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A Fourth Order Approximation of the Neumann Type Overdetermined Elliptic Problem

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Abstract. In this paper, we consider an inverse elliptic problem with Neumann type overdetermination and construct a fourth order of accuracy difference scheme for its solution. Stability, almost coercive stability and coercive stability estimates for the solution of difference problem are proved. Later, we construct a fourth order difference scheme for an inverse problem for multidimensional elliptic equation with Neumann type overdetermination and Dirichlet boundary condition. Finally, we illustrate numerical example with descriptions of numeric realization in a two-dimensional case.

1. Introduction

Inverse problems for partial differential equations with overdetermination are widely used in mathematical modeling of real processes (see [1–3]). Theory and methods of solutions of identification problems of determining the parameter of a partial differential equations have been extensively investigated by numerous authors (see [1–15] and references therein). Details of description for such class of problems for elliptic type differential and difference equations can be found in [16–31].

High order difference schemes for inverse elliptic problem with Dirichlet type overdetermination were studied in [25, 28].

Let A be a self-adjoint positive definite operator A with domain $D(A)$ in an arbitrary Hilbert space H.

We consider the problem of finding an element $p \in H$ and a function $u(\cdot) \in C^2([0,1]; H) \cap C([0,1]; D(A))$ from the following system

$$\begin{cases} -u_{tt}(t) + Au(t) = f(t) + pt, \ t \in (0,1), \\ u_t(0) = \varphi, u_t(\lambda) = \xi, u_t(1) = \psi, \end{cases} \tag{1.1}$$

where φ, ξ, ψ are given elements of H, $\lambda \in (0,1)$ is known number.

For solving inverse problem (1.1), we reduce it to an auxiliary nonlocal problem. Namely, we apply the substitution

$$u(t) = v(t) + A^{-1}(pt), \tag{1.2}$$

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and get the following auxiliary nonlocal problem for obtaining $v(t)$

$$\begin{cases} -v_{tt}(t) + Av(t) = f(t),\ t \in (0,1)\,, \\ v_t(0) - v_t(\lambda) = \varphi - \xi,\ v_t(1) - v_t(\lambda) = \psi - \xi. \end{cases} \tag{1.3}$$

After obtaining the solution of (1.3), we can find $v_t(\lambda)$. Then, by using formula

$$p = A\left(\xi - v_t(\lambda)\right), \tag{1.4}$$

we define an element p. Finally, we can obtain the solution $(u(\cdot), p)$ of problem (1.1) by formulas (1.2) and (1.4).

In Section 2, we study a fourth order of accuracy difference scheme (ADS) for inverse problem (1.1) and establish stability, almost coercive stability and coercive stability inequalities for its approximate solution.

Later, we study a fourth order approximation of the inverse problem for the multidimensional elliptic equation with Neumann type overdetermination and Dirichlet boundary condition

$$\begin{cases} -u_{tt}(t,x) - \sum\limits_{r=1}^{n} (a_r(x)u_{x_r})_{x_r} + \sigma u(t,x) = f(t,x) + p(x)t, x = (x_1, \cdots, x_n) \in \Omega,\ 0 < t < 1, \\ u_t(0,x) = \varphi(x), u_t(1,x) = \psi(x), u_t(\lambda, x) = \xi(x), x \in \overline{\Omega}, \\ u(t,x) = 0,\ 0 \le t \le 1,\ x \in S. \end{cases} \tag{1.5}$$

Here, $\Omega = (0,1) \times \cdots \times (0,1)$ is the open cube in the n-dimensional Euclidean space with boundary S, $\overline{\Omega} = \Omega \cup S, a_r(x)\ (x \in \Omega), \varphi(x), \xi(x), \psi(x)\ (x \in \overline{\Omega}), f(t,x)\ (t \in (0,1),\ x \in \Omega)$ are given smooth functions, $a_r(x) \ge a > 0\ (x \in \overline{\Omega})$, and $\lambda \in (0,1), \sigma > 0$ are known numbers.

Stability estimates for solutions of problems (1.1) and (1.5) were given in [27]. Moreover, the first and second order of ADS for them were presented.

In [29], a fourth order of ADS for inverse elliptic problem with Neumann type overdetermination was presented. Stability estimates for the solution of difference scheme were given without proof.

In Section 3, we study a fourth order of ADS for problem (1.5) and establish stability and almost coercive stability estimates for its solution. Later, we give numerical example with the descriptions of numeric realization in a two-dimensional case.

The remainder of this paper is organized as follows. Section 2 is devoted to proof of Theorems on stability and corcive stability estimates for solutions of a fourth order of ADS for inverse problem (1.1) and auxiliary nonlocal problem (1.3). Stability and almost coercive stability estimates for solution of a fourth order of ADS for problem (1.5) are established in Section 3. The numerical results are given in Section 4. Last Section is conclusion.

2. A Fourth Order of Accuracy Difference Scheme

Let N be a given natural number and $\tau = \frac{1}{N}$. Introduce the set of grid points $\{t_k = k\tau,\ 0 \le k \le N\}$ and the spaces $C_\tau(H), C_\tau^\alpha(H)$, and $C_\tau^{\alpha,\alpha}(H)$ $(0 < \alpha < 1)$ of H-valued grid functions $\{f_k\}_{k=1}^{N-1}$ with the following norms

$$\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} = \max_{1\le k\le N-1}\left\|f_k\right\|_H, \ \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} = \max_{1\le k\le N-1}\left\|f_k\right\|_H + \sup_{1\le k<k+n\le N-1}\frac{\|f_{k+n}-f_k\|_H}{(n\tau)^\alpha},$$
$$\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)} = \max_{1\le k\le N-1}\left\|f_k\right\|_H + \sup_{1\le k<k+n\le N-1}\frac{(k\tau+n\tau)^\alpha(1-k\tau)^\alpha\|f_{k+n}-f_k\|_H}{(n\tau)^\alpha},$$

respectively.

Let $[\cdot]$ be a notation for greatest integer function and $l = \left[\frac{\lambda}{\tau}\right]$.

Applying approximate formulas

$$\begin{aligned} u'(0) &= \tfrac{u(\tau)-u(0)}{\tau} - \tfrac{5\tau^2}{12}u''(0) - \tfrac{\tau^2}{12}u''(\tau) + \tfrac{\tau^3}{12}u'''(0) + o(\tau^5), \\ u'(1) &= \tfrac{u(1)-u(1-\tau)}{\tau} + \tfrac{5\tau^2}{12}u''(1) + \tfrac{\tau^2}{12}u''(1-\tau) + \tfrac{\tau^3}{12}u'''(1) + o(\tau^5), \\ u'(\lambda) &= \tfrac{u(\lambda+\tau)-u(\lambda)}{\tau} - \tfrac{5\tau^2}{12}u''(\lambda) - \tfrac{\tau^2}{12}u''(\lambda+\tau) + \tfrac{\tau^3}{12}u'''(\lambda) + o(\tau^5), \end{aligned} \tag{2.1}$$

and the method of approximation of an abstract elliptic equation ([33] , Section 5.3), we get a fourth order ADS for problem (1.1)

$$\begin{cases} -\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{12}A^2u_k=\theta_k+pt_k, \theta_k=f(t_k)+\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right), \\ t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \\ \left(I-\frac{\tau^2}{12}A\right)u_1-\left(I+\frac{5\tau^2}{12}A\right)u_0=\tau\left(I+\frac{\tau^2}{12}A\right)\varphi-\frac{5\tau^2}{12}f_0-\frac{\tau^2}{12}f_1-\frac{\tau^2}{12}f_0'-\frac{\tau^3}{6}p, \\ \left(I-\frac{\tau^2}{12}A\right)u_{l+1}-\left(I+\frac{5\tau^2}{12}A\right)u_l=\tau\left(I+\frac{\tau^2}{12}A\right)\xi-\frac{5\tau^2}{12}f_l-\frac{\tau^2}{12}f_{l+1}-\frac{\tau^2}{12}f_l'-\frac{\tau^2}{2}t_lp-\frac{\tau^3}{6}p, \\ -\left(I-\frac{\tau^2}{12}A\right)u_{N-1}+\left(I+\frac{5\tau^2}{12}A\right)u_N=\tau\left(I+\frac{\tau^2}{12}A\right)\psi-\frac{5\tau^2}{12}f_N-\frac{\tau^2}{12}f_{N-1}+\frac{\tau^3}{12}f_n'-\frac{\tau^2}{2}p+\frac{\tau^3}{6}p. \end{cases} \tag{2.2}$$

Here I is the identity operator.

