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Vertices of Paths of Minimal Lengths on Suborbital Graphs

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Abstract. The Modular group Γ acts on the set of extended rational numbers $\hat{\mathbb{Q}}$ transitively. Here, our main purpose is to examine some properties of hyperbolic paths of minimal lengths in the suborbital graphs for Γ. We characterize all vertices of these hyperbolic paths in the suborbital graphs which are trees.

1. Introduction

Jones, Singerman and Wicks [2] used the idea of suborbital graphs for finite permutation groups, described by Sims [3], for the congruence subgroup $\Gamma_0(N)$ of the modular group Γ. They studied the action of Γ on the rational projective line $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$ by using suborbital graphs. These are $\Gamma-$invariant directed graphs with vertex set $\hat{\mathbb{Q}}$ and their edge-sets being the orbits of Γ on the cartesian square $\hat{\mathbb{Q}}^2$. For each integer $N \geq 1$ and for each of the $\phi(N)$ units $u \ mod(N)$, there is one suborbital graph $\mathbf{G}_{u,N}$ with edge-set which is the orbit containing the pair $(\infty, u/N)$.

In [2], some properties of the $\mathbf{G}_{u,N}$ graphs were given and the authors conjectured that $\mathbf{G}_{u,N}$ is a forest if and only if it contains no triangles, that is, iff $u^2 \pm u + 1 \not\equiv 0 \ (mod\, N)$. This conjecture was proved in [4]. Then, a few papers on the suborbital graphs for related groups were published [5,6]. Generally, authors examined all circuits in the suborbital graphs in these papers. Clearly, whether the graph contain a circuit or not depends on the choice of u and N. Some subgraph family has just the hyperbolic paths. These subgraphs are also worthwhile to investigate, because it is well-known that some number theoretical results arise from the action of some Fuchsian groups. With this motivation, examining the suborbital graphs of Γ, we obtained some results about the connection between continued fractions and hyperbolic paths of the suborbital graphs [1]. In this paper, we extended our results with standard recurrence relations used in the theory of continued fractions.

Let us summarize the terminology used in [2] briefly.

1.1. *The action of* Γ *on* $\hat{\mathbb{Q}}$

$PSL(2,\mathbb{Z})$ is the set of all Möbius transformations of the form $T : z \longrightarrow \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$, that is group of automorphisms of upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. So the modular

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group Γ is the quotient of the unimodular group $SL(2,\mathbb{Z})$ by its center $\{\pm I\}$. Thus the elements of Γ are of the form as shown with pairs of matrices, that is:

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1.$$

Here we will omit the symbol $\pm$ and identify each matrix with its negative. Γ acts transitively on $\hat{\mathbb{Q}}$ by the transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \longrightarrow \frac{ax+by}{cx+dy},$$

where $x/y \in \hat{\mathbb{Q}}$ is a reduced fraction, that is $(x,y)=1$. Here, $\frac{ax+by}{cx+dy}$ is also a reduced fraction. The group $\Gamma_0(N)$ is the congruence subgroup of Γ with $N|c$.

In [2], authors defined a non-trivial equivalence relation on $\hat{\mathbb{Q}}$, as follows: if $v=r/s$ and $w=x/y$ are elements of $\hat{\mathbb{Q}}$, then $v \approx w$ if and only if there exists u with $(u,N)=1$ such that $x \equiv ur\,(mod\,N)$ and $y \equiv us\,(mod\,N)$. This is $\Gamma-$invariant equivalence relation and so that Γ acts imprimitively on $\hat{\mathbb{Q}}$. Therefore this relation divides $\hat{\mathbb{Q}}$ into blocks. The stabilizer of the block containing $1/0$ is $\Gamma_0(N)$ and the number of blocks is $|\Gamma:\Gamma_0(N)| = \Psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right)$, where p is a prime divisor of N.

1.2. Suborbital graphs for Γ *on* $\hat{\mathbb{Q}}$

Let (G,Ω) be a transitive permutation group. Then G acts on $\Omega\times\Omega$ by

$$g:(\alpha,\beta)\longrightarrow(g(\alpha),g(\beta)),$$

where $g\in G$ and $\alpha,\beta\in\Omega$. The orbits of this action are called *suborbitals* of G, that containing (α,β) being denoted by $O(\alpha,\beta)$. From $O(\alpha,\beta)$ we can form a *suborbital graph* $\mathbf{G}(\alpha,\beta)$: its vertices are the elements of Ω, and there is a *directed edge* from γ to δ if $(\gamma,\delta)\in O(\alpha,\beta)$, denoted by $\gamma\to\delta$. The orbit $O(\beta,\alpha)$ is also a suborbital, and it is either equal to or disjoint from $O(\alpha,\beta)$. In the latter case, $\mathbf{G}(\beta,\alpha)$ is just $\mathbf{G}(\alpha,\beta)$ with the arrows reversed and we call, in this case, $\mathbf{G}(\alpha,\beta)$ and $\mathbf{G}(\beta,\alpha)$ *paired suborbital graphs*. In the former case, $\mathbf{G}(\alpha,\beta)=\mathbf{G}(\beta,\alpha)$ and the graph consists of pairs of oppositely directed edges; it is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call *self-paired*.

$O(\alpha,\alpha) := \{(\gamma,\gamma)|\gamma\in\Omega\}$ is the diagonal of $\Omega\times\Omega$. The corresponding suborbital graph $G(\alpha,\alpha)$, called the trivial suborbital graph, is self-paired: it consists of a loop based at each vertex $\gamma\in\Omega$. We shall be mainly interested in the remaining non-trivial suborbital graphs.

