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Quantitative Estimates for GBS Operators of Chlodowsky-Szász Type

Nurhayat Ispir[a]

[a] *Gazi University, Faculty of Sciences, Department of Mathematics, 06500, Ankara, TURKEY*

Abstract. In this study we construct the GBS (Generalized Boolean Sum) operators associated with combination of Chlodowsky and modified Szász operators and estimate the degree of approximation for these operators in terms of the mixed (Bögel) modulus of smoothness. Furthermore, we improve the measure of smoothness by the mixed K-functional.

1. Introduction

In order to make analysis in multidimensional spaces, Karl Bögel introduced the concepts of B-continuous and B-differentiable function in [10] and [11]. In [12], the important theorems of the real functions in one variable are improved using the concepts of B-continuity and B-differentiability. Approximation theory of the well-known Korovkin theorem is developed for B-continuous functions by C. Badea et.al in [2] and [3]. In [2], the authors proved a Korovkin type theorem for approximation of B-continuous functions using the Boolean sum approach (see also [3], [4]).

The approximation properties of the bivariate Bernstein type operators and corresponding generalized Boolean sum operators were investigated in [7], [8], [25], [26] and [27]. In [16], [17], using the concept of A-statistical convergence, Korovkin type theorems were studied for Bögel continuous functions. In the recent years, several researchers have made significant contributions on this topic. We refer the reader to some of the related papers ([9], [18], [19], [20],[28] and [29]).

In [21], the authors introduced a bivariate operator associated with combination of Chlodowsky and modified Szász type operators as follows

$$L_{n,m}(f;x,y)=\sum_{k=0}^{n}\sum_{j=0}^{\infty}P\left(\frac{x}{\alpha_n}\right)Q_j(\beta_m y)\,f\left(\frac{k}{n}\alpha_n,\frac{j}{\gamma_m}\right) \tag{1}$$

for all $n,m\in\mathbb{N}$, $f\in C(I_{\alpha_n})$ with $I_{\alpha_n}=\{(x,y):0\le x\le\alpha_n,0\le y<\infty\}$. Here (α_n) is an unbounded sequence of positive numbers such that $\lim\limits_{n\to\infty}(\alpha_n/n)=0$ and also $(\gamma_m),(\beta_m)$ denote the unbounded sequences of positive numbers such that $\lim_{m\to\infty}\gamma_m^{-1}=0$, $\beta_m/\gamma_m=1+O(1/\gamma_m)$ and $P_{n,k}(x)=C_n^k x(1-x)^{n-k}$, $Q_j(y)=e^{-y}\left(y^j/j!\right)$.

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Email address: nispir@gazi.edu.tr (Nurhayat Ispir)

It is clear that the operator $L_{n,m}: C(I_{\alpha_n}) \to C(I_{\alpha_n})$ is the tensorial product of ${}_xB_n$ and ${}_yS_m$, i.e., $L_{n,m} = {}_xB_n \circ_y S_m$ where

$$ {}_xB_n(f;x,y) = \sum_{k=0}^{n} C_n^k \left(\frac{x}{\alpha_n}\right)^k \left(1-\frac{x}{\alpha_n}\right)^{n-k} f\left(\frac{k\alpha_n}{n}, y\right) $$

and

$$ {}_yS_m(f;x,y) = e^{-\beta_m y} \sum_{j=0}^{\infty} \frac{(\beta_m y)^j}{j!} f\left(x, \frac{j}{\gamma_m}\right). $$

In [21], the authors studied some approximation properties of the $L_{n,m}$ operators given by (1) in a space of continuous functions on compact subset of I_{α_n} and given the degree of this approximation by means of total and partial modulus of continuity. Furthermore, they investigated the weighted approximation properties of the operators $L_{n,m}$ for continuous functions and having polynomial growth on $[0,\infty)\times[0,\infty)$. Recently, some generalizations of bivariate Chlodowsky polynomials were studied in [13], [14]. The bidimensional case of modified Szász operators ${}_yS_m(f;x,y)$ were studied in [30], [23].

The aim of this study is to introduce GBS (Generalized Boolean Sum) operators of bivariate Chlodowsky and Szász type operators and give the order of approximation in terms of Bögel (mixed) modulus of smoothness for B-continous and B-differentiable functions. Moreover, the smoothness properties are improved by the means of the mixed K-functional for B-continous functions.

Notice that, the degree of approximation of the GBS operator associated with Chlodowsky and Szász type operators to a function f is least as good as that of the bivariate Chlodowsky and Szász type operators given in [21]. The results related to Korovkin type theorem and approximation properties will be true in a wider space of functions because every continuous function is Bögel continuous.

2. Preliminaries

Now, let us give some basic definitions and notations which will be used in this study. The details can be found in [12].