To apply the discrete analogy of algorithm described in Section 1, we will construct the following difference scheme to solve difference scheme (2.2)

$$\begin{aligned} &-\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k+\frac{\tau^2}{12}A^2v_k=\theta_k,\ \theta_k=f(t_k)+\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right), \\ &t_k=k\tau,\ 1\le k\le N-1, N\tau=1, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\left(I-\frac{\tau^2}{12}A\right)v_1-\left(I+\frac{5\tau^2}{12}A\right)v_0-\left(I-\frac{\tau^2}{12}A\right)v_{l+1}+\left(I+\frac{5\tau^2}{12}A\right)v_l \\ &=\tau\left(I+\frac{\tau^2}{12}A\right)(\varphi-\xi)-\frac{5\tau^2}{12}(f_0-f_l)-\frac{\tau^2}{12}(f_1-f_{l+1})-\frac{\tau^2}{12}\left(f_0'-f_l'\right), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &-\left(I-\frac{\tau^2}{12}A\right)v_{N-1}+\left(I+\frac{5\tau^2}{12}A\right)v_N-\left(I-\frac{\tau^2}{12}A\right)v_{l+1}+\left(I+\frac{5\tau^2}{12}A\right)v_l \\ &=\tau\left(I+\frac{\tau^2}{12}A\right)(\psi-\xi)-\frac{5\tau^2}{12}(f_N-f_l)-\frac{\tau^2}{12}(f_{N-1}-f_{l+1})+\frac{\tau^3}{12}\left(f_N'+f_l'\right). \end{aligned} \tag{2.5}$$

Let us $C=A+\frac{\tau^2}{12}A^2, F=\frac{1}{2}(\tau C+\sqrt{4C+\tau^2C^2}), R=(I+\tau F)^{-1}, A\ge\delta I.$

Since A is a self-adjoint positive definite operator , then the operator F is a self-adjoint positive definite operator, too ([35]). In addition, the bounded operator F is defined on the whole space H.

Throught the text, positive constants which can differ in time, hence they are not a subject of precision will be indicated with M. The other side, $M(\delta)$ is used to focus on the fact that the constant depends only on δ .

Lemma 2.1. *The following estimates hold ([33] , p. 298):*

$$\begin{aligned} &\left\|R^k\right\|_{H\to H}\le M(\delta)(1+\delta\tau)^{-k},\ k\tau\left\|FR^k\right\|_{H\to H}\le M(\delta), k\ge1, \delta>0, \left\|(I-R^{2N})^{-1}\right\|_{H\to H}\le M(\delta) \\ &\left\|F^\beta(R^{k+r}-R^k)\right\|_{H\to H}\le M(\delta)\frac{(r\tau)^\alpha}{(k\tau)^{\alpha+\beta}}, 1\le k<k+r\le N, 0\le\alpha,\beta\le1. \end{aligned}$$

Lemma 2.2. *For* $1\le l\le N-1$, *the next operators*

$$\begin{aligned} &S_1=(I-R^{2N})^{-1}\left[\left(I-\frac{\tau^2}{12}A\right)\left(-I+R^{2N}+R-R^{2N-1}+R^{N-1}-R^{N+1}\right)+\frac{\tau^2}{2}A(-I+R^{2N})\right], \\ &S_2=(I-R^{2N})^{-1}\left(I-\frac{\tau^2}{12}A\right)\left(R-R^{2N-1}-R^{l+1}+R^{2N-l-1}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}\right) \\ &+(I-R^{2N})^{-1}\left(I+\frac{5\tau^2}{12}A\right)(-I+R^{2N}+R^l-R^{2N-l}-R^{N-l}+R^{N+l}) \end{aligned}$$

have the inverses

$$G_1=S_1^{-1}, G_2=S_2^{-1}, \tag{2.6}$$

and the following estimates

$$\| G_1 \|_{H\to H}\le M(\delta), \| G_2 \|_{H\to H}\le M(\delta) \tag{2.7}$$

are satisfied.

Proof. Denote by S and Q the next operators

$$\begin{aligned}&S=(I-R^{2N})^{-1}\left[\left(-I+R^{2N}+R-R^{2N-1}+R^{N-1}-R^{N+1}\right)\right],\\ &Q=(I-R^{2N})^{-1}\Big[-I+R^{2N}+R^{l}-R^{2N-l}-R^{N-l}+R^{N+l}+R-R^{2N-1}-R^{l+1}\\ &+R^{2N-l-1}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}\Big].\end{aligned}\tag{2.8}$$

It easy to see that $S=-(I-R^{2N})^{-1}(I-R)\left(I-R^{N-1}\right)\left(I-R^{N+1}\right)$. Hence, there exist $G=S^{-1}$ such that

$$G=-\left(I-R^{N+1}\right)^{-1}\left(I-R^{N-1}\right)^{-1}(I-R)^{-1}(I-R^{2N}),$$

and according to Lemma 2.1, the estimate

$$\| G \|_{H\to H}\le M_1(\delta)\tag{2.9}$$

is valid.

Then, we get

$$G_1-G=G_1GK_1,\tag{2.10}$$

where

$$K_1=-(I-R^{2N})^{-1}\left[-\frac{\tau^2}{12}A(I-R^{2N})^{-1}(I-R)\left(I-R^{N-1}\right)\left(I-R^{N+1}\right)+\frac{\tau^2}{2}A(-I+R^{2N})\right].$$

By using estimates of Lemma 2.1, we can show that

$$\| K_1 \|_{H\to H}\le M_2(\delta)\tau.\tag{2.11}$$

Applying the triangle inequality, formula (2.10), estimates (2.9), (2.11), we obtain

$$\| G_1 \|_{H\to H}\le\| G \|_{H\to H}+\| G_1 \|_{H\to H}\| G \|_{H\to H}\| K_1 \|_{H\to H}\le M_1(\delta)+\| G_1 \|_{H\to H}M_1(\delta)M_2(\delta)\tau$$

for any small positive parameter τ. From that it follows first estimate of (2.7).

Now, we can rewrite Q as $Q=-(I-R^{2N})^{-1}(I-R)\left(I-R^{l}\right)\left(I+R^{N}\right)\left(I-R^{N-l-1}\right)$. So, there exists its inverse $P=Q^{-1}=-(I-R^{N})\left(I-R^{N-l-1}\right)^{-1}\left(I-R^{l}\right)^{-1}(I-R)^{-1}$, and according to the estimates of Lemma 2.1, we can obtain

$$\| P \|_{H\to H}\le M_3(\delta).\tag{2.12}$$

We have

$$G_2-P=G_2PK_2,\tag{2.13}$$

where

$$\begin{aligned}&K_2=-\tfrac{\tau^2}{12}(I-R^{2N})^{-1}A\left(R-R^{2N-1}-R^{l+1}+R^{2N-l-1}-R^{N-1}+R^{N+1}+R^{N-l-1}-R^{N+l+1}\right)\\ &+\tfrac{5\tau^2}{12}(I-R^{2N})^{-1}A(-I+R^{2N}+R^{l}-R^{2N-l}-R^{N-l}+R^{N+l}).\end{aligned}$$

By using Lemma 2.1, we can get that

$$\| K_2 \|_{H\to H}\le M_4(\delta)\tau.\tag{2.14}$$

Applying the triangle inequality, formula (2.13), estimates (2.12), (2.14), we get

$$\| G_2 \|_{H\to H}=\| P \|_{H\to H}+\| G_2 \|_{H\to H}\| P \|_{H\to H}\| K_2 \|_{H\to H}\le M_3(\delta)+\| G_2 \|_{H\to H}M_3(\delta)M_4(\delta)\tau$$

for any small positive parameter τ. So, second estimate of (2.7) is valid. Therefore, proof of Lemma 2.2 is finished.