We now investigate the suborbital graphs for the action of Γ on $\hat{\mathbb{Q}}$. Since Γ acts transitively on $\hat{\mathbb{Q}}$, each suborbital contains a pair $(\infty,\frac{u}{N})$ for some $\frac{u}{N}\in\mathbb{Q}$. We denote this suborbital by $O_{u,N}$ and the corresponding suborbital graph by $\mathbf{G}_{u,N}$ for short. Let us give some results which are proved in [2] as following lemmas.

Lemma 1.1. $\mathbf{G}_{u,N}=\mathbf{G}_{u',N'}$ *iff* $N=N'$ *and* $u\equiv u'\,(mod\,N)$. ∎

Lemma 1.2. $\mathbf{G}_{u,N}$ *is self-paired iff* $u^2\equiv -1\,(mod\,N)$. ∎

Lemma 1.3. *The suborbital graph paired with* $\mathbf{G}_{u,N}$ *is* $\mathbf{G}_{-\bar{u},N}$ *where* $\bar{u}$ *satisfies* $u\bar{u}\equiv 1\,(mod\,N)$. ∎

Lemma 1.4. $\frac{r}{s}\to\frac{x}{y}\in\mathbf{G}_{u,N}$ *if and only if* $x\equiv\mp ur\,(mod\,N)$, $y\equiv\mp us\,(mod\,N)$, $ry-sx=\mp N$. ∎

Since Γ acts on $\hat{\mathbb{Q}}$ transitively, it permutes these blocks transitively. Hence the subgraphs corresponding to the graph whose vertices in the blocks are all isomorphic. We let $\mathbf{F}_{u,N}$ be the subgraph of $\mathbf{G}_{u,N}$ whose vertices form the block $[\infty]=\{x/y\in\hat{\mathbb{Q}}\mid y\equiv 0\,(mod\,N)\}$ containing ∞, so that $\mathbf{G}_{u,N}$ consists of $\Psi(N)$ disjoint copies of $\mathbf{F}_{u,N}$.

Lemma 1.5. $\frac{r}{s}\to\frac{x}{y}\in\mathbf{F}_{u,N}$ *if and only if* $x\equiv\mp ur\,(mod\,N)$, $ry-sx=\mp N$. ∎

The main definitions used in our paper:

(a) Let $v_0, v_1, \ldots, v_m$ be a sequence of different vertices of the graph $\mathbf{F}_{u,N}$. If $m \geq 2$ then the configuration $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_m \rightarrow v_0$ is called a *directed circuit* (*closed path*). If at least one arrow (not all) is reversed in this configuration, it is called an *undirected* (*anti-directed*) *circuit*. If $m = 2$ then the circuit, directed or not, is called a *triangle*. If $m = 1$ then we will call the configuration $v_0 \rightarrow v_1 \rightarrow v_0$ a *self paired edge*.

(b) For visual convenience and because the elements of Γ sends the hyperbolic lines to hyperbolic lines, we have represented the edges of graphs as hyperbolic geodesics in the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \,|\, Im(z) > 0\}$$

that is, as euclidean semi-circles or half-lines perpendicular to $\mathbb{R}$ as in [9].

(c) The configurations $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_m$ and $v_0 \rightarrow v_1 \rightarrow \ldots$ are called a *path* and an *infinite path* in $\mathbf{F}_{u,N}$, respectively.

(d) If $\frac{r}{s} \overset{<}{\rightarrow} \frac{x}{y} \in \mathbf{F}_{u,N}$ $\left(\text{or } \frac{x}{y} \overset{<}{\leftarrow} \frac{r}{s} \in \mathbf{F}_{u,N}\right)$, the *farthest vertex* means that there is no vertex which has greater (or smaller) value than $\frac{x}{y}$ joined with the vertex $\frac{r}{s}$ in the suborbital graph $\mathbf{F}_{u,N}$ by the conditions from Lemma 1.5.

(e) The path $v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_m$ is called of *minimal length* if and only if $v_i \nleftrightarrow v_j$, where $i < j - 1$, $i \in \{0, 1, 2, 3, ..., m-2\}$, $j \in \{2, 3, ..., m\}$ and v_{i+1} must be the farthest vertex which can be joined with the vertex v_i in $\mathbf{F}_{u,N}$.

(f) If $\mathbf{F}_{u,N}$ does not contain any circuits it is called a *forest*. If $\mathbf{F}_{u,N}$ is a connected non-empty graph without circuits it is called a *tree*.

2. Main Calculations

By [1], we know that the transformation $\varphi = \begin{pmatrix} -u & (u^2+ku+1)/N \\ -N & u+k \end{pmatrix}$ is in $\Gamma_0(N)$ and that $\varphi(\infty) = v_0, \varphi(v_0) = v_1$ and so on. For right direction, it gives the vertices of the paths

$$\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u + \frac{1}{k}}{N} \longrightarrow \frac{u + \frac{1}{k - \frac{1}{k}}}{N} \longrightarrow \frac{u + \frac{1}{k - \frac{1}{k - \frac{1}{k}}}}{N} \longrightarrow \cdots$$

of the minimal lengths in $\mathbf{F}_{u,N}$, where $u^2 + ku + 1 \equiv 0 \,(mod\, N)$. Hence,

