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Some Mixed Paranorm Spaces

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Abstract. We generalize the concept of mixed norm spaces and define a class of mixed paranorm spaces, study their fundamental topological properties and determine their first and second duals. Furthermore we obtain the corresponding known results for mixed norm spaces and spaces of sequences that are strongly summable to zero as special cases of our new results.

1. Introduction and Notations

Let $1 \le p \le \infty$. By ω we denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$.
In 1968, Maddox [8] introduced and studied the sets

$$w_0^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\}$$

of sequences that are strongly summable to zero with index p by the Cesàro method of order 1. He also observed that the sections $1/n \sum_{k=1}^{n}$ can be replaced by the dyadic blocks $1/2^\nu \sum_{k=2^\nu}^{2^{\nu+1}-1}$, and that the section and block norms $\|\cdot\|_s$ and $\|\cdot\|_b$ are equivalent where

$$\|x\|_s = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \text{ and } \|x\|_b = \sup_\nu \left(\frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{1/p}.$$

In 1974, Jagers [5] studied the Cesàro sequence spaces

$$\mathrm{ces}_p = \left\{ x \in \omega : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\}$$

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which are Banach spaces with the norm given by

$$\|x\|_{\mathrm{ces}_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{1/p}.$$

It can be found in [3] that an equivalent norm on ces_p is

$$\|x\| = \left(\sum_{\nu=0}^{\infty}2^{\nu(1-p)}\left(\sum_{k=2^{\nu}}^{2^{\nu+1}-1}|x_k|\right)^p\right)^{1/p}.$$

In 1969, Hedlund [4] introduced the mixed norm spaces

$$\ell(r,p) = \left\{x \in \omega : \sum_{\nu=0}^{\infty}\left(\sum_{k=2^{\nu}}^{2^{\nu+1}-1}|x_k|^p\right)^{r/p} < \infty\right\} \text{ (see also Kellog [6]);}$$

obviously the Cesàro sequence spaces ces_p are weighted $\ell(p,1)$ mixed norm spaces. Results on the equivalence of block and section norms on mixed norm spaces can also be found in [3].

In this paper, we generalize the definition of mixed norm spaces to that of mixed paranorm spaces. This is achieved by replacing the dyadic blocks by arbitrary blocks, the constant exponent r by an arbitrary positive sequence $(r_\nu)_{\nu=0}^{\infty}$ and the spaces ℓ_r by $\ell(r)$ and $c_0(r)$. We are going to show among other things that our new spaces are FK spaces with AK if and only if the sequence $(r_\nu)_{\nu=0}^{\infty}$ is bounded. Furthermore we determine their first and second β–duals. Finally, we obtain many known results as special cases.

As usual, we denote by ℓ_∞, c, c_0 and ϕ the sets of all bounded, convergent, null and finite sequences, respectively. If $p = (p_k)_{k=1}^{\infty}$ is a sequence of positive reals then the sets

$$\ell(p) = \left\{x \in \omega : \sum_{k=1}^{\infty}|x_k|^{p_k} < \infty\right\} \text{ and } c_0(p) = \left\{x \in \omega : \lim_{x\to\infty}|x_k|^{p_k} = 0\right\}$$

are generalizations of the sets ℓ_p and c_0. Furthermore, let cs and bs be the sets of all convergent and bounded sequences. We write $\ell_p = \{x \in \omega : \sum_{k=1}^{\infty}|x_k| < \infty\}$ for $1 \le p < \infty$, and e and $e^{(n)}$ $(n = 1, 2, \dots)$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq 0)$, respectively.

An FK space X is a Fréchet sequence space with continuous coordinates $P_n : X \to \mathbb{C}$ where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^{\infty} \in X$ and $n = 1, 2, \dots$. We say that an FK space $X \supset \phi$ has AK if $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)} \to x$ $(m \to \infty)$; $x^{[m]}$ is called the m–section of the sequence x. A normable FK space is said to be a BK space.

If X and Y are subsets of ω and z is a sequence, we write $z^{-1} * Y = \{a \in \omega : a \cdot z = (a_k z_k)_{k=1}^{\infty} \in Y\}$ and $M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : a \cdot x \in Y \text{ for all } x \in X\}$. The special cases $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ are called the α–, β – and γ–duals of X.

2. The Definition of Our Spaces and Their Topological Structures

Throughout, let $(k(\nu))_{\nu=0}^{\infty}$ be a sequence of integers with

$$1 = k(0) < k(1) < \cdots. \tag{1}$$

By $K^{<k(\nu)>}$ $(\nu = 0, 1, \dots)$, we denote the set of all integers k that satisfy the inequality

$$k(\nu) \le k \le k(\nu+1) - 1$$

and we write $\sum_\nu = \sum_{k \in K^{<k(\nu)>}}$ and $\max_\nu = \max_{k \in K^{<k(\nu)>}}$. Given any sequence $x = (x_k)_{k=1}^{\infty} \in \omega$, we define the $K^{<k(\nu)>}$ blocks of x by

$$x^{<\nu>} = \sum_\nu x_k e^{(k)} \text{ for } \nu = 0, 1, \dots.$$

Now we define the mixed paranorm–norm spaces.

Definition 2.1. *Let $(k(\nu))_{\nu=0}^{\infty}$ be a sequence of integers that satisfy the condition in (1). Furthermore, let $(X_i, \|\cdot\|_i)$ be a normed space, (X_o, g_o) be a paranormed space and $\phi \subset X_i, X_o$. We denote the sequences in X_i by $x = (x_k)_{k=1}^{\infty}$ and those in X_o by $y = (y_\nu)_{\nu=0}^{\infty}$ and define the set*

$$Z = [X_o, X_i]^{<k(\nu)>} = \left\{ z = (z_k)_{k=1}^{\infty} \in \omega : \left(\left\| z^{<\nu>} \right\|_i \right)_{\nu=0}^{\infty} \in X_o \right\},$$

and write

$$h(z) = g_o\left(\left(\left\| z^{<\nu>} \right\|_i \right)_{\nu=0}^{\infty} \right) \text{ for all } z \in Z.$$

Remark 2.2. *(a) Since $\phi \subset X_i$, $\|z^{<\nu>}\|_i$ is defined for all $z \in \omega$ and all $\nu = 0, 1, \dots$. Hence the sequence*

$$y = (y_\nu)_{\nu=0}^{\infty} = \left(\left\| z^{<\nu>} \right\|_i \right)_{\nu=0}^{\infty} \tag{2}$$

is defined.
(b) Since $\phi \subset X_i, X_o$, we obviously have $\phi \subset Z$.

We say that a norm $\|\cdot\|$ or a paranorm g on a sequence space X is *monotonous*, if $x, \tilde{x} \in X$ and $|x_k| \le |\tilde{x}_k|$ for all $k = 1, 2, \dots$ together imply $\|x\| \le \|\tilde{x}\|$ or $g(x) \le g(\tilde{x})$. We also recall that a subset X of ω is said to be *normal* if $x \in X$ and $|\tilde{x}_k| \le |x_k|$ for all $k = 1, 2, \dots$ together imply $\tilde{x} \in X$.

Example 2.3. *(a) The natural norms on ℓ_p $(1 \le p < \infty)$, ℓ_∞, c and c_0 are monotonous.*
(b) Let $p = (p_k)_{k=1}^{\infty}$ be a bounded sequence of positive reals and $M(p) = \max\{1, \sup_k p_k\}$. Then the paranorms g_p and $g_{p,0}$ with

$$g_p(x) = \left(\sum_{k=1}^{\infty} |x_k|^{p_k} \right)^{1/M(p)} \text{ and } g_{p,0}(x) = \sup_k |x_k|^{p_k/M(p)}$$

on $\ell(p)$ and $c_0(p)$ are monotonous.
(c) The sets $\ell(p)$, $\ell_\infty(p)$ and $c_0(p)$ are normal, but $c(p)$ is not normal.

Remark 2.4. *Let $(X, \|\cdot\|)$ be a normed space. Then neither does the monotony of $\|\cdot\|$ imply that X is normal, nor does the converse implication hold, in general.*

Proof. If we choose $X = c$ and $\|\cdot\| = \|\cdot\|_\infty$, then $\|\cdot\|$ is monotonous, but c is not normal.
To prove the second part, we choose $X = \ell_1$ and $\|\cdot\| = \|\cdot\|_{bv}$ where $\|x\|_{bv} = \sum_{k=1}^{\infty} |x_k - x_{k-1}|$ with the convention $x_0 = 0$. Then $\|\cdot\|$ obviously is defined on X and X is normal. But, for $x = e^{(1)} + e^{(3)}$ and $\tilde{x} = e^{(1)} + e^{(2)} + e^{(3)}$, we obtain $|x_k| \le |\tilde{x}_k|$ for $k = 1, 2, \dots$ and $\|x\| = 4 > 2 = \|\tilde{x}\|$. □

Now we establish some general results on the structures of the mixed paranorm spaces.

Theorem 2.5. *Let $(X_i, \|\cdot\|_i)$ be a normed space, (X_o, g_o) be a paranormed space, $\phi \subset X_i, X_o$, and $Z = [X_o, X_i]^{<k(\nu)>}$.*
(a) If X_o is normal and $\|\cdot\|_i$ is monotonous, then Z is normal.
(b) If g_o is monotonous, then Z is paranormed with respect to h; if g_o is total, so is h. If, however, g_o is not monotonous, then h does not satisfy the triangle inequality, in general.
(c) If $(X, \|\cdot\|_i)$ and (X_o, g_o) both are FK spaces, g_o is monotonous, and Z is complete, then Z is and FK space.
(d) If X_i is any of the spaces ℓ_p $(1 \le p \le \infty)$ and X_o is any of the spaces $\ell(r)$ or $c_0(r)$ for $r = (r_\nu)_{\nu=0}^{\infty} \in \ell_\infty$, then Z is complete.