Let X and Y be compact real intervals and let $A = X\times Y$. Let $\Delta_{xy}f[x_0,y_0;x,y]$ be mixed difference of f defined by $\Delta_{xy}f[x_0,y_0;x,y] = f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0)$ with $(x,y),(x_0,y_0)\in A$. A function $f: A\to\mathbb{R}$ is called a B-continuous (Bögel continuous) at a point $(x_0,y_0)\in A$ if $\lim_{(x,y)\to(x_0,y_0)}\Delta_{xy}f[x_0,y_0;x,y] = 0$, for any $(x,y)\in A$.

The function $f: A\to\mathbb{R}$ is B-bounded on A if there exists $M>0$ such that $\left|\Delta_{(x,y)}f[t,s;x,y]\right| \le M$ for every $(x,y),(t,s)\in A$. Notice that, if A is a compact subset of $\mathbb{R}^2$ then for each B-continuous function is a B-bounded function on A.

Throughout this paper $B_b(A)$ and $C_b(A)$ denote the spaces of all B-bounded functions and B-continuous functions on A, respectively. As usual $B(A), C(A)$ denote the space of all bounded functions and the space of all continuous (in the usual sense) functions on A endowed with the sup-norm $\|.\|_\infty$. It is known that $C(A)\subset C_b(A)$ ([12], page 52).

Let $L: C_b(A)\to B(A)$ be a linear positive operator. The GBS (Generalized Boolean Sum) operator associated to the operator L is defined by

$$ GL(f;x,y) = L(f(*,y) + f(x,\diamond) - f(*,\diamond);x,y) \tag{2} $$

for every $f\in C_b(A)$ and for each $(x,y)\in A$ with $GL: C_b(A)\to B(A)$ (cf. [2], [3], [7]). Here $f(*,y)$ means that f is considered as function of first variable and analogously $f(x,\diamond)$ means that f is a function respect to second variable.

3. Construction GBS Operator of Chlodowsky-Szász Type

In this section we shall give a generalization of the operator (1) for the B-continuous functions. For this, we shall introduce a GBS operator associated with the bivariate Chlodowsky-Szász type operators and investigate some of its smoothness properties. For $I_{ac} = [0,a] \times [0,c]$, $C_b(I_{ac})$ denotes the space of all B-continuous functions on I_{ac} and let $C(I_{ac})$ be the space of all ordinary continuous functions on I_{ac}.

The operator $L_{n,m}(f;x,y)$ defined by (1) has the following properties:

Lemma ([21]) Let $e_{i,j} = x^i y^j, (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0$, with $i + j \leq 2$ and $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$ be the two dimensional test functions. Then

1. $L_{n,m}(e_{00};x,y) = 1;$
2. $L_{n,m}(e_{1,0};x,y) = x$
3.$L_{n,m}(e_{0,1};x,y) = \frac{\beta_m}{\gamma_m} y$
4. $L_{n,m}(e_{2,0};x,y) = \left(1 - \frac{1}{n}\right) x^2 + \frac{\alpha_n}{n} x,$
5. $L_{n,m}(e_{0,2};x,y) = \frac{\beta_m^2}{\gamma_m^2} y^2 + \frac{\beta_m}{\gamma_m^2} y.;$

Consequently, we have

$$L_{n,m}(t - x;x,y) = 0$$

$$L_{n,m}\left((t - x)^2;x,y\right) = \frac{x(\alpha_n - x)}{n}$$

$$L_{n,m}(s - y;x,y) = \left(\frac{\beta_m}{\gamma_m} - 1\right) y + \frac{1}{2\gamma_m}$$

$$L_{n,m}\left((s - y)^2;x,y\right) = \left(\frac{\beta_m}{\gamma_m} - 1\right)^2 y^2 + \frac{\beta_m}{\gamma_m^2} y.$$

We define the GBS operator associated with the operator $L_{n,m}(f;x,y)$ as follows:

$$GL_{n,m}(f;x,y) \quad : \quad = G^*_{n,m}(f;x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} P_{n,k}\left(\frac{x}{\alpha_n}\right) Q_j(\beta_m y) \times \left[f\left(\frac{k}{n}\alpha_n, y\right) + f\left(x, \frac{j}{\gamma_m}\right) - f\left(\frac{k}{n}\alpha_n, \frac{j}{\gamma_m}\right)\right] \tag{3}$$

where the operator $G^*_{n,m}$ is well-defined from the space $C_b(I_{ac})$ on itself and $f \in C_b(I_{ac})$. It is clear that $G^*_{n,m}$ is a linear positive operator and reproduces linear functions.