Corollary 2.1. *If* $\frac{u+\frac{x}{y}}{N}$ *is the vertex on the path of minimal length in* $\mathbf{F}_{u,N}$*, then the farthest vertex which can be joined with it is* $\varphi\left(\frac{u+\frac{x}{y}}{N}\right) = \frac{u+\frac{y}{ky-x}}{N}$ *and* $v_q = \varphi^q(v_0)$ *for positive integers* q *and* $v_0 = u/N$*. This means that there are infinitely vertices which can be joined with* $\frac{u+\frac{x}{y}}{N}$*, but* $\frac{u+\frac{y}{ky-x}}{N}$ *is the farthest.*

Theorem 2.2. *Assume that* $u^2 + ku + 1 \equiv 0 \,(mod\, N)$ *and* $u^2 - lu + 1 \equiv 0 \,(mod\, N)$ *with* $1 < k, l \leq N$*. If* $\mathbf{F}_{u,N}$ *is self-paired, then* $k = l = N$ *and otherwise* $l = N - k$*.*

Proof. From the equations $u^2 + ku + 1 \equiv 0 \,(mod\, N)$ and $u^2 - lu + 1 \equiv 0 \,(mod\, N)$ with $1 < k, l \leq N$, we have $ku + lu \equiv 0 \,(mod\, N)$. Since $(u, N) = 1$, then $k \equiv -l \,(mod\, N)$. So, there exists an integer y such that $k = Ny - l$. From the inequality $2 < k + l \leq 2N$, we obtain that $2 < Ny \leq 2N$ giving that $2/N < y \leq 2$ by $N > 1$. Hence, y is equal to 1 or 2. On the other hand if $\mathbf{F}_{u,N}$ is self-paired, from Lemma 1.2 as $u^2 + 1 \equiv 0 \,(mod\, N)$ then, $ku \equiv 0 \,(mod\, N)$ and $-lu \equiv 0 \,(mod\, N)$. As $(u, N) = 1$, then $k \equiv 0 \,(mod\, N)$ and $-l \equiv 0 \,(mod\, N)$ giving that $k - l \equiv 0 \,(mod\, N)$. So, there is an integer t such that $k - l = Nt$. If $k = l = N$ obviously $t = 0$. Hence if $1 < k, l < N$ then $1 - N < k - l < N - 1$ giving that $1 - N < Nt < N - 1$. As $N > 1$ then $|t| < 1 - \frac{1}{N}$ and from this t must be zero. So, $k = l$. If graph is self-paired, since $1 < k \leq N$ and $ku \equiv 0 \,(mod\, N)$ then k must be equal to N. So, $k = l = N$. Consequently, if $\mathbf{F}_{u,N}$ is self-paired $y = 2$ and $y = 1$ for other cases. ∎

Theorem 2.3. *In $\mathbf{F}_{u,N}$, let $u^2 - lu + 1 \equiv 0\,(mod\,N)$ and $1 < l \leq N$. Then we have*

(i) *The farthest vertex which can be joined with $\frac{u}{N}$ is $\frac{u-\frac{1}{l}}{N}$ and there is no such a nearest vertex.*

(ii) *The farthest vertex which can be joined with $\frac{u-\frac{1}{l}}{N}$ is $\frac{u-\frac{1}{l-\frac{1}{l}}}{N}$ and there is no such a nearest vertex.* ∎

Proof. In here we will prove (ii). (i) can be proved similarly.

Since $u^2 - lu + 1 \equiv 0\,(mod\,N)$, $\frac{lu-1}{lN}$ is a vertex in $F_{u,N}$. Let

$$\frac{u-\frac{1}{l}}{N} = \frac{lu-1}{lN} \xrightarrow{>} \frac{u-\frac{p}{q}}{N} = \frac{qu-p}{qN}.$$

From Lemma 1.5, we have $qu - p \equiv u(lu-1)\,(mod\,N)$ and $(lu-1)qN - lN(qu-p) = N$, that is $q = pl - 1$, then $u(pl-1) - p \equiv u(lu-1)\,(mod\,N)$, giving $p(lu-1) - lu^2 \equiv 0\,(mod\,N)$. As $lu - 1 \equiv u^2\,(mod\,N)$, we have $pu^2 - lu^2 \equiv 0\,(mod\,N)$, implying $p \equiv l\,(mod\,N)$. Hence $p := l + Nx$ for some $x \in \mathbb{N} \cup \{0\}$. Thus $\frac{p}{q} = \frac{l+Nx}{l(l+Nx)-1}$. We shall define the function f as following

$$f : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{R} \text{ and } f(x) := \frac{u - \frac{l+Nx}{l(l+Nx)-1}}{N}.$$

Then f is strictly increasing function. Actually, looking the derivation, we see that

$$f'(x) = \frac{1}{[l(l+Nx)-1]^2} > 0.$$

Hence the smallest value of the function f is the value at the point $x = 0$, that is $\frac{u-\frac{l}{l^2-1}}{N}$. So we have

$$\frac{lu-1}{lN} \longrightarrow \frac{u-\frac{l}{l^2-1}}{N} = \frac{(l^2-1)u - l}{(l^2-1)N}.$$