Proof. For any sequence $z \in Z$, we define the sequence $y \in X_o$ by (2).
(a) Let $z \in Z$ and $|\tilde{z}_k| \le |z_k|$ for $k = 1, 2, \dots$. Then it follows from the monotony of $\|\cdot\|_i$ that $\tilde{y}_\nu = \|\tilde{z}^{<\nu>}\|_i \le \|z^{<\nu>}\|_i = y_\nu$ for $\nu = 0, 1, \dots$ and, since X_o is normal, this implies $\tilde{y} \in X_o$, hence $\tilde{z} \in Z$.

(b) Obviously, h is defined on Z, $h(0) = 0$, $h(z) \geq 0$ and $h(z) = h(-z)$ for all $z \in Z$. Let $z, \tilde{z} \in Z$ be given. Then we obtain by the triangle inequality for $\|\cdot\|_i$, the monotony of g_o and the triangle inequality for g_o

$$\begin{aligned} h(z+\tilde{z}) &= g_o\left(\left(\left\|(z+\tilde{z})^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) = g_o\left(\left(\left\|z^{<\nu>} + \tilde{z}^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \\ &\leq g_o\left(\left(\left\|z^{<\nu>}\right\|_i + \left\|\tilde{z}^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) = g_o\left(\left(\left\|z^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty} + \left(\left\|\tilde{z}^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \\ &\leq g_o\left(\left(\left\|z^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) + g_o\left(\left(\left\|\tilde{z}^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) = h(z) + h(\tilde{z}). \end{aligned}$$

Finally, let $\lambda_n \to \lambda$ and $h(z^{(n)} - z) \to 0$ $(n \to \infty)$. Then we have

$$0 \leq h(\lambda_n z^{(n)} - \lambda z) \leq S_1(n) + S_2(n) + S_3(n),$$

where

$$\begin{aligned} 0 \leq S_1(n) &= h\left((\lambda_n - \lambda)(z^{(n)} - z)\right) = g_o\left(\left(\left\|(\lambda_n - \lambda)\left(z^{(n)} - z\right)^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \\ &= g_o\left(|\lambda_n - \lambda| \cdot \left(\left\|\left(z^{(n)} - z\right)^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \leq g_o\left(\left(\left\|\left(z^{(n)} - z\right)^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \\ &= h(z^{(n)} - z) \text{ for all sufficiently large } n, \end{aligned}$$

since $|\lambda_n - \lambda| \leq 1$ for all sufficiently large n and g_o is monotonous, hence $\lim_{n\to\infty} S_1(n) = 0$; also

$$\begin{aligned} 0 \leq S_2(n) &= h((\lambda_n - \lambda)z) = g_o\left(\left(\left\|(\lambda_n - \lambda)z^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \\ &= g_o\left(|\lambda_n - \lambda| \cdot \left(\left\|z^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \end{aligned}$$

and putting $\mu_n = |\lambda_n - \lambda|$ and $y^{(n)} = y = (\|z^{<\nu>}\|_i)_{\nu=0}^{\infty}$ for $\nu = 0, 1, \ldots$, we have $\mu_n \to 0$, $g_o(y^{(n)} - y) \to 0$ $(n \to \infty)$, and so, since g_o is a paranorm, it follows that $g_o(\mu_n y^{(n)}) \to 0$ $(n \to \infty)$, that is $\lim_{n\to\infty} S_2(n) = 0$; finally

$$S_3(n) = h\left(\lambda(z^{(n)} - z)\right) = g_o\left(|\lambda| \left(\left\|\left(z^{(n)} - z\right)^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right)$$

and putting $\mu_n = \mu = |\lambda|$ and $y^{(n)} = (\|(z^{(n)} - z)^{<\nu>}\|_i)_{\nu=0}^{\infty}$ for all n, we have $\mu_n \to \mu$, $g_o(y^{(n)}) \to 0$ $(n \to \infty)$, and so, since g_o is a paranorm, $g_o(\mu_n y^{(n)} - 0) \to 0$ $(n \to \infty)$, that is, $\lim_{n\to\infty} S_3(n) = 0$.
Thus we have shown $h(\lambda_n z_n - \lambda z) \to 0$ $(n \to \infty)$.
Furthermore, if g_o is total, then $h(z) = g_o\left((\|z^{<\nu>}\|_i)_{\nu=0}^{\infty}\right) = 0$ if and only if $\|z^{<\nu>}\|_i = 0$ for $\nu = 0, 1, \ldots$, and this is the case if and only if $z_k = 0$ for $k = 1, 2, \ldots$, that is, $z = 0$.
To see the last part, we consider $X_o = bv$ and $g_o = \|\cdot\|_{bv}$. Then g_o is a norm on X_o which is not monotonous, as we have seen in the proof of Remark 2.4. Now we choose $k(\nu) = \nu + 1$ for $\nu = 0, 1, \ldots$, $X_i = \ell_\infty$, $\|\cdot\|_i = \|\cdot\|_\infty$, $z = e^{(1)} + e^{(2)} + e^{(3)}$ and $\tilde{z} = e^{(1)} - e^{(2)} + e^{(3)}$, and obtain $h(z + \tilde{z}) = 2h(e^{(1)} + e^{(3)}) = 8 > 4 = h(z) + h(\tilde{z})$.

(c) We have to show that $z^{(n)} \to z$ $(n \to \infty)$ implies $z_k^{(n)} \to z_k$ $(n \to \infty)$ for each k.
Let

$$h(z^{(n)} - z) = g_o\left(\left(\left\|\left(z^{(n)} - z\right)^{<\nu>}\right\|_i\right)_{\nu=0}^{\infty}\right) \to 0 \text{ as } n \to \infty.$$

Since X_o is an FK space, this implies $\|(z^{(n)} - z)^{<\nu>})\|_i \to 0$ $(n \to \infty)$ for each ν, and since X_i is an FK space, this implies $z_k^{(n)} \to z_k$ for each $k \in K^{<k(\nu)>}$ and $\nu = 0, 1, \ldots$.

(d) We have to show that $Z = [X_o, X_i]^{<k(\nu)>}$ is complete when X_i is any of the spaces ℓ_p $(1 \leq p \leq \infty)$ and X_o is any of the spaces $\ell(r)$ or $c_0(r)$.
Let $(z^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in Z and $\varepsilon > 0$ be given. Then there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that $h(z^{(n)} - z^{(m)}) < \varepsilon/2$ for all $n, m \geq N$. Since X_i and X_o are FK spaces, it follows that $(z_k^{(n)})_{n=1}^{\infty}$ is a Cauchy

sequence of complex numbers for each fixed k, hence convergent by the completeness of $\mathbb{C}$, $z_k = \lim_{m\to\infty} z_k^{(m)}$ $(k = 1, 2, \dots)$, say. Let $\mu \in \mathbb{N}_0$ be given and $z(\mu) = z^{[k(\mu+1)-1]} = \sum_{k=1}^{k(\mu+1)-1} z_k e^{(k)}$. We fix $n \geq N$. Then we have for $X_o = \ell(r)$

$$\begin{aligned}\lim_{m\to\infty} h\left(\left(z^{(n)} - z^{(m)}\right)(\mu)\right) &= \lim_{m\to\infty}\left(\sum_{\nu=0}^{\mu}\left\|\left(z_k^{(n)} - z_k^{(m)}\right)^{<\nu>}\right\|_p^{r_\nu}\right)^{1/M(r)} \\ &= \left(\sum_{\nu=0}^{\mu}\left\|\left(z_k^{(n)} - z_k\right)^{<\nu>}\right\|_p^{r_\nu}\right)^{1/M(r)} \\ &= h\left(\left(z^{(n)} - z\right)(\mu)\right) \leq \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Since $\mu \in \mathbb{N}_0$ was arbitrary, we have

$$h\left(z^{(n)} - z\right) < \varepsilon \text{ for all } n \geq N. \tag{3}$$

Now (3) implies $z^{(N)} - z \in Z = [\ell(r), \ell_p]^{<k(\nu)>}$, and since Z is a linear space, we have $z \in Z$. Thus we have shown that $[\ell(r), \ell_p]^{<k(\nu)>}$ is complete.
Finally let $X_o = c_0(r)$. As before, it can be shown that (3) holds for all $n \geq N$. Since $z^{(N)} \in Z = [c_0(r), \ell_p]^{<k(\nu)>}$, there is an integer ν_0 such that

$$\left\|\left(z^{(N)}\right)^{<\nu>}\right\|_p^{r_\nu/M(r)} < \frac{\varepsilon}{2} \text{ for all } \nu \geq \nu_0, \tag{4}$$

and we obtain from (4) and (3) for all $\nu \geq \nu_0$

$$\left\|z^{<\nu>}\right\|_i^{r_\nu/M(r)} \leq \left\|\left(z - z^{(N)}\right)^{<\nu>}\right\|_p^{r_\nu/M(r)} + \left\|\left(z^{(N)}\right)^{<\nu>}\right\|_p^{r_\nu/M(r)} < \varepsilon,$$

hence $z \in Z$. Thus we have shown that $Z = [c_0(r), \ell_p]^{<k(\nu)>}$ is complete. □

Corollary 2.6. *Let $1 \leq p \leq \infty$ and $r = (r_\nu)_{\nu=0}^\infty$ be a bounded positive sequence. Then $Z = [\ell(r), \ell_p]^{<k(\nu)>}$ and $Z = [c_0(r), \ell_p]^{<k(\nu)>}$ are FK spaces with AK with respect to the total paranorms h defined by*

$$h(z) = \begin{cases}\left(\sum\limits_{\nu=0}^{\infty}\|z^{<\nu>}\|_p^{r_\nu}\right)^{1/M(r)} & (z \in Z = [\ell(r), \ell_p]^{<k(\nu)>}) \\ \sup\limits_\nu \|z^{<\nu>}\|_p^{r_\nu/M(r)} & (z \in Z = [c_0(r), \ell_p]^{<k(\nu)>}).\end{cases} \tag{5}$$