Theorem 2.3. *Suppose that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{f_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)$ $(0<\alpha<1)$. Then, for any $\{f_k\}_{k=1}^{N-1}$, φ,ψ,ξ the solution of difference problem (2.3)–(2.5) exists and for its solution in $C_\tau(H)$ obeys the following stability and almosty coercive stability estimates:*

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\le M\left[\left\|\varphi\right\|_H+\left\|\psi\right\|_H+\|\xi\|_H+\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.15}$$

$$\begin{gathered}\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|\left\{\left(A+\tfrac{\tau^2}{12}A^2\right)v_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\\ \le M\left[\min\left\{\ln\tfrac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|F\varphi\right\|_H+\left\|F\psi\right\|_H+\|F\xi\|_H\right],\end{gathered} \tag{2.16}$$

where M is independent of $\tau,\alpha,\varphi,\psi,\xi$, and $\{f_k\}_{k=1}^{N-1}$.

Proof. It is known that , the direct difference problem

$$\begin{cases}-\tau^{-2}(v_{k+1}-2v_k+v_{k-1})+Av_k+\frac{\tau^2}{12}A^2v_k=\theta_k,\theta_k=f(t_k)+\frac{\tau^2}{12}\left(\frac{f(t_{k+1})-2f(t_k)+f(t_{k-1})}{\tau^2}+Af(t_k)\right),\\ t_k=k\tau,\ 1\le k\le N-1,N\tau=1,\\ v_0,v_N \text{ are given}\end{cases} \tag{2.17}$$

has a solution, and its solution is represented by formula ([32])

$$\begin{gathered}v_k=(I-R^{2N})^{-1}[\left(\left(R^k-R^{2N-k}\right)v_0+\left(R^{N-k}-R^{N+k}\right)v_N\right)-\left(R^{N-k}-R^{N+k}\right)(I+\tau F)(2I+\tau F)^{-1}F^{-1}\\ \times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau]+(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right)\theta_i\tau.\end{gathered} \tag{2.18}$$

Applying (2.18) to (2.4) and (2.5), we get the following system equation to define v_0 and v_N :

$$\begin{gathered}(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(R-R^{2N-1}-R^{l+1}+R^{2N-l-1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)(-I+R^{2N}+R^l-R^{2N-l})\right]v_0\\ +(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{N-l}-R^{N+l}\right)\right]v_N\\ -(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{N-l}-R^{N+l}\right)\right]\\ \times(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau+(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\right.\\ \left.\times\left(R^{|1-i|}-R^{1+i}-R^{|l+1-i|}+R^{l+1+i}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{|l-i|}-R^{l+i}\right)\right]\theta_i\tau\\ =\tau\left(I+\tfrac{\tau^2}{12}A\right)(\varphi-\xi)\ -\tfrac{5\tau^2}{12}(f_0-f_l)-\tfrac{\tau^2}{12}(f_1-f_{l+1})-\tfrac{\tau^3}{12}\left(f_0'-f_l'\right),\end{gathered} \tag{2.19}$$

$$\begin{gathered}(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(-R^{N-1}+R^{N+1}-R^{l+1}+R^{2N-l-1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^l-R^{2N-l}\right)\right]v_0+(I-R^{2N})^{-1}\\ \times\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(-R+R^{2N-1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)(I-R^{2N}+R^{N-l}-R^{N+l})\right]v_N-(I-R^{2N})^{-1}\\ \times\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(-R+R^{2N-1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)(R^{N-l}-R^{N+l})\right](I+\tau F)(2I+\tau F)^{-1}F^{-1}\\ \times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau\ +(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(-R^{|N-1-i|}+R^{N-1+i}-R^{|l+1-i|}+R^{l+1+i}\right)\right.\\ \left.+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{|l-i|}-R^{l+i}\right)\right]\theta_i\tau\ =\tau\left(I+\tfrac{\tau^2}{12}A\right)(\psi-\xi)-\tfrac{5\tau^2}{12}(f_N-f_l)-\tfrac{\tau^2}{12}(f_{N-1}-f_{l+1})+\tfrac{\tau^3}{12}\left(f_N'+f_l'\right)\end{gathered} \tag{2.20}$$

Solving this system equation, we obtain

$$\begin{aligned}
v_0 &= G_2(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{N-l}-R^{N+l}\right)\right]\\
&\times\Big\{G_1(I-R^{2N})^{-1}\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}+R-R^{2N-1}\right)(I+\tau F)(2I+\tau F)^{-1}F^{-1}\\
&\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau-G_1(I+\tau F)(2I+\tau F)^{-1}F^{-1}\\
&\times\sum_{i=1}^{N-1}\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{|1-i|}-R^{1+i}+R^{|N-1-i|}-R^{N-1+i}\right)\theta_i\tau\\
&+\tau\left(I+\tfrac{\tau^2}{12}A\right)(\varphi-\psi)-\tfrac{5\tau^2}{12}(f_0+f_N)-\tfrac{\tau^2}{12}(f_1+f_{N-1})-\tfrac{\tau^3}{12}\left(f'_0-f'_N\right)\Big\}+G_2(I-R^{2N})^{-1}\\
&\times\left[\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{N-l}-R^{N+l}\right)\right](I+\tau F)(2I+\tau F)^{-1}F^{-1}\\
&\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau-G_2(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\Big[\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{|1-i|}-R^{1+i}-R^{|l+1-i|}+R^{l+1+i}\right)\\
&+\left(I+\tfrac{5\tau^2}{12}A\right)\left(R^{|l-i|}-R^{l+i}\right)\Big]\theta_i\tau+\tau G_2\left(I+\tfrac{\tau^2}{12}A\right)(\varphi-\xi)-\tfrac{5\tau^2}{12}G_2(f_0-f_l)\\
&-\tfrac{\tau^2}{12}G_2(f_1-f_{l+1})-\tfrac{\tau^3}{12}G_2\left(f'_0-f'_l\right),
\end{aligned}\tag{2.21}$$

$$\begin{aligned}
v_N &= -v_0+G_1(I-R^{2N})^{-1}\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{N-1}-R^{N+1}+R-R^{2N-1}\right)(I+\tau F)(2I+\tau F)^{-1}F^{-1}\\
&\times\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau-G_1(I+\tau F)(2I+\tau F)^{-1}F^{-1}\sum_{i=1}^{N-1}\left(I-\tfrac{\tau^2}{12}A\right)\left(R^{|1-i|}-R^{1+i}+R^{|N-1-i|}-R^{N-1+i}\right)\theta_i\tau\\
&+\tau\left(I+\tfrac{\tau^2}{12}A\right)(\varphi-\psi)-\tfrac{5\tau^2}{12}(f_0+f_N)-\tfrac{\tau^2}{12}(f_1+f_{N-1})-\tfrac{\tau^3}{12}\left(f'_0-f'_{N'}\right).
\end{aligned}\tag{2.22}$$

where G_1 and G_2 are defined by (2.6).