Now, our aim is to show that $((l^2-1)u - l, (l^2-1)N) = 1$. First, suppose $((l^2-1)u - l, l^2 - 1) = m$. If $m|(l^2-1)$, then $m|u(l^2-1)$. Since $m|(l^2-1)u - l$, we have $m| - l$. As $m|l^2 - 1$, we obtain that $m| - 1$ and so $m = \pm 1$. Suppose $((l^2-1)u - l, N) = N_0$. Clearly, we have

$$(l^2-1)u - l = l(lu-1) - u \equiv 0\,(mod\,N_0).$$

Since $lu - 1 \equiv u^2\,(mod\,N)$, then $lu^2 - u \equiv 0\,(mod\,N_0)$, giving $lu - 1 \equiv 0\,(mod\,N_0)$, a contradiction. Because we have $N_0|u^2$ by $u^2 \equiv 0\,(mod\,N_0)$. Since $(u, N) = 1$, then $N_0 = 1$. Hence, with

$$((l^2-1)u - l, (l^2-1)N) = 1,$$

$\frac{u-\frac{l}{l^2-1}}{N}$ is a vertex in $F_{u,N}$ and is also the farthest vertex which can be joined with $\frac{lu-1}{lN}$. Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{u-\frac{l+Nx}{l(l+Nx)-1}}{N} = \frac{u-\frac{1}{l}}{N}$, there is no such a nearest vertex. Because there are infinitely many vertex bigger than such a nearest vertex. ∎

Now we can give the transformation $\omega = \begin{pmatrix} -u & (u^2-lu+1)/N \\ -N & u-l \end{pmatrix}$ is in $\Gamma_0(N)$ and that $\omega(\infty) = v_0, \omega(v_0) = v_1$ and so on. For left direction, it gives vertices of paths of minimal length in $\mathbf{F}_{u,N}$. Hence,

Corollary 2.4. *If $\frac{u-\frac{x}{y}}{N}$ is the vertex on the path of the minimal length in $\mathbf{F}_{u,N}$, then the farthest vertex which can be joined with it is $\omega\left(\frac{u-\frac{x}{y}}{N}\right) = \frac{u-\frac{y}{ly-x}}{N}$ and $v_q = \omega^q(v_0)$ for positive integers q and $v_0 = u/N$.*

Some result below have immediate analogues in the case of right direction, so we state them without proof.

Corollary 2.5. *If $u^2 - u + 1 \equiv 0\,(mod\,N)$, then $\mathbf{F}_{u,N}$ has a triangle of the form $\frac{1}{0} \longleftarrow \frac{u-1}{N} \longleftarrow \frac{u}{N} \longleftarrow \frac{1}{0}$.* ■

Corollary 2.6. *If $u^2 - lu + 1 \equiv 0\,(mod\,N)$ and $1 < l \leq N$, then there is an infinite path of the minimal length*

$$\cdots \longleftarrow \frac{u - \frac{1}{l-\frac{1}{l-\frac{1}{l}}}}{N} \longleftarrow \frac{u - \frac{1}{l-\frac{1}{l}}}{N} \longleftarrow \frac{u - \frac{1}{l}}{N} \longleftarrow \frac{u}{N} \longleftarrow \frac{1}{0}$$

whose vertices are in the set

$$M := \bigcup_{m=0}^{\infty} \left\{ \frac{u - T_m(0)}{N} : T_m = t_0 t_1 t_2 \ldots t_m,\ t_0(z) = z, t_m(z) := t(z) = \frac{-1}{-l+z} \right\} \cup \{\infty\}$$

in $\mathbf{F}_{u,N}$. ■

2.1. Continued fractions with recurrence relations for the vertices of suborbital graphs

It is well known that a continued fraction may be regarded as a sequence of Möbius maps. We saw that the set M of vertices were obtained by a sequence of Möbius maps. So, there is naturally a connection between the two in here. The purpose of this section is to provide some formulas which give the vertices of the subgraph more practically by this connection. We know any continued fraction can be expressed as the symbol $b_0 + \mathrm{K}_{m=1}^{\infty}(a_m/b_m)$ by [7]. Using the terminology in [7], the n^{th} numerator A_n and the n^{th} denominator B_n of a continued fraction $b_0 + \mathrm{K}(a_m/b_m)$ are defined by the recurrence relations (second order linear difference equations)

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} := b_n \begin{bmatrix} A_{n-1} \\ B_{n-1} \end{bmatrix} + a_n \begin{bmatrix} A_{n-2} \\ B_{n-2} \end{bmatrix}, \tag{1}$$

where $n = 1, 2, 3, \ldots$ with initial conditions $A_{-1} := 1, B_{-1} := 0, A_0 := b_0, B_0 := 1$. The modified approximant $T_n(z_n)$ can then be written as $T_n(z_n) = \frac{A_n + A_{n-1} z_n}{B_n + B_{n-1} z_n}$, where $n = 0, 1, 2, 3, \ldots$ and hence for the n^{th} approximant f_n we have $f_n = T_n(0) = \frac{A_n}{B_n}, f_{n-1} = T_n(\infty) = \frac{A_{n-1}}{B_{n-1}}$.

For the left direction, since $a_n := -1 \neq 0$ and $b_n := -l$, for all $n \geq 1$, then from recurrence relations we get $B_n = -A_{n+1}$ and then since from set M, a vertex on the path of minimal length of the graph $\mathbf{F}_{u,N}$ is $\frac{u - T_n(0)}{N}$, then this vertex can be given with

$$\frac{A_{n+1} u + A_n}{A_{n+1} N} \tag{2}$$

as n^{th} vertex, where $T_n = -\frac{A_n}{A_{n+1}}$.