Proof. In view of Theorem 2.5 and Example 2.3, the spaces Z are *FK* spaces in both cases with respect to h defined in (5); hence we have to show that they have *AK*. First we observe that $\phi \subset Z$ by Remark 2.2 (b). We consider the case of $Z = [\ell(r), \ell_p]^{<k(\nu)>}$. Let $z \in Z$ and $\varepsilon > 0$ be given. Then there exists $\nu_0 \in \mathbb{N}_0$ such that

$$\left(\sum_{\nu=\nu_0}^{\infty}\left\|z^{<\nu>}\right\|_p^{r_\nu}\right)^{1/M(r)} < \varepsilon. \tag{6}$$

We choose $m_0 = k(\nu_0) \in \mathbb{N}$. For $m \geq m_0$, let $\nu(m)$ denote the integer such that $m \in K^{<\nu(m)>}$. Then we have for all $m \geq m_0$

$$h(z^{[m]} - z) = \left(\sum_{\nu=0}^{\infty}\left\|\left(z^{[m]} - z\right)^{<\nu>}\right\|_p^{r_\nu}\right)^{1/M(r)}$$

$$= \left(\left\| \left(z^{[m]} - z\right)^{<\nu(m)>} \right\|_p^{r_\nu} + \sum_{\nu=\nu(m+1)}^{\infty} \left\| z^{<\nu>} \right\|_p^{r_\nu} \right)^{1/M(r)}$$
$$\leq \left(\sum_{\nu=\nu_0}^{\infty} \left\| z^{<\nu>} \right\|_p^{r_\nu} \right)^{1/M(r)} < \varepsilon,$$

and so $z = \lim_{m\to\infty} z^{[m]} = \sum_{k=1}^{\infty} x_k e^{(k)}$.
It is easy to see that this representation is unique.
The case of $[c_0(p), \ell_r]^{<k(\nu)>}$ is proved analogously. □

Remark 2.7. *Let r be a sequence of positive reals and $1 \leq p \leq \infty$. Then $[\ell(r), \ell_p]^{<k(\nu)>}$ and $[c_0(r), \ell_p]^{<k(\nu)>}$ are linear spaces if and only if $r \in \ell_\infty$.*

Proof. The sufficiency of the condition $r \in \ell_\infty$ is clear by Corollary 2.6.
To show the necessity of the condition we assume that $r \notin \ell_\infty$. Then there exists a strictly increasing sequence $(\nu(i))_{i=0}^{\infty}$ of positive integers such that $r_{\nu(i)} > i$ for $i = 0, 1, \dots$. We define the sequence x by $x_{k(\nu(i))} = 1/2$ and $x_k = 0$ for $k \neq (k(\nu(i)))$ $(i = 0, 1, \dots)$. Then we have $\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_p^{r(\nu)} = \sum_{i=0}^{\infty} 2^{r_{\nu(i)}} < \sum_{i=0}^{\infty} 2^{-i} < \infty$, that is, $x \in [\ell(r), \ell_p]^{<k(\nu)>}$, but

$$\left\| 2 \cdot x^{<\nu(i)>} \right\|_p^{r_{\nu(i)}} = 1 \text{ for all } i, \text{ that is, } 2x \notin [c_0(r), \ell_p]^{<k(\nu)>}.$$

The statement now is clear, since obviously $[\ell(r), \ell_p]^{<k(\nu)>} \subset [c_0(r), \ell_p]^{<k(\nu)>}$. □

The following figures show the projections on (x_2, x_3, x_4) of the unit balls in $[\ell(r), \ell_p]^{<k(\nu)>}$ when $k(\nu) = 2^\nu$ $(\nu = 0, 1, \dots)$

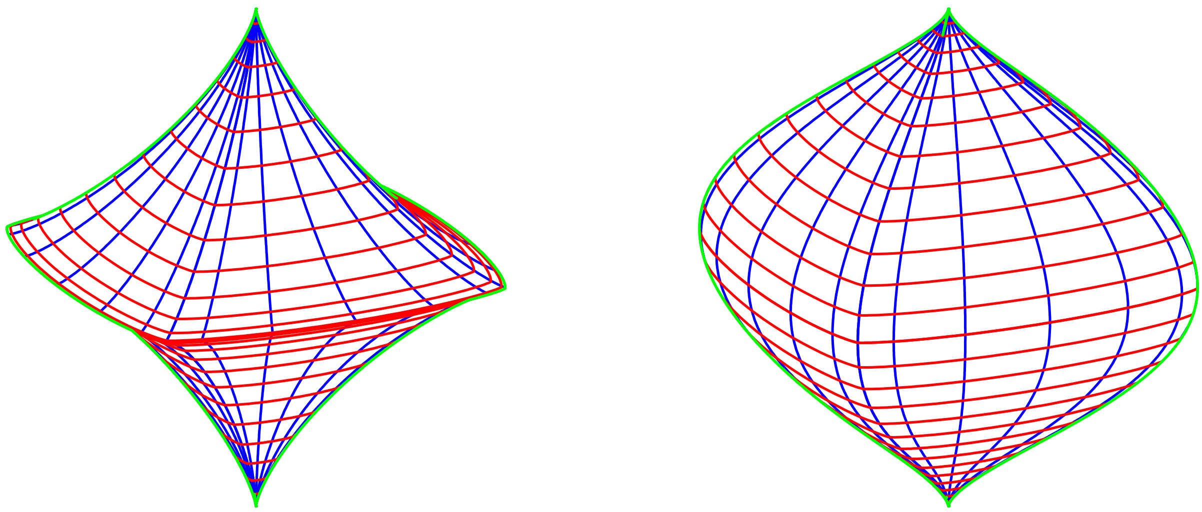

Figure 1: Left $r_1 = 3/4$, $r_2 = 4/5$, $p = 5/4$. Right $r_1 = 2/3$, $r_2 = 2$, $p = 5/4$.

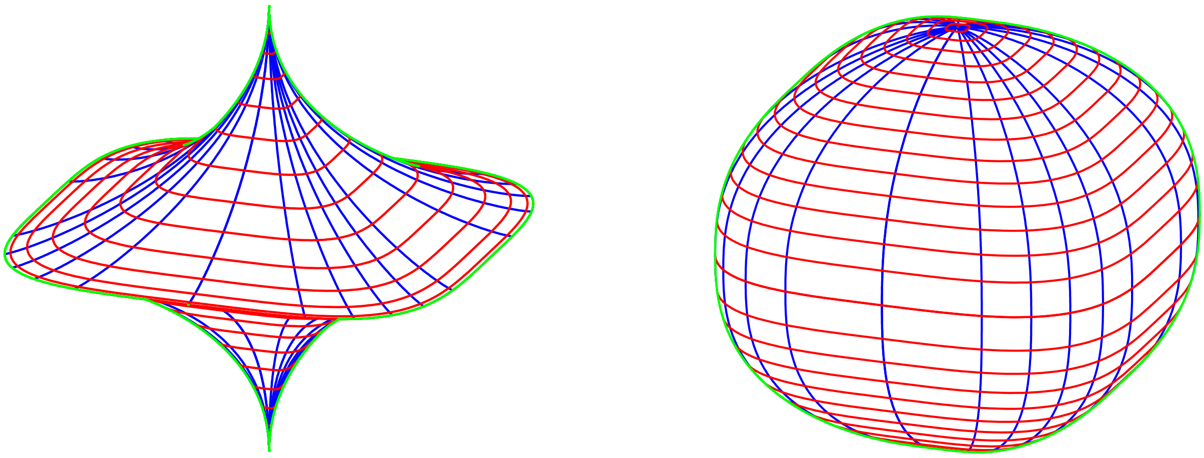

Figure 2: Left $r_1 = 1/2$, $r_2 = 4/5$, $p = 4$. Right $r_1 = 3/2$, $r_2 = 5/2$, $p = 7/2$.

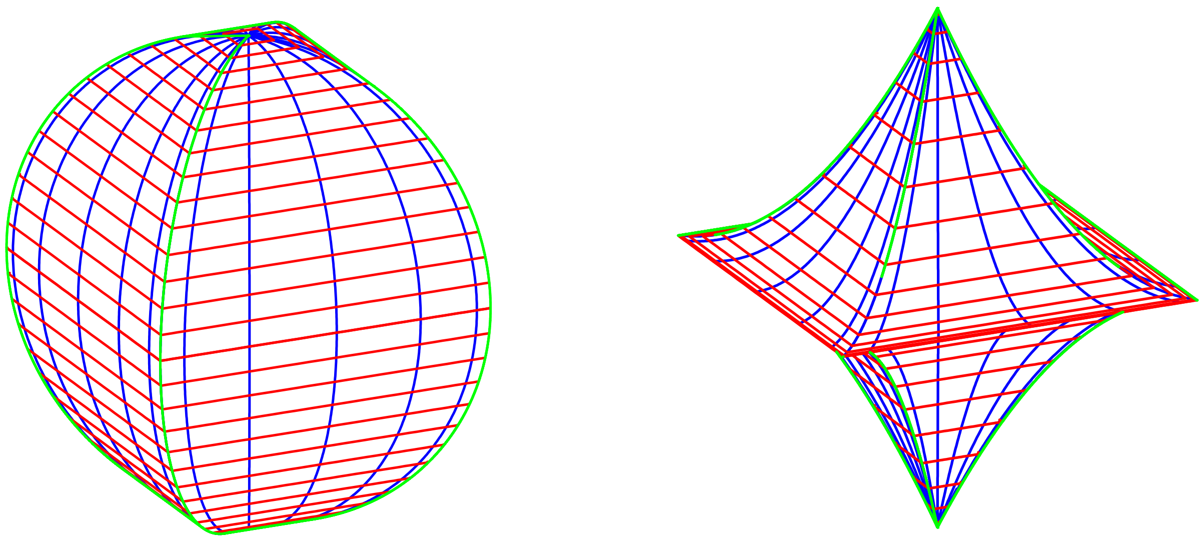

Figure 3: Left $r_1 = 2$, $r_2 = 2$, $p = 1$. Right $r_1 = 1$, $r_2 = 1/2$, $p = 1$.

