $\sigma_{ta}(L) = \mathbb{D} \neq \emptyset$ and $\mathbb{S}$ coincides with the boundary of $\sigma(L)$ we infer from Lemma 5.4(iii) that $\mathbb{S} \subsetneq \sigma^e_{ta}(L)$. Therefore $\sigma^e_{ta}(L) = \mathbb{S}$. □

Proposition 5.2. *(i)* $\sigma_{td}(L^{-1}) = \sigma^e_{td}(L^{-1}) = \sigma^e_{ta}(L^{-1}) = \mathbb{S}$.

(ii) $\sigma_{ta}(L^{-1}) = \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$.

Proof. Combine Lemma 5.3 with Proposition 5.1 □

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