Similarly for the right direction, from recurrence relations (1), n^{th} vertex $\frac{u + T_n(0)}{N}$ on the path of minimal length of the graph $\mathbf{F}_{u,N}$ can be given with

$$\frac{A_{n+1} u - A_n}{A_{n+1} N} \tag{3}$$

where a_n and b_n defined as -1 and $-k$ respectively for all $n \geq 1$.

Corollary 2.7. *If $l \geq 2$, then from the linear equations* (1)*, we have recurrence relation as $lA_{n+1} + A_{n+2} + A_n = 0$.* ■

Also from using the vertices of the suborbital graph $\mathbf{F}_{u,N}$, since n^{th} vertex on the path of the minimal length of the graph $\mathbf{F}_{u,N}$ is given with $\frac{u - \frac{A_n}{B_n}}{N} = \frac{u + \frac{A_n}{A_{n+1}}}{N}$, then from $\omega\left(\frac{u - \frac{x}{y}}{N}\right) = \frac{u - \frac{y}{ly - x}}{N}$, where $x = A_n$ and $y = B_n = -A_{n+1}$, the farthest vertex which can be joined with it is can be given as $(n+1)^{th}$ vertex by

$$\omega\left(\frac{u + \frac{A_n}{A_{n+1}}}{N}\right) = \frac{u - \frac{A_{n+1}}{lA_{n+1} + A_n}}{N} = \frac{ulA_{n+1} + uA_n - A_{n+1}}{NlA_{n+1} + NA_n}.$$

On the other hand, from (2) this vertex can be given with $\frac{A_{n+2}u+A_{n+1}}{A_{n+2}N}$. Hence, from both expressions above, since $A_{n+1} \neq 0$ for all $n \geq 0$, then

$$\frac{ulA_{n+1} + uA_n - A_{n+1}}{NlA_{n+1} + NA_n} = \frac{A_{n+2}u + A_{n+1}}{A_{n+2}N}$$

and so by this equation Corollary 2.7 holds.

Corollary 2.8. *For right direction, whereas, for $k \geq 2$, then we have recurrence relation as $kA_{n+1} + A_{n+2} + A_n = 0$.* ∎

Theorem 2.9. *If $k = 2$, then $A_n = (-1)^n n$ and if $k > 2$, then*

$$A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} (k + \sqrt{k^2-4})^{n-t} (k - \sqrt{k^2-4})^{t-1}. \tag{4}$$

Proof. From Corollary 2.8 we get the recurrence relation as

$$A_n = -kA_{n-1} - A_{n-2}. \tag{5}$$

If $k = 2$, then we must solve the recurrence relation $A_n + 2A_{n-1} + A_{n-2} = 0$, where initial conditions are from (1). The characteristic equation for this relation is $s^2 + 2s + 1 = 0$, which gives only one real root as $s = -1$. Then any solution of (5) has the form

$$A_n = \alpha(-1)^n + \beta n(-1)^n. \tag{6}$$

Therefore,

$$\begin{aligned} A_0 = \alpha &= 0 \\ A_1 = -\alpha - \beta &= -1. \end{aligned}$$

Solving this system, we get $\alpha = 0$ and $\beta = 1$. So, from (6), $A_n = (-1)^n n$ is obtained.

On the other hand, for the second case, if $k > 2$, then we must solve the recurrence relation $A_n + kA_{n-1} + A_{n-2} = 0$, where initial conditions are from also (1). The characteristic equation for this relation is $s^2 + ks + 1 = 0$, which gives two distinct real roots as $s = \frac{-k+\sqrt{k^2-4}}{2}$ and $s = \frac{-k-\sqrt{k^2-4}}{2}$. Then any solution of (5) has the form

$$A_n = \alpha \left(\frac{-k + \sqrt{k^2-4}}{2} \right)^n + \beta \left(\frac{-k - \sqrt{k^2-4}}{2} \right)^n. \tag{7}$$

Therefore,

$$\begin{aligned} A_0 = \alpha + \beta &= 0 \\ A_1 = \alpha\left(-k + \sqrt{k^2-4}\right) + \beta\left(-k - \sqrt{k^2-4}\right) &= -2. \end{aligned}$$

Solving this system, we get $\alpha = -\frac{1}{\sqrt{k^2-4}}$ and $\beta = \frac{1}{\sqrt{k^2-4}}$. So, from (7),

$$A_n = \left(-\frac{1}{2}\right)^n \frac{1}{\sqrt{k^2-4}} \left[(k + \sqrt{k^2-4})^n - (k - \sqrt{k^2-4})^n \right] \tag{8}$$

is obtained. From $(k + \sqrt{k^2-4})^n - (k - \sqrt{k^2-4})^n$ we get

$$(2\sqrt{k^2-4}) \sum_{t=1}^{n} (k + \sqrt{k^2-4})^{n-t} (k - \sqrt{k^2-4})^{t-1}. \tag{9}$$

So, from (8) and (9),

$$A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} (k + \sqrt{k^2-4})^{n-t}(k - \sqrt{k^2-4})^{t-1}$$

is obtained. ■

From Corollary 2.6 and by using Corollary 2.7, the following result can be obtained similarly

Corollary 2.10. *If $l = 2$, then $A_n = (-1)^n n$ and if $l > 2$, then*

$$A_n = (-1)^n 2^{1-n} \sum_{t=1}^{n} (l + \sqrt{l^2-4})^{n-t}(l - \sqrt{l^2-4})^{t-1}. \tag{10}$$