We close this section by considering some special cases.

Example 2.8. *(a) Let $r_\nu = r \geq 1$ $(\nu = 0, 1, \dots)$ and $1 \leq p \leq \infty$. Then the spaces $[\ell(r), \ell_p]^{<k(\nu)>}$ reduce to the mixed norm spaces $\ell(r,p) = [\ell_r, \ell_p]^{<k(\nu)>}$ ([3, 5]).*
(b) If $k(\nu) = 2^\nu$ and $r_\nu = 1$ for $\nu = 0, 1, \dots$, and the sequence $u = (u_k)_{k=1}^\infty$ is defined by $u_k = 2^{-\nu/p}$ $(k \in K^{<k(\nu)>}; \nu = 0, 1, \dots)$, then we obtain the sets

$$w_0^p = u^{-1} * \left[c_0, \ell_p\right]^{<k(\nu)>} = \left\{x \in \omega : \lim_{\nu \to \infty} \frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p = 0\right\}$$

of sequences that are strongly summable C_1 to 0, with index p ([8]).
(c) By Corollary 2.6 and [14, Theorem 4.3.6], the sets $\ell(r,p)$ and w_0^p are BK space with AK with respect to the norms $\|\cdot\|_{(r,p)}$ and $\|\cdot\|_{(0,p)}$ defined by

$$\|x\|_{(r,p)} = \left\|\left(\left\|x^{<\nu>}\right\|_p\right)_{\nu=0}^\infty\right\|_r = \left(\sum_{\nu=0}^\infty \left\|x^{<\nu>}\right\|_p^r\right)^{1/r}$$

and

$$\|x\|_{(0,p)} = \left\|\left(\|(u \cdot x)^{<\nu>}\|_p\right)_{\nu=0}^\infty\right\|_\infty = \sup_\nu \left(\frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p\right)^{1/p} \quad \text{([8], [10, Proposition 3.44]).}$$

Example 2.9. *Let $r = (r_\nu)_{\nu=0}^\infty$ be any sequence of positive reals and $1 \leq p \leq \infty$. If $k(\nu) = \nu + 1$ for $\nu = 0, 1, \dots$, then we obtain $[\ell(r), \ell_p]^{<k(\nu)>} = \ell(r)$ and $[c_0(r), \ell_p]^{<k(\nu)>} = c_0(r)$. It follows from Corollary 2.6 and Remark 2.7 that $\ell(r)$ and $c_0(p)$ are FK spaces with AK if and only if $r \in \ell_\infty$ ([2] and [1, Remark]).*

3. The Dual Spaces

Here we determine the dual spaces of $[c_0(p), \ell_r]^{<k(\nu)>}$ and $[\ell(p), \ell_r]^{<k(\nu)>}$.

Since both $[c_0(p), \ell_r]^{<k(\nu)>}$ and $[\ell(p), \ell_r]^{<k(\nu)>}$ are normal by Theorem 2.5 (a), the α–, β– and γ–duals coincide. Also if $r = (r_\nu)_{\nu=0}^\infty \in \ell_\infty$, then $[c_0(p), \ell_r]^{<k(\nu)>}$ and $[\ell(p), \ell_r]^{<k(\nu)>}$ are *FK* spaces with *AK* by Corollary 2.6 and so the β– and functional duals conicide ([14, Theorem 7.2.7 (ii)]).

Throughout, let $r = (r_\nu)_{\nu=0}^\infty$ be a sequence of positive reals, not necessarily bounded, $1 \leq p \leq \infty$ and q the conjugate number of p, that is, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$; also let $s_\nu = r_\nu/(r_\nu - 1)$ for $r_\nu > 1$. We define the following sets

$$M_0((r), p) = \bigcup_{N>1} \left\{a \in \omega : \sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_q \cdot N^{-1/r_\nu} < \infty\right\},$$

$$M((r), p) = \begin{cases} \bigcup_{N>1} \left\{a \in \omega : \sum_{\nu=0}^\infty \|a^{<\nu>}\|_q^{s_\nu} \cdot N^{-s_\nu/r_\nu} < \infty\right\} & \text{if } r_\nu > 1 \text{ for all } \nu \\ [\ell_\infty(r), \ell_q]^{<k(\nu)>} & \text{if } r_\nu \leq 1 \text{ for all } \nu. \end{cases}$$

Theorem 3.1. *We have*

$$\left([c_0(r), \ell_p]^{<k(\nu)>}\right)^\beta = M_0((r), p). \tag{7}$$

Proof. First we show

$$M_0((r), p) \subset \left([c_0(r), \ell_p]^{<k(\nu)>}\right)^\beta. \tag{8}$$

We assume $a \in M_0((r), p)$. Then there is $N > 1$ such that $\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q \cdot N^{-1/r_\nu} < \infty$. Let $x \in [c_0(r), \ell_p]^{<k(\nu)>}$ be given. Then there exists $\nu_0 \in \mathbb{N}_0$ such that we have for all $\nu \geq \nu_0$

$$\left\|x^{<\nu>}\right\|_p^{r_\nu} \leq \frac{1}{N}, \text{ that is, } \left\|x^{<\nu>}\right\|_p \leq N^{-1/r_\nu}.$$

Since

$$\textstyle\sum_\nu |a_k x_k| \leq \left\|a^{<\nu>}\right\|_q \cdot \left\|x^{<\nu>}\right\|_p \text{ for all } \nu \text{ and for } 1 \leq p \leq \infty, \tag{9}$$

(the case $1 < p < \infty$ by Hölder's inequality), we obtain

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{\nu=0}^{\infty} \textstyle\sum_\nu |a_k x_k| \leq \displaystyle\sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_q \cdot \left\|x^{<\nu>}\right\|_p \leq \sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_q \cdot N^{-1/r_\nu} < \infty,$$

that is, $a \in ([c_0(r), \ell_p]^{<k(\nu)>})^\beta$. Thus we have shown (8).
Now we show

$$\left([c_0(r), \ell_p]^{<k(\nu)>}\right)^\beta \subset M_0((r), p). \tag{10}$$

We assume that $a \notin M_0((r), p)$. Then $\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q \cdot N^{-1/r_\nu} = \infty$ for all $N > 1$, and consequently we can determine a sequence $(\nu(n))_{n=0}^{\infty}$ of integers $0 = \nu(0) < \nu(1) < \nu(2) < \dots$ such that

$$M_n = \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q \cdot (n+1)^{-1/r_\nu} > 1 \text{ for } n = 0, 1, \dots.$$

If $1 < p < \infty$, we define the sequence x by

$$x_k = \operatorname{sgn}(a_k)|a_k|^{q-1} \left\|a^{<\nu>}\right\|_q^{-q/p} (n+1)^{-1/r_\nu} M_n^{-1}$$
$$(k \in K^{<k(\nu)>}; \nu(n) \leq \nu \leq \nu(n+1) - 1; n = 0, 1, \dots).$$

Then we have for $n = 0, 1, \dots$

$$\begin{aligned}\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \textstyle\sum_\nu a_k x_k &= \displaystyle\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{q(1-1/p)} (n+1)^{-1/r_\nu} M_n^{-1} \\ &= \frac{1}{M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q (n+1)^{-1/r_\nu} = 1,\end{aligned}$$

hence $\sum_{k=1}^{\infty} a_k x_k$ diverges. But, since $M_n > 1$ for all n, we have for $\nu(n) \leq \nu \leq \nu(n+1) - 1$

$$\left\|x^{<\nu>}\right\|_p^{r_\nu} = \left(\textstyle\sum_\nu |a_k|^{qp-p} \left\|a^{<\nu>}\right\|_q^{-q} (n+1)^{-p/r_\nu} M_n^{-p}\right)^{r_\nu/p} = \dfrac{M_n^{-r_\nu}}{n+1} \left\|a^{<\nu>}\right\|_q^{(q-q)r_\nu/p} \leq \dfrac{1}{n+1},$$

that is, $x \in [c_0(r), \ell_p]^{<k(\nu)>}$. Thus we have shown (10) when $1 < p < \infty$.
If $p = 1$, we define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}\left(a_{k_0(\nu)}\right)(n+1)^{-1/r_\nu} M_n^{-1} & (k = k_0(\nu)) \\ \text{where } k_0(\nu) \text{ is the smallest integer in } K^{<k(\nu)>} & \\ \text{ with } |a_{k_0(\nu)}| = \|a^{<\nu>}\|_\infty & \\ 0 & (k \neq k_0(\nu)) \\ \qquad (k \in K^{<k(\nu)>}; \nu(n) \leq \nu \leq \nu(n+1) - 1; n = 0, 1, \dots). & \end{cases}$$