Hence, the difference problem (2.3)–(2.5) has solution (2.18), where v_0 and v_N are defined by (2.21), (2.22).

Applying (2.18), (2.21), (2.22), and Lemmas 2.1-2.2, we can show that for the solution of difference problem (2.3)–(2.5) the following inequalities hold:

$$\|Rv_0\|_{C_\tau(H)} \leq M\left[\left\|\varphi\right\|_H+\left\|\psi\right\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.23}$$

$$\|Rv_N\|_{C_\tau(H)} \leq M\left[\left\|\varphi\right\|_H+\left\|\psi\right\|_H+\|\xi\|_H+\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right] \tag{2.24}$$

In the [33], the estimates

$$\left\|\{v_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \leq M\left[\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|Rv_0\|_{C_\tau(H)}+\|Rv_N\|_{C_\tau(H)}\right], \tag{2.25}$$

$$\begin{aligned}
&\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)v_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\\
&\leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\|CRv_0\|_H+\|CRv_N\|_H\right]
\end{aligned}\tag{2.26}$$

are proved for the solution of difference problem (2.17).

So, estimate (2.15) follows from estimates (2.23), (2.24), and (2.25).

By using (2.18), (2.21), (2.22), and Lemmas 2.1-2.2, we can get for the solution of difference problem (2.3)–(2.5) the following estimates

$$\|CRv_0\|_H \leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|F\varphi\right\|_H+\left\|F\psi\right\|_H+\|F\xi\|_H\right], \tag{2.27}$$

$$\|CRv_N\|_H \leq M\left[\min\left\{\ln\frac{1}{\tau},1+|\ln\|F\|_{H\to H}|\right\}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}+\left\|F\varphi\right\|_H+\left\|F\psi\right\|_H+\|F\xi\|_H\right] \tag{2.28}$$

are valid. Therefore, inequality (2.16) follows from estimates (2.26), (2.27), and (2.28). The proof of Theorem 2.3 is finished.

Let us $0<\alpha<1$. Denote by $E_\alpha = E_\alpha(D(F),H)$, the Banach space of those functions $v\in H$ for which the norm $\|v\|_{E_\alpha} = \sup_{z>0} z^{1-\alpha}\left\|Fe^{-zF}v\right\|_H + \|v\|_H$ is finite.

Theorem 2.4. *Suppose that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(F)$ and $\{f_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)$ $(0<\alpha<1)$. Then, the solution $\{v_k\}_{k=1}^{N-1}$ of difference problem (2.3)–(2.5) obeys the following coercive inequality*

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)v_k\right\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}$$
$$\le M\left[\frac{1}{\alpha(1-\alpha)}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|F\varphi\right\|_{E_\alpha} + \left\|F\psi\right\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \tag{2.29}$$

where M is independent of $\tau,\alpha,\varphi,\psi,\xi$, and $\{f\}_{k=1}^{N-1}$.

Proof. By using formulas (2.18), (2.21), (2.22), Lemmas 2.1 and 2.2, and definitions of norm of spaces E_α and $C_\tau^\alpha(H)$, it can be showed that the following inequalities hold:

$$\|CRv_0-\theta_1\|_{E_\alpha} \le M\left[\frac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|F\varphi\right\|_{E_\alpha} + \left\|F\psi\right\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \tag{2.30}$$

$$\|CRv_N-\theta_{N-1}\|_{E_\alpha} \le M\left[\frac{1}{\alpha(1-\alpha)}\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|F\varphi\right\|_{E_\alpha} + \left\|F\psi\right\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right]. \tag{2.31}$$

In the [33], for the solution of difference problem (2.17) estimate

$$\left\|\left\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|\left\{\left(A+\frac{\tau^2}{12}A^2\right)v_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)}$$
$$\le M\left[\left\|\{\theta_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \|CRv_0-\theta_1\|_{E_\alpha} + \|CRv_N-\theta_{N-1}\|_{E_\alpha}\right]$$

is established. Therefore, (2.29) is valid.

From Theorems 2.3 and 2.4, (1.2), (1.4), and triangle inequality follow the following Theorems on stability, almost corcive stability and coercive stability estimates for the solution $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (2.2).

Theorem 2.5. *Suppose that A is a self-adjoint positive definite operator, $\varphi,\psi,\xi\in D(A)$ and $\{f_k\}_{k=1}^{N-1}\in C_\tau^{\alpha,\alpha}(H)$ $(0<\alpha<1)$. Then, for the solution $\left(\{u_k\}_{k=1}^{N-1},p\right)$ of difference problem (2.2) in $C_\tau(H)\times H$ obeys the following stability estimates:*

$$\left\|\{u_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} \le M\left[\left\|\varphi\right\|_H + \left\|\psi\right\|_H + \|\xi\|_H + \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.32}$$

$$\left\|A^{-1}p\right\|_H \le M\left[\left\|\varphi\right\|_H + \left\|\psi\right\|_H + \|\xi\|_H + \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)}\right], \tag{2.33}$$

$$\left\|p\right\|_H \le M\left[\left\|A\varphi\right\|_H + \left\|A\psi\right\|_H + \|A\xi\|_H + \frac{1}{\alpha(1-\alpha)}\left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^{\alpha,\alpha}(H)}\right], \tag{2.34}$$

where M is independent of $\tau,\alpha,\varphi,\psi,\xi$, and $\{f_k\}_{k=1}^{N-1}$.

Theorem 2.6. *Assume that $\varphi, \psi, \xi \in D(F)$ and $\{f_k\}_{k=1}^{N-1} \in C_\tau(H)$. Then, solution $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference scheme (2.2) in $C_\tau(H) \times H$ obeys the almost coercive stability estimate*

$$\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|\left\{\left(A + \tfrac{\tau^2}{12}A^2\right) u_k\right\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|p\right\|_H$$
$$\leq M\left[\min\left\{\ln \tfrac{1}{\tau}, 1 + |\ln \|F\|_{H\to H}|\right\} \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau(H)} + \left\|F\varphi\right\|_H + \left\|F\psi\right\|_H + \|F\xi\|_H\right],$$

where M does not depend on $\tau, \alpha, \varphi, \psi, \xi$, and $\{f_k\}_{k=1}^{N-1}$.

Theorem 2.7. *Suppose that A is a self-adjoint positive definite operator, $\varphi, \psi, \xi \in D(A)$ and $\{f_k\}_{k=1}^{N-1} \in C_\tau^\alpha(H)\ (0 < \alpha < 1)$. Then, the solutions $\left(\{u_k\}_{k=1}^{N-1}, p\right)$ of difference problem (2.2) obeys the following coercive inequality*

$$\left\|\left\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\right\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|\left\{\left(A + \tfrac{\tau^2}{12}A^2\right) u_k\right\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|p\right\|_H$$
$$\leq M\left[\tfrac{1}{\alpha(1-\alpha)} \left\|\{f_k\}_{k=1}^{N-1}\right\|_{C_\tau^\alpha(H)} + \left\|F\varphi\right\|_{E_\alpha} + \left\|F\psi\right\|_{E_\alpha} + \|F\xi\|_{E_\alpha}\right], \tag{2.35}$$

where M is independent of $\tau, \alpha, \varphi, \psi, \xi$, and $\{f_k\}_{k=1}^{N-1}$.