■

Lemma 2.11. *For the suborbital graphs $\mathbf{F}_{u+bN,(b+1)N}$, where $N = u + 1$ and $b = 0, 1, 2, 3, \cdots$, the infinite paths of the minimal lengths for right direction can be given as*

$$\frac{1}{0} \longrightarrow \frac{u+bN}{N(b+1)} \longrightarrow \frac{u+N(2b+1)}{2N(b+1)} \longrightarrow \frac{u+N(3b+2)}{3N(b+1)} \longrightarrow \frac{u+N(4b+3)}{4N(b+1)} \longrightarrow \cdots$$

Proof. If $N = u + 1$, then the suborbital graphs $\mathbf{F}_{u+bN,(b+1)N}$ can be given as $\mathbf{F}_{\alpha,\alpha+1}$, where $b = 0, 1, 2, 3, \cdots$ and $\alpha = u(1 + b) + b$. So, from the congruence $\alpha^2 + k\alpha + 1 \equiv 0\,(mod\,(\alpha + 1))$, k must be 2 as minimal positive integer. Hence, the infinite paths of the minimal lengths of the suborbital graphs $\mathbf{F}_{u+bN,(b+1)N}$ for right direction are

$$\frac{1}{0} \longrightarrow \frac{u+bN}{(b+1)N} \longrightarrow \frac{u+bN+\frac{1}{2}}{(b+1)N} \longrightarrow \frac{u+bN+\frac{1}{2-\frac{1}{2}}}{(b+1)N} \longrightarrow \frac{u+bN+\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}{(b+1)N} \longrightarrow \cdots$$

From this we get same paths as

$$\frac{1}{0} \longrightarrow \frac{u+bN}{N(b+1)} \longrightarrow \frac{2u+2bN+1}{2N(b+1)} \longrightarrow \frac{3u+3bN+2}{3N(b+1)} \longrightarrow \frac{4u+4bN+3}{4N(b+1)} \longrightarrow \cdots$$

If one u fixed in the numerator of all fractions as vertices of these paths and other every u replaced with $u = N - 1$, then the infinite paths of the minimal lengths obtained as desired hyperbolic paths. ■

Corollary 2.12. *If $l = 2$ or $k = 2$ then the n^{th} vertex on the path of the minimal length starting with the vertex $\frac{u}{N}$ is $\frac{(n+1)u-n}{(n+1)N}$ and $\frac{(n+1)u+n}{(n+1)N}$, respectively.*

Proof. For all $n \geq 0$, since $b_n = -l = -2$ and $a_n = -1$, from the Corollary 2.10,

$$A_n = (-1)^n n. \tag{11}$$

So, by (2), the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ for the left direction is

$$\frac{(-1)^{n+1}(n+1)u + (-1)^n n}{(-1)^{n+1}(n+1)N}$$

and this clearly gives demanded vertex as $\frac{(n+1)u-n}{(n+1)N}$. For right direction assumption can be proved similarly. ■

Corollary 2.13. *If $l = 3$ or $k = 3$ then the n^{th} vertex on the path of the minimal length starting with the vertex $\frac{u}{N}$ is $\frac{F_{2n+2}u-F_{2n}}{F_{2n+2}N}$ and $\frac{F_{2n}+F_{2n+2}u}{F_{2n+2}N}$, respectively, where, for each $m \in \mathbb{N} \cup \{0\}$,*

$$F_m = \begin{cases} 0, & if\ m=0; \\ 1, & if\ m=1; \\ F_{m-1} + F_{m-2}, & if\ m>1 \end{cases}$$

is the m^{th} Fibonacci number.

Proof. For all $n \geq 0$, since $b_n = -l = -3$ and $a_n = -1$, from the Corollary 2.10,

$$A_n = (-1)^n F_{2n}. \tag{12}$$

So, by (2), the n^{th} vertex on the path of the minimal length of $\mathbf{F}_{u,N}$ for the left direction is

$$\frac{(-1)^{n+1}F_{2n+2}u + (-1)^n F_{2n}}{(-1)^{n+1}F_{2n+2}N}$$

and this clearly gives demanded vertex as $\frac{F_{2n}-F_{2n+2}u}{F_{2n+2}N}$. For right direction assumption can be proved similarly. ∎

For right direction, from the matrix relation for recurrence relations we can give

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -k \end{pmatrix}^n \tag{13}$$

As n^{th} vertex of the suborbital graph $\mathbf{F}_{u,N}$ which is on the path of minimal length is $\frac{A_{n+1}u-A_n}{A_{n+1}N}$, we also find this vertex from the matrix equation (13).

Similarly, we can find the n^{th} vertex for left direction by $\frac{A_{n+1}u+A_n}{A_{n+1}N}$, using the matrix relation

$$\begin{pmatrix} A_{n-1} & A_n \\ -A_n & -A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -l \end{pmatrix}^n.$$

3. Scale and Complexity of Subgraph

The Farey graph-$\mathcal{F}$, which is referred to as $\mathbf{F}_{1,1}$, is a connected, undirected graph with triangular circuits. From [2], by means of connectedness, the Farey distance $d(v_1, v_2)$ between any two vertices $v_1, v_2 \in \hat{\mathbb{Q}}$ to be the minimum number of edges in any path from v_1 to v_2 in $\mathcal{F}$; thus d is a metric on $\hat{\mathbb{Q}}$. Also, from [2], a shortest path in the $\mathcal{F}$ from ∞ to any vertex v can be found by expressing v as a continued fraction

$$v = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \ddots - \frac{1}{c_m}}} \tag{14}$$

($c_i \in \mathbb{Z}$), the distance $d(\infty, v)$ being equal to m. Hence, $d(\infty, v)$ can be regard as a measure of the *complexity* of v. Accordingly, the complexity of integers are 1, while the rationals $p + q^{-1}$ ($p, q \in \mathbb{Z}, |q| \geq 2$) have the complexity 2 in $\mathcal{F}$.