Then we have for $n = 0, 1, \dots$

$$\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \sum_\nu a_k x_k = \sum_{\nu=\nu(n)}^{\nu(n+1)-1} |a_{k_0(\nu)}|(n+1)^{-1/r_\nu} M_n^{-1} = \frac{1}{M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_\infty (n+1)^{-1/r_\nu} = 1,$$

hence $\sum_{k=1}^\infty a_k x_k$ diverges. But, since $M_n > 1$ for all n, we have for $\nu(n) \le \nu \le \nu(n+1) - 1$

$$\|x^{<\nu>}\|_1^{r_\nu} \le \frac{M_n^{-r_\nu}}{n+1} \le \frac{1}{n+1},$$

hence $x \in [c_0(r), \ell_1]^{<k(\nu)>}$. Thus we have shown (10) when $p = 1$.
Finally, if $p = \infty$, we define the sequence x by

$$x_k = \operatorname{sgn}(a_k)(n+1)^{-1/r_\nu} M_n^{-1} \ (k \in K^{<k(\nu)>}; \nu(n) \le \nu \le \nu(n+1) - 1; n = 0, 1, \dots).$$

Then we have for $n = 0, 1, \dots$

$$\begin{aligned}\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \sum_\nu a_k x_k &= \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \sum_\nu |a_k|(n+1)^{-1/r_\nu} M_n^{-1} \\ &= \frac{1}{M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_1 (n+1)^{-1/r_\nu} = 1,\end{aligned}$$

hence $\sum_{k=1}^\infty a_k x_k$ diverges. But, since $M_n > 1$ for all n, we have for $\nu(n) \le \nu \le \nu(n+1) - 1$

$$\|x^{<\nu>}\|_\infty^{r_\nu} \le \frac{M_n^{-r_\nu}}{n+1} \le \frac{1}{n+1},$$

hence $x \in [c_0(r), \ell_1]^{<k(\nu)>}$. Thus we have shown (10), when $p = \infty$.
Consequently, we have established (10) for $1 \le p \le \infty$, and finally, (10) and (8) yield (7). □

Theorem 3.2. *We have*

$$\left([\ell(r), \ell_p]^{<k(\nu)>}\right)^\beta = M((r), p). \tag{11}$$

Proof. **Case 1.** $r_\nu > 1$ for all $\nu = 0, 1, \dots$. First we show

$$M((r), p) \subset \left([\ell(r), \ell_p]^{<k(\nu)>}\right)^\beta. \tag{12}$$

Let $a \in M((r), p)$ be given. Then there exists $N > 1$ such that $\sum_{\nu=0}^\infty \|a^{<\nu>}\|_q^{s_\nu} N^{-s_\nu/r_\nu} < \infty$. Let $x \in [\ell(r), \ell_p]^{<k(\nu)>}$ be given. Using (9) and applying the well–known inequality $|b_\nu y_\nu| \le |b_\nu|^{s_\nu} + |y_\nu|^{r_\nu}$ with $b_\nu = \|a^{<\nu>}\|_q N^{-1/r_\nu}$ and $y = \|x^{<\nu>}\|_p N^{1/r_\nu}$ $(\nu = 0, 1, \dots)$, we obtain

$$\sum_\nu |a_k x_k| \le \left\|a^{<\nu>}\right\|_q \cdot \left\|x^{<\nu>}\right\|_p \le \left\|a^{<\nu>}\right\|_q^{s_\nu} N^{-s_\nu/r_\nu} + N \left\|x^{<\nu>}\right\|_p^{r_\nu} \text{ for } \nu = 0, 1, \dots,$$

hence

$$\begin{aligned}\sum_{k=1}^\infty |a_k x_k| &\le \sum_{\nu=0}^\infty \left(\left\|a^{<\nu>}\right\|_q^{s_\nu} N^{-s_\nu/r_\nu} + N \left\|x^{<\nu>}\right\|_p^{r_\nu} \right) \\ &= \sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_q^{s_\nu} N^{-s_\nu/r_\nu} + N \sum_{\nu=0}^\infty \left\|x^{<\nu>}\right\|_p^{r_\nu} < \infty.\end{aligned}$$

Thus we have shown (12).
Now we show

$$\left([\ell(r),\ell_p]^{<k(\nu)>}\right)^{\beta} \subset M((r),p). \tag{13}$$

We assume $a \notin M((r),p)$. Then $\sum_{\nu=0}^{\infty} \|a^{<\nu>}\|_q^{-s_\nu} \cdot N^{-s_\nu/r_\nu} = \infty$ for all $N > 1$, and consequently we can determine a sequence $(\nu(n))_{n=0}^{\infty}$ of integers $0 = \nu(0) < \nu(1) < \nu(2) < \ldots$ such that

$$M_n = \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{s_\nu} \cdot (n+1)^{-s_\nu/r_\nu} > 1 \text{ for } n = 0,1,\ldots.$$

If $1 < p < \infty$ then we define the sequence x by

$$x_k = \operatorname{sgn}(a_k)|a_k|^{q-1} \left\|a^{<\nu>}\right\|_q^{s_\nu - q} \cdot (n+1)^{-s_\nu} M_n^{-1}$$
$$(k \in K^{<k(\nu)>}; \nu(n) \leq \nu \leq \nu(n+1)-1; n = 0,1,\ldots).$$

Since $1 - s_\nu = -s_\nu/r_\nu$ for all ν, we obtain for all n

$$\begin{aligned}\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \textstyle\sum_\nu a_k x_k &= \sum_{\nu=\nu(n)}^{\nu(n+1)-1} |a_k|^q \left\|a^{<\nu>}\right\|_q^{s_\nu - q} \cdot (n+1)^{-s_\nu} M_n^{-1} \\ &= \frac{1}{(n+1)M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{s_\nu} \cdot (n+1)^{-s_\nu/r_{(\nu)}} = \frac{1}{n+1},\end{aligned}$$

and so $\sum_{k=1}^{\infty} a_k x_k$ diverges. But, since $r_\nu(s_\nu - q + q/p) = s_\nu$, we have for all ν with $\nu(n) \leq \nu \leq \nu(n+1) - 1$ and for all n

$$\begin{aligned}\left\|x^{<\nu>}\right\|_p^{r_\nu} &= \left(\textstyle\sum_\nu |a_k|^{pq-p}\right)^{r_\nu/p} \left\|a^{<\nu>}\right\|_q^{r_\nu(s_\nu - q)} (n+1)^{-s_\nu r_\nu} M_n^{-r_\nu} \\ &= \left\|a^{<\nu>}\right\|_q^{r_\nu(s_\nu - q + q/p)} (n+1)^{-s_\nu r_\nu} M_n^{-r_\nu} = \left\|a^{<\nu>}\right\|_q^{s_\nu} (n+1)^{-s_\nu r_\nu} M_n^{-r_\nu}.\end{aligned}$$

Furthermore, since $M_n > 1$ for all n, $r_\nu > 1$ and $s_\nu = 1 + s_\nu/r_\nu$, it follows that for all n

$$\begin{aligned}\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|x^{<\nu>}\right\|_p^{r_\nu} &\leq \frac{1}{M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{s_\nu} (n+1)^{-s_\nu - r_\nu} \\ &\leq \frac{1}{(n+1)M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{s_\nu} (n+1)^{-s_\nu} \\ &= \frac{1}{(n+1)^2 M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|a^{<\nu>}\right\|_q^{s_\nu} (n+1)^{-s_\nu/r_\nu} = \frac{1}{(n+1)^2},\end{aligned}$$

that is,

$$\sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_p^{r_\nu} = \sum_{n=0}^{\infty} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \left\|x^{<\nu>}\right\|_p^{r_\nu} \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

and so $x \in [\ell(r),\ell_p]^{<k(\nu)>}$. Thus we have shown (13) when $1 < p < \infty$.

If $p = 1$ then we define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}\left(a_{k_0(\nu)}\right) \|a^{<\nu>}\|_\infty^{s_\nu - 1}(n+1)^{-s_\nu} M_n^{-1} & (k = k_0(\nu)) \\ \text{where } k_0(\nu) \text{ is the smallest integer in } K^{<k(\nu)>} & \\ \text{ with } |a_{k_0(\nu)}| = \|a^{<\nu>}\|_\infty & \\ 0 & (k \neq k_0(\nu)) \\ \qquad (k \in K^{<k(\nu)>}; \nu(n) \leq \nu \leq \nu(n+1) - 1; n = 0, 1, \dots). & \end{cases}$$

Again, since $1 - s_\nu = -s_\nu / r_\nu$ for all ν, we obtain for all n

$$\begin{aligned} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \sum\nolimits_\nu a_k x_k &= \sum_{\nu=\nu(n)}^{\nu(n+1)-1} |a_{k_0(\nu)}| \cdot \|a^{<\nu>}\|_\infty^{s_\nu - 1}(n+1)^{-s_\nu} M_n^{-1} \\ &= \frac{1}{M_n(n+1)} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_\infty^{s_\nu}(n+1)^{-s_\nu/r_\nu} = \frac{1}{n+1}, \end{aligned}$$

and so $\sum_{k=1}^\infty a_k x_k$ diverges. But, since $s_\nu r_\nu - r_\nu = s_\nu$, we have for all ν with $\nu(n) \leq \nu \leq \nu(n+1) - 1$ and for all n

$$\|x^{<\nu>}\|_1^{r_\nu} \leq \left(\|a^{<\nu>}\|_\infty^{s_\nu - 1}\right)^{r_\nu} (n+1)^{-s_\nu r_\nu} M_n^{-r_\nu} = \|a^{<\nu>}\|_\infty^{s_\nu}(n+1)^{-s_\nu - r_\nu} M_n^{-r_\nu}.$$