3. Difference Scheme for the Problem (1.5)

Now, we consider problem (1.5). The differential expression

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x)u_{x_r}(x))_{x_r} + \sigma u(x) \tag{3.1}$$

generated by problem (1.5) defines a self-adjoint strongly positive definite operator A^x acting on $L_2(\overline{\Omega})$ with the domain

$$D(A^x) = \left\{u(x) \in W_2(\overline{\Omega}),\ u(x) = 0,\ \ x \in S\right\}.$$

The discretization of problem (1.5) is carried out in two steps. Let $M_1, ..., M_n$ be given natural numbers. Denote $m = (m_1, \cdots, m_n)$ and $h = (h_1, \cdots, h_n)$. In the first step, we define the grid spaces

$$\widetilde{\Omega}_h = \{x = (h_1m_1, \cdots, h_nm_n);\ m_r = 0, \cdots, M_r,\ h_rM_r = 1, r = 1, \cdots, n\}, \Omega_h = \widetilde{\Omega}_h \cap \Omega, S_h = \widetilde{\Omega}_h \cap S.$$

Let $L_{2h} = L_2(\widetilde{\Omega}_h)$, $W_{2h}^1 = W_2^1(\widetilde{\Omega}_h)$ and $W_{2h}^2 = W_2^2(\widetilde{\Omega}_h)$ be spaces of the grid functions $\zeta^h(x) = \{\zeta(h_1m_1, \cdots, h_nm_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\|\zeta\|_{L_{2h}} = \left(\sum_{x\in\widetilde{\Omega}_h} |\zeta^h(x)|^2 h_1 \cdots h_n\right)^{1/2}, \left\|\zeta^h\right\|_{W_{2h}^1} = \left\|\zeta^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^n \left|(\zeta^h(x))_{x_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2},$$
$$\left\|\zeta^h\right\|_{W_{2h}^2} = \left\|\zeta^h\right\|_{L_{2h}} + \left(\sum_{x\in\widetilde{\Omega}_h} \sum_{r=1}^n \left|(\zeta^h(x))_{x_r\overline{x}_r, m_r}\right|^2 h_1 \cdots h_n\right)^{1/2}.$$

To the differential operator A^x (3.1), assign the difference operator A_h^x , defined by the formula,

$$A_h^x u^h(x) = -\sum_{i=1}^{n} \left(a_i(x)u_{\overline{x}_i}^h(x)\right)_{x_i, m_i} + \sigma u^h(x) \tag{3.2}$$

acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x) = 0$ for all $x \in S_h$. It is known that ([33]) A_h^x is a self-adjoint positive define operator in $L_2\left(\widetilde{\Omega}_h\right)$.

For $u^h(t,x)$ and $p^h(x)$ functions, we get a system of ordinary differential equations

$$\begin{cases} -\frac{d^2u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x) + tp^h(x),\ 0 < t < T,\ x \in \widetilde{\Omega}_h, \\ u_t^h(0,x) = \varphi(x),\ u_t^h(\lambda,x) = \xi(x),\ \ u_t^h(T,x) = \psi(x), x \in \widetilde{\Omega}_h. \end{cases} \tag{3.3}$$

In the second step of discretization, applying discrete analogy of (2.1), system equations (3.3) is replaced by a fourth order of ADS

$$\begin{cases} -\tau^{-2}(u_{k+1}^h - 2u_k^h + u_{k-1}^h) + A_h^x u_k^h + \frac{\tau^2}{12}\left(A_h^x\right)^2 u_k^h = \theta_k^h + p^h t_k, \\ \theta_k^h = f^h(t_k) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1})-2f^h(t_k)+f(t_{k-1})}{\tau^2} + Af^h(t_k)\right), t_k = k\tau,\ 1 \le k \le N-1, N\tau = 1, \\ \left(I - \frac{\tau^2}{12}A_h^x\right)u_1^h - \left(I + \frac{5\tau^2}{12}A_h^x\right)u_0^h = \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\varphi^h - \frac{5\tau^2}{12}f_0^h - \frac{\tau^2}{12}f_1^h - \frac{\tau^2}{12}f_0'^h - \frac{\tau^3}{6}p^h, \\ \left(I - \frac{\tau^2}{12}A_h^x\right)u_{l+1}^h - \left(I + \frac{5\tau^2}{12}A_h^x\right)u_l^h = \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\xi^h - \frac{5\tau^2}{12}f_l^h - \frac{\tau^2}{12}f_{l+1}^h - \frac{\tau^2}{12}f_l'^h - \frac{\tau^2}{2}t_l p^h - \frac{\tau^3}{6}p^h, \\ -\left(I - \frac{\tau^2}{12}A_h^x\right)u_{N-1}^h + \left(I + \frac{5\tau^2}{12}A_h^x\right)u_N^h = \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\psi^h - \frac{5\tau^2}{12}f_N^h - \frac{\tau^2}{12}f_{N-1}^h + \frac{\tau^3}{12}f_N'^h - \frac{\tau^2}{2}p^h + \frac{\tau^3}{6}p^h. \end{cases} \tag{3.4}$$

By using discrete analogy of (1.2), we get auxiliary difference problem for function $\left\{v_k^h\right\}_{k=0}^N$

$$\begin{cases} -\frac{v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)}{\tau^2} + A_h^x v_k^h(x) + \frac{\tau^2}{12}\left(A_h^x\right)^2 v_k^h(x) = \theta_k^h(x), \\ \theta_k^h(x) = f^h(t_k,x) + \frac{\tau^2}{12}\left(\frac{f^h(t_{k+1},x)-2f^h(t_k,x)+f^h(t_{k-1},x)}{\tau^2} + A_h^x f^h(t_k,x)\right), 1 \le k \le N-1, x \in \widetilde{\Omega}_h, \\ \left(I - \frac{\tau^2}{12}A_h^x\right)v_1^h(x) - \left(I + \frac{5\tau^2}{12}A_h^x\right)v_0^h(x) - \left(I - \frac{\tau^2}{12}A_h^x\right)v_{l+1}^h(x) \\ +\left(I + \frac{5\tau^2}{12}A_h^x\right)v_l^h(x) = \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\left(\varphi^h(x) - \xi^h(x)\right) - \frac{5\tau^2}{12}\left[f_0^h(x) - f_l^h(x)\right] \\ -\frac{\tau^2}{12}\left[f_1^h(x) - f_{l+1}^h(x)\right] - \frac{\tau^2}{12}\left[f_0'^h - f_l'^h\right]\ ,\ x \in \widetilde{\Omega}_h, \\ \left(I + \frac{5\tau^2}{12}A_h^x\right)v_N^h(x) - \left(I - \frac{\tau^2}{12}A_h^x\right)v_{N-1}^h(x) - \left(I - \frac{\tau^2}{12}A_h^x\right)v_{l+1}^h(x) \\ +\left(I + \frac{5\tau^2}{12}A_h^x\right)v_l^h(x) = \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\left(\psi^h(x) - \xi^h(x)\right) - \frac{5\tau^2}{12}\left[f_N^h(x) - f_l^h(x)\right] \\ -\frac{\tau^2}{12}\left[f_{N-1}^h(x) - f_{l+1}^h\ (x)\right] - \frac{5\tau^2}{12}\left[f_N^h(x) - f_l^h\ (x)\right] + \frac{\tau^3}{12}\left[f_N'^h + f_l'^h\right],\ x \in \widetilde{\Omega}_h. \end{cases} \tag{3.5}$$