4. Degree of Approximation by $G^*_{n,m}$

We begin by recalling the definition of Bögel (mixed) modulus of smoothness of $f \in C_b(I_{ac})$. The Bögel (mixed) modulus of smoothness of $f \in C_b(I_{ac})$ is defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) := \omega_B(f;\delta_1,\delta_2) = \sup\left\{\left|\Delta_{(x,y)} f[t,s;x,y]\right| : |x - t| < \delta_1, \left|y - s\right| < \delta_2\right\}$$

for all $(x,y), (t,s) \in I_{ac}$ and for any $(\delta_1,\delta_2) \in (0,\infty) \times (0,\infty)$ with $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ [4]. This modulus will be useful to evaluating the approximation order of B-continuous functions using GBS operators. The basic properties of ω_B were obtained by Badea et.al in [3] and [4] which are similar to properties of usual modulus of continuity. For example, if $f \in C_b(I_{ac})$ then f is uniform B-continuous on A_{ab} and

$$\lim_{n,m\to\infty} \omega_B(f;\delta_n,\delta_m) = 0$$

as $\delta_n \to 0^+$ and $\delta_m \to 0^+$.

We shall estimate the rate of convergence of the sequences of the operators (3) to $f \in C_b(I_{ac})$ using the Bögel modulus of smoothness. For this estimation, we use the well-known Shisha-Mond theorem for B-continuous functions established by Gonska ([22]), Badea and Cottin ([3]).

Theorem 4.1. *For every $f \in C_b(I_{ac})$, in each point $(x,y) \in I_{ac}$, the operator (3) verifies the following inequality*

$$\left|G^*_{n,m}(f;x,y) - f(x,y)\right| \le M\omega_B(f;,\alpha_n n^{-1}, \gamma_m^{-1}) \tag{4}$$

where M is a constant independent of n, m.

Proof. Using the definition of $\omega_B(f;\delta_1,\delta_2)$ and by the elementary inequality

$$\omega_B(f;\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\,\omega_B(f;\delta_1,\delta_2);\ \lambda_1,\lambda_2 > 0$$

we can write,

$$\begin{aligned}\left|\Delta_{(x,y)} f[t,s;x,y]\right| &\le \omega_B(f;|t-x|,|s-y|)\\ &\le \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_B(f;\delta_1,\delta_2)\end{aligned} \tag{5}$$

for every $(x,y),(t,s) \in I_{ac}$ and for any $\delta_1,\delta_2 > 0$.

From the definition of $\Delta_{(x,y)} f[t,s;x,y]$, we get

$$f(x,s) + f(t,y) - f(t,s) = f(x,y) - \Delta_{(x,y)} f[t,s;x,y].$$

Applying the operator $G^*_{n,m}(f;x,y)$ to this equality we can write

$$G^*_{n,m}(f;x,y) = f(x,y)\,L_{n,m}(f;x,y)(e_{00};x,y) - L_{n,m}(f;x,y)\left(\Delta_{(x,y)} f[t,s;x,y];x,y\right).$$

Since $L_{n,m}(e_{00};x,y) = 1$, considering the inequality (5), using the linearity of the operator $L_{n,m}$ and applying Cauchy-Schwarz inequality we obtain,

$$\begin{aligned}&\left|G^*_{n,m}(f;x,y) - f(x,y)\right| \le L_{n,m}\left(\left|\Delta_{(x,y)} f[t,s;x,y]\right|;x,y\right)\\ &\le \Big(L_{n,m}(e_{00};x,y) + \delta_1^{-1}\sqrt{L_{n,m}\left((t-x)^2;x,y\right)}\\ &+\delta_2^{-1}\sqrt{L_{n,m}\left((s-y)^2;x,y\right)}\\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{L_{n,m}\left((t-x)^2;x,y\right)\left((s-y)^2;x,y\right)}\Big)\,\omega_B(f;\delta_1,\delta_2).\end{aligned}$$

From Lemma, for all $(x,y) \in I_{ac}$, we have the following inequalities;

$$\begin{aligned}L_{n,m}\left((t-x)^2;x,y\right) &= \frac{x(\alpha_n - x)}{n}\\ &\le \frac{\alpha_n}{n}\left(x^2+x\right) \le \frac{\alpha_n}{n}\left(a^2+a\right)\\ &\le M_1\frac{\alpha_n}{n}\end{aligned} \tag{6}$$

where $M_1 = \max\left\{a, a^2\right\}$ and similarly

$$L_{n,m}((s-y)^2; x, y) = O\left(\gamma_m^{-1}\right)(y^2 + y) \leq M_2 \gamma_m^{-1}, \tag{7}$$

$M_2 = \left\{c, c^2\right\}$.Therefore, taking $\delta_1 = \alpha_n n^{-1}$, $\delta_2 = \gamma_m^{-1}$ and $M = \max\{M_1, M_2\}$ we reach the desired inequality (4). □

Corollary 4.2. *If $f \in C_b(I_{ac})$, then*

$$\lim_{n,m\to\infty} G^*_{n,m}(f; x, y) = f(x, y)$$

uniformly on I_{ac}.