In [8], the authors defined a new kind of continued fraction namely $\mathcal{F}_{1,2}$-continued fraction, which is related to the suborbital graph $\mathbf{F}_{1,2}$. A finite continued fraction of the form

$$\frac{1}{0+}\frac{2}{b+}\frac{\epsilon_1}{a_1+}\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n} \quad (n \geq 0) \tag{15}$$

or an infinite continued fraction of the form

$$\frac{1}{0+}\frac{2}{b+}\frac{\epsilon_1}{a_1+}\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n+}\cdots, \tag{16}$$

where b is an odd integer, $a_1, a_2, \ldots$ are even positive integers, and $\epsilon_1, \epsilon_2, \ldots \in \{\pm 1\}$, is called an $\mathcal{F}_{1,2}$-continued fraction. There is a connection between this continued fraction and the suborbital graph $\mathbf{F}_{1,2}$ as, each finite $\mathcal{F}_{1,2}$-continued fraction is shown to correspond naturally to a path in $\mathbf{F}_{1,2}$ from ∞ to its value. Also, there is a connection between $\mathcal{F}_{1,2}$-continued fractions and vertices of the suborbital graph $\mathbf{F}_{1,2}$, which are on the paths of minimal lengths.

From Corollary 2.8. [8], if vertex set of $\mathbf{F}_{1,2}$ is

$$\mathcal{X} = \left\{ \frac{p}{2q} : p, q \in Z, q > 0, (p, 2q) = 1 \right\} \cup \{\infty\},$$

then there is a unique path (of edges in $\mathbf{F}_{1,2}$) from ∞ to every point in $\mathcal{X}$.

Theorem 3.1 (See [8], Theorem 3.1.)**.**
(A) *The path in the suborbital graph* $\mathbf{F}_{1,2}$ *(which is subgraph of the Farey graph* $\mathcal{F}$*) from* ∞ *to* $x \in \mathcal{X}$ *defines a finite* $\mathcal{F}_{1,2}$*-continued fraction of x.*
(B) *The value of every finite* $\mathcal{F}_{1,2}$*-continued fraction belongs to* $\mathcal{X}$ *and the continued fraction defines a path in* $\mathbf{F}_{1,2}$ *from* ∞ *to its value with the convergents as the vertices.* ■

Theorem 3.2 (See [8], Theorem 4.1.)**.**
Given any $x \in \mathcal{X}$*, the* $\mathcal{F}_{1,2}$*-continued fraction expansion*

$$x = \frac{1}{0+}\,\frac{2}{b+}\,\frac{\epsilon_1}{a_1+}\,\frac{\epsilon_2}{a_2+}\cdots\frac{\epsilon_n}{a_n},$$

is obtained as follows: $b = 2\lfloor x \rfloor + 1$ *and for* $(1 \leq i \leq n)$*, setting* $y_1 = 2x - b$*,*

(1) $a_i = 2\left\lfloor \frac{1}{2}\left(1 + \frac{1}{|y_i|}\right)\right\rfloor$,
(2) $\epsilon_i = sign(y_i)$,
(3) $y_{i+1} = \frac{1}{|y_i|} - a_i$.

In fact, n is the smallest non-negative integer for which $y_{n+1} = 0$. ■

Example 3.3. *Let* $x = \frac{17}{18} \in \mathcal{X}$*. From* Theorem 3.2 *we get the* $\mathcal{F}_{1,2}$*-continued fraction of x as,*

$$x = \frac{17}{18} = \frac{1}{0+}\,\frac{2}{1+}\,\frac{1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2+}\,\frac{-1}{2}.$$

This gives the unique shortest path from ∞ *to x in the suborbital graph* $\mathbf{F}_{1,2}$ *as,*

$$\infty = \frac{1}{0} \to \frac{1}{0+\frac{2}{1}} \to \frac{1}{0+\frac{2}{1+\frac{1}{2}}} \to \frac{1}{0+\frac{2}{1+\frac{1}{2-\frac{1}{2}}}} \to \cdots \to \frac{1}{0+\frac{2}{1+\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}}}}}},$$

that is a path of minimal length to right direction in the suborbital graph $\mathbf{F}_{1,2}$ *from* ∞ *to* $v_8 = \frac{17}{18}$ *(which is* 8^{th} *vertex from* Corollary 2.12*) as follows,*

$$\infty = \frac{1}{0} \to \frac{1}{2} \to \frac{3}{4} \to \frac{5}{6} \to \frac{7}{8} \to \frac{9}{10} \to \frac{11}{12} \to \frac{13}{14} \to \frac{15}{16} \to \frac{17}{18}. \blacksquare$$