Furthermore, since $M_n > 1$, $r_\nu > 1$ and $-s_\nu = -1 - s_\nu / r_\nu$, it follows that for all n

$$\begin{aligned} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|x^{<\nu>}\|_1^{r_\nu} &\leq \frac{1}{(n+1)M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_\infty^{s_\nu}(n+1)^{-s_\nu} \\ &= \frac{1}{(n+1)^2 M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_\infty^{s_\nu}(n+1)^{-s_\nu/r_\nu} = \frac{1}{(n+1)^2}, \end{aligned}$$

that is,

$$\sum_{\nu=0}^\infty \|x^{<\nu>}\|_1^{r_\nu} = \sum_{n=0}^\infty \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|x^{<\nu>}\|_1^{r_\nu} \leq \sum_{n=0}^\infty \frac{1}{(n+1)^2} < \infty,$$

and so $x \in [\ell(r), \ell_1]^{<k(\nu)>}$. Thus we have shown (13) when $p = 1$.
Finally if $p = \infty$ then we define the sequence x by

$$x_k = \operatorname{sgn}(a_k) \|a^{<\nu>}\|_1^{s_\nu - 1}(n+1)^{-s_\nu} M_n^{-1} \qquad (k \in K^{<k(\nu)>}; \nu(n) \leq \nu \leq \nu(n+1) - 1; n = 0, 1, \dots).$$

Again, since $1 - s_\nu = -s_\nu / r_\nu$, we obtain for all n

$$\begin{aligned} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \sum\nolimits_\nu a_k x_k &= \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_1^{s_\nu - 1} \left(\sum\nolimits_\nu |a_k|\right)(n+1)^{-s_\nu} M_n^{-1} \\ &= \frac{1}{(n+1)M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_1^{s_\nu}(n+1)^{-s_\nu/r_\nu} = \frac{1}{n+1}, \end{aligned}$$

and so $\sum_{k=1}^\infty a_k x_k$ diverges. But, since $s_\nu r_\nu - r_\nu = s_\nu$, we have for all ν with $\nu(n) \leq \nu \leq \nu(n+1) - 1$ and for all n

$$\|x^{<\nu>}\|_\infty^{r_\nu} \leq \left(\|a^{<\nu>}\|_1^{s_\nu - 1}\right)^{r_\nu} (n+1)^{-s_\nu r_\nu} M_n^{-r_\nu} = \|a^{<\nu>}\|_1^{s_\nu}(n+1)^{-s_\nu - r_\nu} M_n^{-r_\nu}.$$

Furthermore, since $M_n > 1$, $r_\nu > 1$ and $1 - s_\nu = -s_\nu/r_\nu$, it follows that for all n

$$\sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|x^{<\nu>}\|_\infty^{r_\nu} \leq \frac{1}{(n+1)M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_1^{-s_\nu}(n+1)^{-s_\nu}$$
$$= \frac{1}{(n+1)^2 M_n} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|a^{<\nu>}\|_1^{s_\nu}(n+1)^{-s_\nu/r_\nu} = \frac{1}{(n+1)^2},$$

that is,

$$\sum_{\nu=0}^{\infty} \|x^{<\nu>}\|_\infty^{r_\nu} = \sum_{n=0}^{\infty} \sum_{\nu=\nu(n)}^{\nu(n+1)-1} \|x^{<\nu>}\|_\infty^{r_\nu} \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

and so $x \in [\ell(r), \ell_\infty]^{<k(\nu)>}$. Thus we have shown (13) when $p = \infty$.
Consequently, we have established (13) for $1 \leq p \leq \infty$, and finally, (13) and (12) yield (11).
This concludes the proof of **Case 1** .

Case 2. $r_\nu \leq 1$ for all $\nu = 0, 1, 2 \ldots$. First we show (12).
Let $a \in M((r), p) = [\ell_\infty(r), \ell_q]^{<k(\nu)>}$ be given. Then there exists $N \in \mathbb{N}$ such that $\sup_\nu \|a^{<\nu>}\|_q^{r_\nu} \leq N$. Let

$$\bar{B}_{1/N} = \left\{ x \in \omega : \sum_{\nu=1}^{\infty} \left\|x^{<\nu>}\right\|_p^{r_\nu} \leq \frac{1}{N} \right\}$$

be the closed ball in $[\ell(r), \ell_p]^{<k(\nu)>}$ with radius $1/N$ and centre in the origin. Then, by (9), we have $(\sum_\nu |a_k x_k|)^{r_\nu} \leq \|a^{<\nu>}\|_q^{r_\nu} \cdot \|x^{<\nu>}\|_p^{r_\nu} \leq 1$ for all $x \in \bar{B}_{1/N}$ and for all ν. But we have $\sum_\nu |a_k x_k| \leq (\sum_\nu |a_k x_k|)^{r_\nu}$, since $r_\nu \leq 1$ for all ν, and so

$$\sum_{k=1}^{\infty} |a_k x_k| = \sum_{\nu=0}^{\infty} \textstyle\sum_\nu |a_k x_k| \leq \displaystyle\sum_{\nu=0}^{\infty} (\textstyle\sum_\nu |a_k x_k|)^{r_\nu} \leq \displaystyle\sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_q^{r_\nu} \cdot \left\|x^{<\nu>}\right\|_p^{r_\nu}$$
$$\leq \sup_\nu \left\|a^{<\nu>}\right\|_q^{r_\nu} \cdot \sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_p^{r_\nu} \leq 1. \tag{14}$$

This shows $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for all $x \in \bar{B}_{1/N}$.
Now let $x \in [\ell(r), \ell_p]^{<k(\nu)>}$ be arbitrary. Since $[\ell(r), \ell_p]^{<k(\nu)>}$ is a paranormed space by Corollary 2.6, $\bar{B}_{1/N}$ is absorbing ([13, Fact (ix), p. 53]), and so there exists a positive constant C such that $y = C^{-1}x \in \bar{B}_{1/N}$. Now (14) yields $\sum_{k=1}^{\infty} |a_k x_k| = C \sum_{k=1}^{\infty} |a_k y_k| \leq C$. Thus $\sum_{k=1}^{\infty} |a_k x_k| < \infty$ for all $x \in [\ell(r), \ell_p]^{<k(\nu)>}$, and we have shown (12).
Now we show (13). We assume $a \notin M((r), p) = [\ell_\infty(r), \ell_q]^{<k(\nu)>}$. Then there exists a sequence $(\nu(n))_{n=0}^{\infty}$ of integers $\nu(0) < \nu(1) < \nu(2) < \ldots$ such that $\|a^{<\nu(n)>}\|_q^{r_{\nu(n)}} > (n+1)^2$ for $n = 0, 1, \ldots$.
If $1 < p < \infty$, then we define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}(a_k)|a_k|^{q-1} \left\|a^{<\nu(n)>}\right\|_q^{-q} & (k \in K^{<\nu(n)>}) \\ 0 & (k \notin K^{<\nu(n)>}) \end{cases} \quad (n = 0, 1, \ldots).$$

Then we have

$$\sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_p^{r_\nu} = \sum_{n=0}^{\infty} \left(\textstyle\sum_{\nu(n)} |x_k|^p\right)^{r_{\nu(n)}/p} = \displaystyle\sum_{n=0}^{\infty} \left\|a^{<\nu(n)>}\right\|_q^{r_{\nu(n)} q(1/p-1)}$$
$$= \sum_{n=0}^{\infty} \frac{1}{\left\|a^{<\nu(n)>}\right\|_q^{r_{\nu(n)}}} < \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

that is, $x \in [\ell(r), \ell_p]^{<k(\nu)>}$, but

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{n=0}^{\infty} \left(\sum_{\nu(n)} |a_k|^q\right) \left\|a^{<\nu>}\right\|_q^{-q} = \sum_{n=0}^{\infty} 1 = \infty.$$

Thus we have shown (13) when $1 < p < \infty$.
If $p = 1$, then we define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}\left(a_{k_{\nu(n)}}\right) \left\|a^{<\nu(n)>}\right\|_\infty^{-1} & (k = k_{\nu(n)}) \\ \text{where } k_{\nu(n)} \text{ is the smallest integer in } K^{<\nu(n)>} & \\ \text{ with } |a_{k_{\nu(n)}}| = \left\|a^{<\nu(n)>}\right\|_\infty & \\ 0 & (k \neq k_{\nu(n)} \text{ or } k \notin K^{<\nu(n)>}) \end{cases}$$

$$(n = 0, 1, \dots).$$

Then we have

$$\sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_1^{r_\nu} \leq \sum_{n=0}^{\infty} \left\|a^{<\nu(n)>}\right\|_\infty^{-r_{\nu(n)}} < \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

that is, $x \in [\ell(r), \ell_p]^{<k(\nu)>}$, but

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{n=0}^{\infty} |a_{k_{\nu(n)}}| \cdot \left\|a^{<\nu(n)>}\right\|_\infty^{-1} = \sum_{n=0}^{\infty} 1 = \infty.$$

Thus we have shown (13) when $p = 1$.
Finally, let $p = \infty$. We define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}(a_k) \left\|a^{<\nu(n)>}\right\|_1^{-1} & (k \in K^{<\nu(n)>}) \\ 0 & (k \notin K^{<\nu(n)>}) \end{cases} \quad (n = 0, 1, \dots).$$

Then we obtain

$$\sum_{\nu=0}^{\infty} \left\|x^{<\nu>}\right\|_\infty^{r_\nu} \leq \sum_{n=0}^{\infty} \left\|a^{<\nu(n)>}\right\|_1^{-r_\nu(n)} < \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty,$$

that is, $x \in [\ell(r), \ell_\infty]^{<k(\nu)>}$. But we have

$$\sum_{k=1}^{\infty} a_k x_k = \sum_{n=0}^{\infty} \sum_{\nu(n)} |a_k| \cdot \left\|a^{<\nu(n)>}\right\|_1^{-1} = \sum_{n=0}^{\infty} 1 = \infty.$$

Thus we have shown (13) when $p = \infty$.
Consequently, we have established (13) for $1 \leq p \leq \infty$, and finally, (13) and (12) yield (11).
This concludes the proof of **Case 2** . □

Now we obtain the β–duals of the sets considered in examples 2.8 and 2.9 as an immediate consequence of Theorems 3.1 and 3.2.