Difference problem (3.5) has solution

$$\begin{aligned} v_k^h \ &= \ (I - R^{2N})^{-1}[\left(\left(R^k - R^{2N-k}\right)v_0^h + \left(R^{N-k} - R^{N+k}\right)v_N^h\right) - \left(R^{N-k} - R^{N+k}\right)(I + \tau F_h^x)(2I + \tau F_h^x)^{-1} \\ &\times\left(F_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)\theta_i^h\tau] + (I + \tau F_h^x)(2I + \tau F_h^x)^{-1}\left(F_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|} - R^{k+i}\right)\theta_i^h\tau, \end{aligned} \tag{3.6}$$

where,

$$\begin{aligned} &v_N^h = -\,v_0^h + G_1(I - R^{2N})^{-1}\left(I - \frac{\tau^2}{12}A_h^x\right)\left(R^{N-1} - R^{N+1} + R - R^{2N-1}\right)(I + \tau F_h^x)(2I + \tau F_h^x)^{-1}\left(F_h^x\right)^{-1} \\ &\times\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)\theta_i^h\tau - G_1(I + \tau F_h^x)(2I + \tau F_h^x)^{-1}\left(F_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(I - \frac{\tau^2}{12}A_h^x\right)\left(R^{|1-i|} - R^{1+i}\right. \\ &\left.+R^{|N-1-i|} - R^{N-1+i}\right)\theta_i^h\tau + \tau\left(I + \frac{\tau^2}{12}A_h^x\right)\left(\varphi^h - \psi^h\right)\ - \frac{5\tau^2}{12}\left(f_0^h + f_N^h\right) - \frac{\tau^2}{12}\left(f_1^h + f_{N-1}^h\right) - \frac{\tau^3}{12}\left(f_0'^h - f_N'^h\right). \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&v_0^h = G_2(I-R^{2N})^{-1}\left[\left(I-\tfrac{\tau^2}{12}A_h^x\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A_h^x\right)\left(R^{N-l}-R^{N+l}\right)\right]\\
&\times\left\{G_1(I-R^{2N})^{-1}\left(I-\tfrac{\tau^2}{12}A_h^x\right)\left(R^{N-1}-R^{N+1}+R-R^{2N-1}\right)(I+\tau F_h^x)(2I+\tau F_h^x)^{-1}\right.\\
&\times\left(F_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i\tau-G_1(I+\tau F_h^x)(2I+\tau F_h^x)^{-1}\left(F_h^x\right)^{-1}\\
&\times\sum_{i=1}^{N-1}\left(I-\tfrac{\tau^2}{12}A_h^x\right)\left(R^{|1-i|}-R^{1+i}+R^{|N-1-i|}-R^{N-1+i}\right)\theta_i^h\tau\\
&\left.+\tau\left(I+\tfrac{\tau^2}{12}A_h^x\right)\left(\varphi^h-\psi^h\right)-\tfrac{5\tau^2}{12}\left(f_0^h+f_N^h\right)-\tfrac{\tau^2}{12}\left(f_1^h+f_{N-1}^h\right)-\tfrac{\tau^3}{12}\left(f_0'^h-f_N'^h\right)\right\}+G_2(I-R^{2N})^{-1}\\
&\times\left[\left(I-\tfrac{\tau^2}{12}A_h^x\right)\left(R^{N-1}-R^{N+1}-R^{N-l-1}+R^{N+l+1}\right)+\left(I+\tfrac{5\tau^2}{12}A_h^x\right)\left(R^{N-l}-R^{N+l}\right)\right]\\
&\times(I+\tau F_h^x)(2I+\tau F_h^x)^{-1}\left(F_h^x\right)^{-1}\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)\theta_i^h\tau-G_2(I+\tau F_h^x)(2I+\tau F_h^x)^{-1}\left(F_h^x\right)^{-1}\\
&\times\sum_{i=1}^{N-1}\left[\left(I-\tfrac{\tau^2}{12}A_h^x\right)\left(R^{|1-i|}-R^{1+i}-R^{|l+1-i|}+R^{l+1+i}\right)+\left(I+\tfrac{5\tau^2}{12}A_h^x\right)\left(R^{|l-i|}-R^{l+i}\right)\right]\\
&\times\theta_i^h\tau+\tau G_2\left(I+\tfrac{\tau^2}{12}A_h^x\right)\left(\varphi^h-\xi^h\right)-\tfrac{5\tau^2}{12}G_2\left(f_0^h-f_l^h\right)-\tfrac{\tau^2}{12}G_2\left(f_1^h-f_{l+1}^h\right)-\tfrac{\tau^3}{12}G_2\left(f_0'^h-f_l'^h\right),
\end{aligned}
\tag{3.8}
$$

After solving difference problem (3.5), we define function $p^h(x)$ by formula

$$p^h(x)=A_h^x\left[\xi(x)-\frac{1}{12\tau}\left(2v_{l+4}^h-9v_{l+3}^h+18v_{l+2}^h-11v_{l+1}^h\right)\right], x\in\widetilde{\Omega}_h. \tag{3.9}$$

Finally, the solution of difference problem (3.4) will be calculated by formula

$$u_k^h=v_k^h+\left(A_h^x\right)^{-1}(p^h t_k). \tag{3.10}$$

It is well known that ([34])

$$\min\left\{\ln\frac{1}{\tau},1+\left|\ln\left\|F_h^x\right\|_{C_h\to C_h}\right|\right\}\le M\ln\frac{1}{\tau+|h|}. \tag{3.11}$$

Theorem 3.1. *The solution of difference scheme (3.4) obeys the following stability estimates:*

$$
\begin{aligned}
&\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\le M\,[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}],\\
&\left\|(A^x)^{-1}p^h\right\|_{L_{2h}}\le M\left[\left\|\varphi^h\right\|_{L_{2h}}+\left\|\psi^h\right\|_{L_{2h}}+\left\|\xi^h\right\|_{L_{2h}}+\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_\tau(L_{2h})}\right],
\end{aligned}
$$

where M does not depend on $\tau,\alpha,h,\varphi^h(x),\psi^h(x),\xi^h(x)$, *and* $\left\{f_k^h(x)\right\}_1^{N-1}$.

Theorem 3.2. *For the solution of difference scheme (3.4) the following almost coercive stability estimate is valid:*

$$
\begin{aligned}
&\max_{1\le k\le N-1}\left\|\tfrac{u_{k+1}^h-2u_k^h+u_{k-1}^k}{\tau^2}\right\|_{L_{2h}}+\max_{1\le k\le N-1}\left\|\left(u_k^h\right)_{x_r\overline{x}_r,\,m_r}\right\|_{L_{2h}}+\left\|p^h\right\|_{L_{2h}}\\
&\le M\left[\ln\left(\tfrac{1}{\tau+h}\right)\left\|\left\{f_k^h\right\}_1^N\right\|_{C_\tau(L_{2h})}+\left\|\varphi^h\right\|_{W_{2h}^1}+\left\|\psi^h\right\|_{W_{2h}^1}+\left\|\xi^h\right\|_{W_{2h}^1}\right]
\end{aligned}
$$

is valid, where M is independent of $\tau,\alpha,h,\varphi^h(x),\psi^h(x),\xi^h(x)$ *and* $\left\{f_k^h(x)\right\}_1^{N-1}$.

The proofs of Theorems 3.1 and 3.2 are based on the results of abstract Theorems 2.5 - 2.6, formulas (3.6)–(3.11), symmetry properties of operator A_h^x in L_{2h} and the following theorem.

Theorem 3.3. *([36]) For the solution of the elliptic difference problem*

$$
\begin{cases}
A_h^x u^h(x)=\omega^h(x),\ x\in\widetilde{\Omega}_h,\\
u^h(x)=0,\ x\in S_h,
\end{cases}
$$

the following coercivity inequality holds :

$$\sum_{r=1}^{n}\left\|(u_k^h)_{\bar{x}_r x_r, j_r}\right\|_{L_{2h}} \le M\|\omega^h\|_{L_{2h}},$$

where M does not depend on h and ω^h.

4. Numerical Results

In this section, by using a fourth order approximation, we obtain numerical solution of the inverse problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = \exp(-\pi t)\sin(\pi x) + tp(x), 0 < x < 1, 0 < t < 1, \\ u_t(0,x) = (-\pi + 1)\sin(\pi x), 0 \le x \le 1, \\ u_t(1,x) = (-\pi\exp(-\pi) + 1)\sin(\pi x), 0 \le x \le 1, \\ u_t(\frac{1}{2},x) = \left(-\pi\exp\left(-\frac{\pi}{2}\right) + 1\right)\sin(\pi x),\ 0 \le x \le 1, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1 \quad (\lambda = \frac{1}{2}) \end{cases} \tag{4.1}$$

for the elliptic equation. Note that $u(t,x) = (\exp(-\pi t) + t)\sin(\pi x)$ and $p(x) = \left(\pi^2 + 1\right)\sin(\pi x)$ are the exact solutions of (4.1).