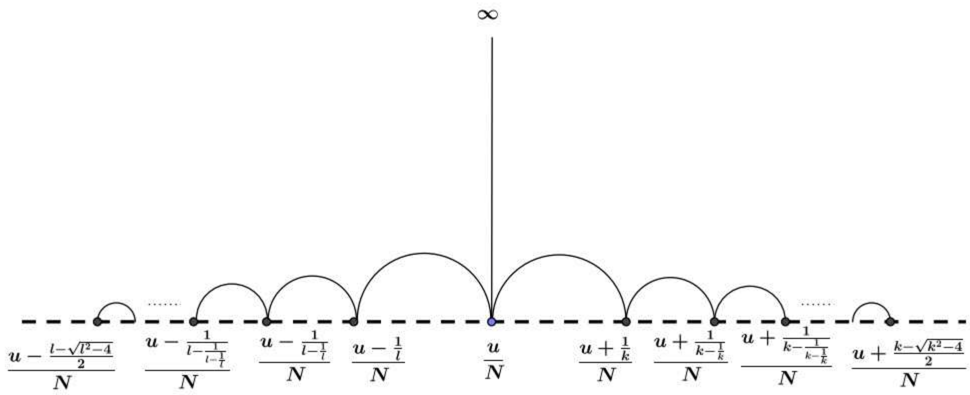

Figure 1: Self-paired subgraph

In our case, infinite paths of minimal lengths for left and right direction are of the form

$$\begin{array}{c} \infty \\ \downarrow \\ \ldots \longleftarrow \omega^2(v_0) \longleftarrow \omega(v_0) \longleftarrow v_0 \longrightarrow \varphi(v_0) \longrightarrow \varphi^2(v_0) \longrightarrow \ldots \end{array}$$

where $v_0 = \frac{u}{N}$. Clearly, a path of the minimal length is a tree of valency 2 by joining each term to its immediate predecessor and successor. The union of these trees with both of direction is the subgraph $\mathbf{F}_{u,N}$. Naturally, $\mathbf{F}_{u,N}$ is connected and the vertices of the paths of the minimal length in $\mathbf{F}_{u,N}$ are the set $K_1 \cup K_2 \subset \hat{\mathbb{Q}}$ where $K_1 := \left\{ \dfrac{u + T_n(0)}{N} : n \in \mathbb{N} \right\}$ and $K_2 := \left\{ \dfrac{u - T_n(0)}{N} : n \in \mathbb{N} \right\}$. So, we can easily see that $d_* : K_1 \times K_1 \mapsto \mathbb{N}$ (or $d_* : K_2 \times K_2 \mapsto \mathbb{N}$), $d_*(v_i, v_j) = \begin{cases} |j-i| & \text{for } i \neq j \\ j+1 & \text{for } v_i = \infty \end{cases}$ which is the number of edges between v_i and v_j is also a metric for the vertices $v_i, v_j \in K_1 (or K_2), i, j = 0, 1, 2, \ldots$. In here, the complexity of any vertex is equal to the number $d_*(\infty, v_i) = i + 1$.

If $\mathbf{F}_{u,N}$ is self-paired, the figure of subgraph with left-direction will be symmetric to that of right-direction as in Figure 1 by Theorem 2.2. Since figures are identical, the subgraph can be regarded as *scale-free*.

Case $\mathbf{F}_{2,9}$**.** We know that the path $\infty \to 2/9 \to 5/18 \to \ldots$ never becomes a circuit in $\mathbf{F}_{2,9}$ [4]. We can easily verify that this is also a path of minimal length in fact. From $u^2 + ku + 1 \equiv 0\,(mod\,N)$ for $u = 2, N = 9$, we have $k = 2$. This value gives us the path of the minimal length to right direction as

$$\infty \to \frac{2}{9} \to \frac{2+\frac{1}{2}}{9} = \frac{5}{18} \to \frac{2+\frac{1}{2-\frac{1}{2}}}{9} = \frac{8}{27} \to \ldots$$

Symmetrically,

$$\ldots \leftarrow 4/27 \leftarrow 3/18 \leftarrow 2/9 \leftarrow \infty$$

is not a path to left direction by Lemma 1.5. By Theorem 2.2, for $l = 7$ we have path of the minimal length to left direction as follows

$$\begin{array}{c} \infty \\ \downarrow \\ \ldots \longleftarrow \frac{89}{432} = \frac{2-\frac{1}{7-\frac{1}{7}}}{9} \longleftarrow \frac{13}{63} = \frac{2-\frac{1}{7}}{9} \longleftarrow \frac{2}{9} \longrightarrow \frac{5}{18} = \frac{2+\frac{1}{2}}{9} \longrightarrow \frac{8}{27} = \frac{2+\frac{1}{2-\frac{1}{2}}}{9} \longrightarrow \ldots \end{array}$$

Case $\mathbf{F}_{1,2}$**.** Since $u^2 \equiv -1\,(mod\,N)$, suborbital graph $\mathbf{F}_{1,2}$ is a self-paired. From $u^2 + ku + 1 \equiv 0\,(mod\,N)$ for $u = 1, N = 2$, we have $k \equiv 0\,(mod\,2)$. By Theorem 2.2, for $k = l = 2$ we have paths of the minimal lengths for both direction as follows

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\infty \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\downarrow \\ \ldots \longleftarrow \frac{1}{6} = \frac{1-\frac{1}{2-\frac{1}{2}}}{2} \longleftarrow \frac{1}{4} = \frac{1-\frac{1}{2}}{2} \longleftarrow \frac{1}{2} \longrightarrow \frac{3}{4} = \frac{1+\frac{1}{2}}{2} \longrightarrow \frac{5}{6} = \frac{1+\frac{1}{2-\frac{1}{2}}}{2} \longrightarrow \ldots \end{array}$$

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