Example 3.3. *(a) Let $r_\nu = r > 0$ for all ν and $1 \leq p \leq \infty$. Then we obviously have by Theorem 3.1*

$$\left(\left[c_0, \ell_p\right]^{<k(\nu)>}\right)^\beta = \ell(1, q) = \left[c_0^\beta, \ell_p^\beta\right]^{<k(\nu)>},$$

and by Theorem 3.2, for $1 < r < \infty$ and $s = r/(r-1)$,

$$(\ell(r,p))^\beta = \left\{a \in \omega : \sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_q^s < \infty\right\} = \ell(s,q),$$

and for $r \leq 1$, $(\ell(r,p))^\beta = \ell(\infty,q)$, that is,

$$(\ell(r,p))^\beta = \left[\ell_r^\beta, \ell_p^\beta\right]^{<k(\nu)>} \text{ in both cases.}$$

(b) If $k = 2^\nu$, $r_\nu = 1$ for all ν, $1 \leq p < \infty$, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and u is the sequence with $u_k = 2^{-\nu/p}$ ($k \in K^{<k(\nu)>}; \nu = 0,1,\dots$), then we obtain by Part (a)

$$\begin{aligned}\left(w_0^p\right)^\beta &= \left(u^{-1} * [c_0, \ell_p]^{<k(\nu)>}\right)^\beta = (1/u)^{-1} * \ell(1,q) \\ &= \begin{cases} \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1}-1} |a_k| < \infty\right\} & (p=1) \\ \left\{a \in \omega : \sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_{k=2^\nu}^{2^{\nu+1}-1} |a_k|^q\right)^{1/q} < \infty\right\} & (1 < p < \infty) \end{cases} \quad ([8]).\end{aligned}$$

Example 3.4. *Let $r = (r_\nu)_{\nu=0}^\infty$ be any sequence of positive reals, $1 \leq p \leq \infty$ and $k(\nu) = \nu + 1$ for all ν. Then we obtain from Theorems 3.1 and 3.2*

$$(c_0(r))^\beta = M_0(r) = \bigcup_{N>1} \left\{a \in \omega : \sum_{k=1}^{\infty} |a_k| N^{-1/r_k} < \infty\right\} \text{ ([9, Theorem 6])}$$

and

$$(\ell(r))^\beta = M(r) = \begin{cases} \bigcup_{N>1} \left\{a \in \omega : \sum_{k=1}^{\infty} |a_k|^{s_k} N^{-s_k/r_k} < \infty\right\} & (r_k > 1) \text{ ([9, Theorem 1])} \\ \ell_\infty(r) & (r_k \leq 1) \text{ ([12, Theorem 10]).} \end{cases}$$

Remark 3.5. *We obtain from* Examples 3.3 *and* 3.4

$$\left([c_0(r), \ell_p]^{<k(\nu)>}\right)^\beta = \left[(c_0(r))^\beta, \ell_p^\beta\right]^{<k(\nu)>} \text{ and } \left([\ell(r), \ell_p]^{<k(\nu)>}\right)^\beta = \left[(\ell(r))^\beta, \ell_p^\beta\right]^{<k(\nu)>}.$$

Now we determine the second β–duals of the sets $[c_0(r), \ell_p]^{<k(\nu)>}$ and $[\ell(r), \ell_p]^{<k(\nu)>}$. We put

$$M_\infty((r),p) = \bigcap_{N>1} \left\{a \in \omega : \sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_q N^{1/r_\nu} < \infty\right\},$$

$$M_0^{(2)}((r),p) = \bigcap_{N>1} \left\{a \in \omega : \sup_\nu \left\|a^{<\nu>}\right\|_p N^{1/r_\nu} < \infty\right\}$$

and

$$M^{(2)}((r),p) = \begin{cases} M((s),q) & \text{if } r_\nu > 1 \text{ for all } \nu \\ M_\infty((r),q) & \text{if } r_\nu \leq 1 \text{ for all } \nu. \end{cases}$$

We need the following result for the determination of the second β–dual of $[\ell(r), \ell_p]^{<k(\nu)>}$ in the case of $r_\nu \leq 1$ for all ν.

Lemma 3.6. *Let $r = (r_\nu)_{\nu=0}^\infty$ be a sequence of positive reals and $1 \le p \le \infty$. Then we have*

$$\left([\ell_\infty(r), \ell_p]^{<k(\nu)>}\right)^\beta = M_\infty((r), p). \tag{15}$$

Proof. First we show

$$M_\infty((r), p) \subset \left([\ell_\infty(r), \ell_p]^{<k(\nu)>}\right)^\beta. \tag{16}$$

Let $a \in M_\infty((r), p)$ and $x \in [\ell_\infty(r), \ell_p]^{<k(\nu)>}$ be given. Then there exists $N_0 \in \mathbb{N} \setminus \{1\}$ such that

$$\max\left\{1, \sup_\nu \left\|x^{<\nu>}\right\|_p^{r_\nu}\right\} < N_0, \text{ hence } \sup_\nu \left\|x^{<\nu>}\right\|_p < N_0^{1/r_\nu},$$

and we obtain as in (9)

$$\sum_{k=1}^\infty |a_k x_k| \le \sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_q \left\|x^{<\nu>}\right\|_p \le \sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_q \cdot N_0^{1/r_\nu} < \infty.$$

Thus we have shown (16).
Now we show

$$\left([\ell_\infty(r), \ell_p]^{<k(\nu)>}\right)^\beta \subset M_\infty((r), p). \tag{17}$$

We assume $a \notin M_\infty((r), p)$. Then there exists $N_0 \in \mathbb{N} \setminus \{1\}$ such that $\sum_{\nu=0}^\infty \|a^{<\nu>}\|_q N_0^{1/r_\nu} = \infty$.
If $1 < p < \infty$, then we define the sequence x by

$$x_k = \operatorname{sgn}(a_k)|a_k|^{q-1} \left\|a^{<\nu>}\right\|_q^{-q/p} N_0^{1/r_\nu} \ (k \in K^{<k(\nu)>}; \nu = 0, 1, \dots).$$

Then we have for all ν

$$\left\|x^{<\nu>}\right\|_p^{r_\nu} = \left((\textstyle\sum_\nu |a_k|^q)^{1/p} \left\|a^{<\nu>}\right\|_q^{-q/p} N_0^{1/r_\nu}\right)^{1/r_\nu} = \left(\left\|a^{<\nu>}\right\|_q^{q/p - q/p}\right)^{r_\nu} N_0 = N_0,$$

that is, $x \in [\ell_\infty(r), \ell_p]^{<k(\nu)>}$, but for all ν

$$\textstyle\sum_\nu a_k x_k = (\sum_\nu |a_k|^q) \left\|a^{<\nu>}\right\|_q^{-q/p} N_0^{1/r_\nu} = \left\|a^{<\nu>}\right\|_q^{q(1-1/p)} N_0^{1/r_\nu} = \left\|a^{<\nu>}\right\|_q N_0^{1/r_\nu},$$

hence $\sum_{k=1}^\infty a_k x_k = \infty$.
Thus we have shown (17) for $1 < p < \infty$.
If $p = 1$, then we define the sequence x by

$$x_k = \begin{cases} \operatorname{sgn}(a_{k_\nu}) N_0^{1/r_\nu} & (k = k_\nu) \\ \text{where } k_\nu \text{ is the smallest integer in } K^{<k(\nu)>} & \\ \text{with } |a_{k_\nu}| = \|a^{<\nu>}\|_\infty & \\ 0 & (k \ne k_\nu) \end{cases} \quad (\nu = 0, 1, \dots).$$

Then we have $\|x^{<\nu>}\|_1^{r_\nu} \le N_0$ for all ν, that is, $x \in [\ell_\infty(r), \ell_1]^{<k(\nu)>}$, but $\sum_\nu a_k x_k = |a_{k_\nu}| N_0^{1/r_\nu} = \|a^{<\nu>}\|_\infty N_0^{1/r_\nu}$ for all ν, hence $\sum_{k=1}^\infty a_k x_k = \infty$.
Thus we have shown (17) for $p = 1$.
Finally, if $p = \infty$, then we define the sequence x by $x_k = \operatorname{sgn}(a_k) N_0^{1/r_\nu}$ $(k \in K^{<k(\nu)>}; \nu = 0, 1, \dots)$. Then we have $\|x^{<\nu>}\|_\infty^{r_\nu} \le N_0$ for all ν, that is, $x \in [\ell_\infty(r), \ell_\infty]^{<k(\nu)>}$, but for all ν $\sum_\nu a_k x_k = \sum_\nu |a_k| N_0^{1/r_\nu} = \|a^{<\nu>}\|_1 N_0^{1/r_\nu}$, hence $\sum_{k=1}^\infty a_k x_k = \infty$.
Thus we have shown (17) for $p = 1$.
Consequently, we have established (17) for $1 \le p \le \infty$, and finally, (17) and (16) yield (15). □

Theorem 3.7. *We have*

$$\left([c_0(r), \ell_p]^{<k(\nu)>}\right)^{\beta\beta} = M_0^{(2)}((r), p); \tag{a}$$

$$\left([\ell(r), \ell_p]^{<k(\nu)>}\right)^{\beta\beta} = M^{(2)}((r), p). \tag{b}$$

Proof. We write for each $N \in \mathbb{N} \setminus \{1\}$

$$S_N(0) = S_N(0, (r), p) = \left\{a \in \omega : \left(\left\|a^{<\nu>}\right\|_q N^{-1/r_\nu}\right)_{\nu=0}^{\infty} \in \ell_1\right\}$$

and

$$S_N = S_N((r), p) = \left\{a \in \omega : \left(\left\|a^{<\nu>}\right\|_q N^{-1/r_\nu}\right)_{\nu=0}^{\infty} \in \ell(s)\right\} \ (r_\nu > 1 \text{ for all } \nu).$$