Let N and M be natural numbers, $\tau = \frac{1}{N}$ and $h = \frac{1}{M}$. Introduce the set of grid points

$$[0,1]_\tau \times [0,1]_h = \{(t_k, x_i) : t_k = k\tau, k = 1,\cdots,N-1, x_i = ih,\ j = 1,\cdots,M-1\}.$$

Applying (3.5) , we get, a fourth order of ADS

$$\begin{cases} -\frac{v_i^{k+1}-2v_i^k+v_i^{k-1}}{\tau^2} - \frac{v_{i+1}^k-2v_i^k+v_{i-1}^k}{h^2} + v_i^k - \frac{\tau^2}{12}[-\frac{1}{h^2}\left(-\frac{v_{i+2}^k-2v_{i+1}^k+v_i^k}{h^2} + v_{i+1}^k\right) + \frac{2}{h^2}\left(-\frac{v_{i+1}^k-2v_i^k+v_{i-1}^k}{h^2} + v_i^k\right) \\ -\frac{1}{h^2}\left(\frac{v_i^k-2v_{i-1}^k+v_{i-2}^k}{h^2} + v_{i-1}^k\right) - \frac{v_{i+1}^k-2v_i^k+v_{i-1}^k}{h^2} + v_i^k] = \exp(-\pi t)\sin(\pi x_i)\left[1 + \frac{\tau^2}{12}\left(2\pi^2+1\right)\right] \\ +\left(\pi^2+1\right)\left(1 + \frac{\tau^2}{12}\left(\pi^2+1\right)\right)\sin(\pi x_i),\ k = 1,\cdots,N-1,\ i = 1,\cdots,M-2, \\ v_0^k = v_M^k = 0, v_1^k = \frac{4}{5}v_2^k - \frac{1}{5}v_3^k, v_{M-1}^k = \frac{4}{5}v_{M-2}^k - \frac{1}{5}v_{M-3}^k, k = 0,\cdots,N, \\ -v_i^0 - \frac{5\tau^2}{12}\left[-\frac{v_{i+1}^0-2v_i^0+v_{i-1}^0}{h^2} + v_i^0\right] + v_i^1 - \frac{\tau^2}{12}\left[-\frac{v_{i+1}^1-2v_i^1+v_{i-1}^1}{h^2} + v_i^1\right] - v_i^{l+1} \\ +\frac{\tau^2}{12}\left[-\frac{v_{i+1}^{l+1}-2v_i^{l+1}+v_{i-1}^{l+1}}{h^2} + v_i^{l+1}\right] + v_i^l + \frac{5\tau^2}{12}\left[-\frac{v_{i+1}^l-2v_i^l+v_{i-1}^l}{h^2} + v_i^l\right] = \tau\left[-\pi\left(1 - e^{-\lambda}\right)\right. \\ \left.-\frac{\tau^2}{12}\pi\left(\pi^2+1\right)\left(1-e^{-\lambda}\right) - \frac{5\tau^2}{12}\left(1 - e^{-\pi l\tau}\right) - \frac{\tau^2}{12}\left(e^{-\pi\tau} - e^{-\pi(l+1)\tau}\right) + \frac{\tau^2}{12}\pi\left(1 - e^{-\pi l\tau}\right)\right] \\ \times\sin(\pi x_i),\ i = 1,\cdots,M-1, \\ v_i^N + \frac{5\tau^2}{12}\left[-\frac{v_{i+1}^N-2v_i^N+v_{i-1}^N}{h^2} + v_i^N\right] - v_i^{N-1} + \frac{\tau^2}{12}\left[-\frac{v_{i+1}^{N-1}-2v_i^{N-1}+v_{i-1}^{N-1}}{h^2} + v_i^{N-1}\right] \\ -v_i^{l+1} + \frac{\tau^2}{12}\left[-\frac{v_{i+1}^{l+1}-2v_i^{l+1}+v_{i-1}^{l+1}}{h^2} + v_i^{l+1}\right] + v_i^l + \frac{5\tau^2}{12}\left[-\frac{v_{i+1}^l-2v_i^l+v_{i-1}^l}{h^2} + v_i^l\right] \\ = \tau\left[-\pi\left(e^{-\pi} - e^{-\lambda}\right) - \frac{\tau^2}{12}\pi\left(\pi^2+1\right)\left(e^{-\pi} - e^{-\lambda}\right) - \frac{5\tau^2}{12}\left(e^{-\pi} - e^{-\pi l}\right)\right. \\ \left.-\frac{\tau^2}{12}\left(e^{-(1-\tau)\pi} - e^{-\pi(l+1)}\right) - \frac{\tau^2}{12}\pi\left(e^{-\pi} + e^{-\pi l\tau}\right)\right]\sin(\pi x_i),\ i = 1,\cdots,M-1 \end{cases} \tag{4.2}$$

for the approximate solutions of the corresponding nonlocal boundary value problem. Applying (3.9), and second order of accuracy in x approximation of A, we get formula for p function in grid points

$$p_i = -\frac{1}{12h^2\,\tau}\ \left[\xi_{i+1} - \left(2v_{l+4}^{i+1} - 9v_{l+3}^{i+1} + 18v_{l+2}^{i+1} - 11v_{l+1}^{i+1}\right) - 2\left(\xi_i - \left(2v_{l+4}^{i} - 9v_{l+3}^{i} + 18v_{l+2}^{i} - 11v_{l+1}^{i}\right)\right) + \xi_{i-1}\right.$$
$$\left.-\left(2v_{l+4}^{i-1} - 9v_{l+3}^{i-1} + 18v_{l+2}^{i-1} - 11v_{l+1}^{i-1}\right)\right] + \xi_i - \frac{2v_{l+4}^{i} - 9v_{l+3}^{i} + 18v_{l+2}^{i} - 11v_{l+1}^{i}}{12\tau},\ i = 1,\cdots,M-1.$$

Now, we can rewrite difference scheme (4.2) in the matrix form

$$\begin{aligned} &AV_{i+2} + BV_{i+1} + CV_i + BV_{i-1} + AV_{i-2} = I\theta_i,\ i = 2, \cdots, M-2, \\ &V_0 = \overrightarrow{0}, V_M = \overrightarrow{0}, V_1 = \tfrac{4}{5}V_2 - \tfrac{1}{5}V_3\ , V_{M-1} = \tfrac{4}{5}V_{M-2} - \tfrac{1}{5}V_{M-3}. \end{aligned} \tag{4.3}$$

Here, I is the $(N+1)\times(N+1)$ identity matrix, θ_i, $i = 2, \cdots, M-2$ are $(N+1)\times 1$ column matrices, A, B, C are $(N+1)\times(N+1)$ square matrices,

$$V_s = \begin{bmatrix} v_s^0 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1}, s = i-1, i, i+1; \theta_i = \begin{bmatrix} \theta_i^0 \\ \vdots \\ \theta_i^N \end{bmatrix},$$

$$\begin{aligned} &\theta_i^k = e^{-t_k}\left[1 + \tfrac{\tau^2}{12}\left(2\pi^2+1\right) + \left(\pi^2+1\right)\left(1 + \tfrac{\tau^2}{12}\left(\pi^2+1\right)\right)\right]\sin(x_i), \\ &k = 1, \cdots, N-1,\ i = 1, \cdots, M-1, \\ &\theta_i^0 = \tau\left[-\pi\left(1-e^{-\lambda}\right) - \tfrac{\tau^2}{12}\pi\left(\pi^2+1\right)\left(1-e^{-\lambda}\right) - \tfrac{5\tau^2}{12}\left(1-e^{-\pi l\tau}\right)\right. \\ &\left.-\tfrac{\tau^2}{12}\left(e^{-\pi\tau} - e^{-\pi(l+1)\tau}\right) + \tfrac{\tau^2}{12}\pi\left(1-e^{-\pi l\tau}\right)\right]\sin(\pi x_i), \\ &\theta_i^N = \tau\left[-\pi\left(e^{-\pi} - e^{-\lambda}\right) - \tfrac{\tau^2}{12}\pi\left(\pi^2+1\right)\left(e^{-\pi} - e^{-\lambda}\right) - \tfrac{5\tau^2}{12}\left(e^{-\pi} - e^{-\pi l}\right)\right. \\ &\left.-\tfrac{\tau^2}{12}\left(e^{-(1-\tau)\pi} - e^{-\pi(l+1)}\right) - \tfrac{\tau^3}{12}\pi\left(e^{-\pi} + e^{-\pi l\tau}\right)\right]\sin(\pi x_i), i = 1, \cdots, M-1. \end{aligned}$$

Nonzero elements of these matrices are defined by

$$\begin{aligned} &a_{i,i} = \tfrac{\tau^2}{12h^4},\ b_{i,i} = -\tfrac{1}{h^2} - \tfrac{\tau^2}{3h^4} - \tfrac{\tau^2}{6h^2}, c_{i,i} = 1 + \tfrac{2}{\tau^2} + \tfrac{2}{h^2} + \tfrac{\tau^2}{12}\left(\tfrac{6}{h^4} + \tfrac{4}{h^2} + 1\right), \\ &c_{i-1,i} = c_{i,i+1} = -\tfrac{1}{\tau^2}, i = 2, \ldots, N; c_{1,1} = -1 - \tfrac{5\tau^2}{6h^2} - \tfrac{5\tau^2}{12},\ c_{1,2} = 1 - \tfrac{\tau^2}{6h^2} - \tfrac{\tau^2}{12}, \\ &c_{1,l} = c_{N+1,l} = 1 + \tfrac{5\tau^2}{6h^2} + \tfrac{5\tau^2}{12}, c_{1,l+1} = c_{N+1,l+1} = -1 + \tfrac{\tau^2}{6h^2} + \tfrac{\tau^2}{12}, \\ &c_{N+1,N} = -1 + \tfrac{\tau^2}{6h^2} + \tfrac{\tau^2}{12}, c_{N+1,N+1} = 1 + \tfrac{5\tau^2}{6h^2} + \tfrac{5\tau^2}{12}, \\ &b_{1,1} = \tfrac{5\tau^2}{12h^2},\ b_{1,2} = \tfrac{\tau^2}{12h^2}, b_{1,l} = b_{N+1,l} = -\tfrac{5\tau^2}{12h^2}, \\ &b_{1,l+1} = b_{N+1,l+1} = -\tfrac{\tau^2}{12h^2}, b_{N+1,N} = -\tfrac{\tau^2}{12h^2}, b_{N+1,N+1} = -\tfrac{5\tau^2}{12h^2}. \end{aligned}$$

We search solution of system equation (4.3) by formula

$$V_i = \alpha_i V_{i+1} + \beta_i V_{i+2} + \gamma_i,\ i = M-2, \cdots, 0. \tag{4.4}$$

Here, α_i and β_i $(i = 1, \ldots, M)$ are $(N+1)\times(N+1)$ square matrices, γ_i $(i = 1, \ldots, M)$ are $(N+1)\times 1$ column matrices. For their calculation, we get formulas

$$\begin{aligned} &F_i = \left(C + B\alpha_{i-1} + A\beta_{i-2} + A\alpha_{i-2}\alpha_{i-1}\right), \beta_i = -F_i^{-1}A, \alpha_i = -F_i^{-1}\left(B + B\beta_{i-1} + A\alpha_{i-2}\beta_{i-1}\right), \\ &\gamma_i = -F_i^{-1}\left(I\theta_i - B\gamma_{i-1} - A\alpha_{i-2}\gamma_{i-1} - A\gamma_{i-2}\right), i = 2, \cdots, M-2 \end{aligned}$$

with $\gamma_0 = \gamma_1 = \overrightarrow{0}$, $\alpha_0 = \beta_0$ are $(N+1)\times(N+1)$ zero matrices, $\alpha_1 = -4, \beta_1 = \frac{4}{5}I$.

Vectors V_M and V_{M-1} are defined by formulas

$$V_M = \overrightarrow{0},\ V_{M-1} = \left[(\beta_{M-2} + 5I) - (4I - \alpha_{M-2})\,\alpha_{M-1}\right]^{-1}\left[(4I - \alpha_{M-2})\,\gamma_{M-1} - \gamma_{M-2}\right].$$

Now, we give the results of the numerical analysis using by MATLAB programs. The numerical solutions are recorded for different values of N, M, and u_n^k represents the numerical solutions of these difference schemes at grid points of (t_k, x_n), and p_n represents the numerical solutions at x_n. For their comparison with exact solutions, the errors computed by

$$Ev_M^N = \max_{1\le k\le N-1} (\sum_{n=1}^{M-1} \left|v(t_k, x_n) - v_n^k\right|^2 h)^{\frac{1}{2}},\ Eu_M^N = \max_{1\le k\le N-1} (\sum_{n=1}^{M-1} \left|u(t_k, x_n) - u_n^k\right|^2 h)^{\frac{1}{2}},\ Ep_M = (\sum_{n=1}^{M-1} \left|p(x_n) - p_n\right|^2 h)^{\frac{1}{2}}.$$

Tables 1-3 are constructed for $N = 10, M = 300, N = 20, M = 1200$. Table 1 gives the error between the exact solution and solutions derived by difference schemes for nonlocal problem. Table 2 include error between the exact p solution and approximate p derived by difference schemes. Table 3 gives the error between the exact u solution and solutions derived by difference schemes.

Table 1. Error Ev_M^N

Difference Schemes for v	N=10,M=300	N=20,M=1200
First order ADS	0.1096	0.052929
Second order ADS	0.0217	5.84×10^{-3}
Fourth order ADS	4.61×10^{-5}	3.09×10^{-6}

Table 2. Error Ep_M

Calculation of p	N=10,M=300	N=20,M=1200
First order ADS	0.21335	0.095715
Second order ADS	0.25584	0.065631
Fourth order ADS	1.29×10^{-3}	7.85×10^{-5}

Table 3. Error Eu_M^N

Difference Schemes for u	N=10,M=300	N=20,M=1200
First order ADS	0.1096	0.052929
Second order ADS	0.0234	6.13×10^{-3}
Fourth order ADS	1.15×10^{-4}	7.00×10^{-6}

5. Conclusion

In [29], a fourth order of ADS for inverse elliptic problem with Neumann type overdetermination was presented. Stability estimates for the solution of difference scheme were given without proof.

In the present study, stability, almost coercive stability and coercive stability estimates for the solution of a fourth order ADS for Neumann type overdetermined inverse elliptic problem are established. Then, we study a fourth order approximation of the inverse problem for multidimensional elliptic equation with Neumann type overdetermination and Dirichlet boundary condition. Stability and almost coercive stability estimates for the solution of this difference problem are obtained.

Finally, we illustrate numerical example with the descriptions of numeric realization in a two-dimensional case. The results of computer calculations show that a fourth order of ADS is more accurate comparing with the first and second order of ADS proposed in [27].

Of couse established abstract results can be applied to contstruct stable high order of ADS for multidimensional elliptic equations with mixed boundary conditions.

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