It follows by a well–known result ([7, Lemma 1 (iv)]) that

$$\left(\bigcup_{N>1} S_N(0)\right)^{\beta} = \bigcap_{N>1} (S_N(0))^{\beta} \text{ and } \left(\bigcup_{N>1} S_N\right)^{\beta} = \bigcap_{N>1} S_N^{\beta}. \tag{18}$$

We define the sequence $v(N) = (v_k(N))_{k=1}^{\infty}$ by $v_k(N) = N^{-1/r_\nu}$ for $k \in K^{<k(\nu)>}$ and $\nu = 0, 1, \dots$. Then we have $S_N(0) = (v(N))^{-1} * [\ell_1, \ell_q]^{<k(\nu)>}$ and $S_N = (v(N))^{-1} * [\ell(s), \ell_q]^{<k(\nu)>}$. We conclude

$$(S_N(0))^{\beta} = (1/v(N))^{-1} * \left([\ell_1, \ell_q]^{<k(\nu)>}\right)^{\beta} \tag{19}$$

and

$$S_N^{\beta} = (1/v(N))^{-1} * \left([\ell(s), \ell_q]^{<k(\nu)>}\right)^{\beta}. \tag{20}$$

(a) It follows from Theorem 3.1, (18), (19) and Example 3.3 (a)

$$\begin{aligned}
\left([c_0(r), \ell_p]^{<k(\nu)>}\right)^{\beta\beta} &= (M_0((r), p))^{\beta} = \left(\bigcup_{N>1} S_N(0)\right)^{\beta} = \bigcap_{N>1} (S_N(0))^{\beta} \\
&= \bigcap_{N>1} (1/v(N))^{-1} * \left(\left[\ell_1, \ell_q\right]^{<k(\nu)>}\right)^{\beta} \\
&= \bigcap_{N>1} (1/v(N))^{-1} * \left[\ell_\infty, \ell_p\right]^{<k(\nu)>} = M_0^{(2)}((r), p).
\end{aligned}$$

(b) If $r_\nu > 1$ for all ν, then it follows, similarly as in the proof of Part (a), from Theorem 3.2, (18), (20) and Theorem 3.2 with r_ν and p interchanged with s_ν and q that

$$\begin{aligned}
\left([\ell(r), \ell_p]^{<k(\nu)>}\right)^{\beta\beta} &= (M((r), p))^{\beta} = \left(\bigcup_{N>1} S_N\right)^{\beta} = \bigcap_{N>1} S_N^{\beta} \\
&= \bigcap_{N>1} (1/v(N))^{-1} * \left(\left[\ell(s), \ell_q\right]^{<k(\nu)>}\right)^{\beta} \\
&= \bigcap_{N>1} (1/v(N))^{-1} * \left(\bigcup_{M>1} S_M((s), q)\right)
\end{aligned}$$

$$= \bigcap_{N>1}\bigcup_{M>1}\left\{a\in\omega : \left\|a^{<\nu>}N^{1/r_\nu}\right\|_p^{r_\nu} M^{-r_\nu/s_\nu} < \infty\right\}$$
$$= \bigcup_{M>1}\left\{a\in\omega : \left\|a^{<\nu>}\right\|_p^{r_\nu} M^{-r_\nu/s_\nu} < \infty\right\} = M((s),q)$$
$$= M^{(2)}((r),p).$$

Finally, if $r_\nu \le 1$ for all ν, then it follows from Theorem 3.2 and (15) with q instead of p that

$$\left([\ell(r),\ell_p]^{<k(\nu)>}\right)^{\beta\beta} = (M((r),p))^\beta = \left([\ell_\infty(r),\ell_q]^{<k(\nu)>}\right)^\beta = M_\infty((r),q) = M^{(2)}((r),p).$$

□

Corollary 3.8. *Let $r=(r_\nu)_{\nu=0}^\infty$ be a sequence of positive reals with $r_\nu > 1$ for all ν, and $1\le p\le\infty$. Then $[\ell(r),\ell_p]^{<k(\nu)>}$ is β–perfect if and only if $r\in\ell_\infty$.*

Proof. First, we assume $1 < r_\nu \le M(r) = \sup_\nu r_\nu < \infty$ for all ν. Then we have

$$s_\nu = \frac{1}{1-1/r_\nu} \ge \frac{1}{1-1/M(r)} = \frac{M(r)}{M(r)-1} \text{ for } \nu = 0,1,\dots. \tag{21}$$

Theorem 3.2 implies

$$[\ell(s),\ell_q]^{<k(\nu)>} \subset M((r),p) = \left([\ell(r),\ell_p]^{<k(\nu)>}\right)^\beta,$$

and so by [14, Theorem 7.2.2 (iii)] and Theorem 3.2 with r and p replaced by s and q

$$\left([\ell(r),\ell_p]^{<k(\nu)>}\right)^{\beta\beta} \subset \left([\ell(s),\ell_q]^{<k(\nu)>}\right)^\beta = M((s),q). \tag{22}$$

It also follows from (21) that $m(s) = \inf_\nu s_\nu > 0$, hence $N^{-r_\nu/s_\nu} \ge N^{-M(r)/m(s)}$ for all ν and each $N > 1$. So we obtain

$$\sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_p^{r_\nu} \le N^{M(r)/m(s)} \sum_{\nu=0}^\infty \left\|a^{<\nu>}\right\|_p^{r_\nu} N^{-r_\nu/s_\nu},$$

whence

$$M((s),q) \subset [\ell(r),\ell_p]^{<k(\nu)>}. \tag{23}$$

Now it follows from [14, Theorem 7.2.2 (i)], (22) and (23) that

$$[\ell(r),\ell_p]^{<k(\nu)>} \subset \left([\ell(r),\ell_p]^{<k(\nu)>}\right)^{\beta\beta} \subset M((s),q) \subset [\ell(r),\ell_p]^{<k(\nu)>},$$

that is,

$$\left([\ell(r),\ell_p]^{<k(\nu)>}\right)^{\beta\beta} = [\ell(r),\ell_p]^{<k(\nu)>}.$$

Conversely, if $[\ell(r),\ell_p]^{<k(\nu)>}$ is perfect, then it is a linear space, being the β–dual of a set, and so we have $r\in\ell_\infty$ by Remark 2.7. □

Finally we obtain the β–duals of the sets considered in Examples 2.8 and 2.9 as an immediate consequence of Theorem 3.7 (a) and (b).

Example 3.9. *(a) Let $r_\nu = r > 0$ for all ν and $1 \le p \le \infty$. Then we obviously have by Theorem 3.7 (a)*

$$\left(\left[c_0, \ell_p\right]^{<k(\nu)>}\right)^{\beta\beta} = \ell(\infty, p) = \left[c_0^{\beta\beta}, \ell_p^{\beta\beta}\right]^{<k(\nu)>}.$$

Also Theorem 3.7 (b) yields for $1 < r < \infty$ and $s = r/(r-1)$

$$(\ell(r,p))^{\beta\beta} = \left\{a \in \omega : \sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_p^r < \infty\right\} = \ell(r,p),$$

and for $r \le 1$,

$$(\ell(r,p))^{\beta\beta} = \left\{a \in \omega : \sum_{\nu=0}^{\infty} \left\|a^{<\nu>}\right\|_p < \infty\right\} = \ell(1,p),$$

Thus we conclude

$$(\ell(r,p))^{\beta\beta} = \left[\ell_r^{\beta\beta}, \ell_p^{\beta\beta}\right]^{<k(\nu)>} \text{ in both cases.}$$

Consequently $\ell(r,p)$ is β perfect, if and only if $r \ge 1$.
(b) If $k = 2^\nu$, $r_\nu = 1$ for all ν, $1 \le p < \infty$, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and u is the sequence with $u_k = 2^{-\nu/p}$ ($k \in K^{<k(\nu)>}$; $\nu = 0, 1, \dots$), then we obtain by Part (a)

$$\begin{aligned}\left(w_0^p\right)^{\beta\beta} &= \left(u^{-1} * [c_0, \ell_p]^{<k(\nu)>}\right)^{\beta\beta} = u^{-1} * \left([c_0, \ell_p]^{<k(\nu)>}\right)^{\beta\beta} \\ &= u^{-1} * \ell(\infty, p) = \left\{a \in \omega : \sup_\nu \frac{1}{2^\nu} \sum_{k=2^\nu}^{2^{\nu+1}-1} |a_k|^p < \infty\right\} \text{ ([11, Theorem 5.8]).}\end{aligned}$$

Example 3.10. *Let $r = (r_\nu)_{\nu=0}^\infty$ be any sequence of positive reals, $1 \le p \le \infty$ and $k(\nu) = \nu + 1$ for all ν. Then we obtain from Theorem 3.7 (a) and (b)*

$$(c_0(r))^{\beta\beta} = M_0^{(2)}(r) = \bigcap_{N>1} \left\{a \in \omega : \sup_k |a_k| N^{1/r_k} < \infty\right\} \text{ ([7, Theorem 2])}$$

and

$$(\ell(r))^{\beta\beta} = M^{(2)}(r) = \begin{cases} \bigcup_{N>1} \left\{a \in \omega : \sum_{k=1}^\infty |a_k|^{r_k} N^{-r_k/s_k} < \infty\right\} & \text{if } r_k > 1 \text{ for all } k \\ \bigcap_{N>1} \left\{a \in \omega : \sum_{k=1}^\infty |a_k| N^{1/r_k} < \infty\right\} & \text{if } r_k \le 1 \text{ for all } k; \end{cases}$$

also $\ell(r)$ for $r_k > 1$ is β perfect if and only if $r \in \ell_\infty$ ([7, Theorem 4 (i)]).

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