Proof. Since $f \in C_b(I_{ac})$, f is uniform B-continuous on I_{ac} and then

$$\lim_{n,m\to\infty} \omega_B\left(f; \alpha_n n^{-1}, \gamma_m^{-1}\right) = 0.$$

Hence, from (4), the desired result is obtained. □

Now, we recall the concept of a B-differentiable function. A function $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ is called a B-differentiable (Bögel differentiable) function at the point $(x_0, y_0) \in A$ if the limit, $\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta_{xy} f[x_0, y_0; x, y]}{(x-x_0)(y-y_0)}$ exists and is finite. The limit is said the B-differential of f at the point (x_0, y_0) and it is denoted by $D_{xy} f(x_0, y_0) := D_B(f; x_0, y_0)$. $D_b(A)$ will be denoted the space of all B-differentiable functions. The partial derivatives are following:

$$D_x f(x_0, y_0) := D_B^{1,0}(f; x_0, y_0) = \lim_{x\to x_0} \frac{\Delta_x f\{[x_0, x]; y_0\}}{x - x_0}$$

and

$$D_y(f; x_0, y_0) := D_B^{0,1}(f; x_0, y_0) = \lim_{y\to y_0} \frac{\Delta_y f\{x_0; [y_0, y]\}}{y - y_0}$$

where $\Delta_x f\{[x_0, x]; y_0\} = f(x, y_0) - f(x_0, y_0)$ and $\Delta_y f\{x_0; [y_0, y]\} = f(x_0, y) - f(x_0, y_0)$. The second order partial derivatives are analogous to the ordinary derivatives. For example, the derivative of $D_x(f; x_0, y_0)$ with respect to the variable y at point (x_0, y_0) is defined by

$$D_y D_x(f; x_0, y_0) := D_B^{0,1} D_B^{1,0}(f; x_0, y_0) = \lim_{y\to y_0} \frac{\Delta_y(D_x f)\{x_0; [y_0, y]\}}{y - y_0}.$$

We would like to give an estimate for the rate of the convergence of the B-differentiable functions by the operator $G^*_{n,m}(f; x, y)$. Notice that, an important theorem estimating the rate of convergence of the B-differentiable functions was proved by O. T. Pop [26].

Theorem 4.3. *Let the function $f \in D_b(I_{ac})$ with $D_B f \in B(I_{ac})$. Then, for each $(x, y) \in I_{ac}$, we have*

$$\left|G^*_{n,m}(f; x, y) - f(x, y)\right| \leq \left[\left\|D_B f\right\|_\infty + M\omega_B\left(D_B f; \alpha_n n^{-1}, \gamma_m^{-1}\right)\right]\left(\alpha_n n^{-1}\gamma_m^{-1}\right)^{-1/2}. \tag{8}$$

Proof. Since $f \in D_b(I_{ac})$, we have the identity $\Delta_{(x,y)} f[t,s;x,y] = (t-x)(s-y) D_B f(\zeta,\eta)$ with $x < \zeta < t, y < \eta < s$ (cf. [12], page 62). It is clear that

$$D_B f(\zeta,\eta) = \Delta_{(x,y)} D_B f(\zeta,\eta) + D_B f(\zeta,y) + D_B f(x,\eta) - D_B f(x,y).$$

Since $D_B f \in B(I_{ac})$, by above relations, we can write

$$\left|L_{n,m}\left(\Delta_{(x,y)} f[t,s;x,y];x,y\right)\right| = \left|L_{n,m}\left((t-x)(s-y) D_B f(\zeta,\eta);x,y\right)\right|$$

$$\leq L_{n,m}\left(|t-x|\left|s-y\right|\left|\Delta_{(x,y)} D_B f(\zeta,\eta)\right|;x,y\right)$$

$$+L_{n,m}\left(|t-x|\left|s-y\right|\left(\left|D_B f(\zeta,y)\right| + \left|D_B f(x,\eta)\right| + \left|D_B f(x,y)\right|\right);x,y\right)$$

$$\leq L_{n,m}\left(|t-x|\left|s-y\right|\omega_B\left(D_B f;|\zeta-x|,\left|\eta-y\right|\right);x,y\right)$$

$$+3\left\|D_B f\right\|_\infty L_{n,m}\left(|t-x|\left|s-y\right|;x,y\right).$$

Since the Bögel modulus of smoothness ω_B is nondecreasing, we have

$$\begin{aligned}\omega_B\left(D_B f;|\zeta-x|,\left|\eta-y\right|\right) &\leq \omega_B\left(D_B f;|t-x|,\left|s-y\right|\right)\\ &\leq \left(1+\delta_1^{-1}|t-x|\right)\left(1+\delta_2^{-1}\left|s-y\right|\right)\omega_B(f;\delta_1,\delta_2).\end{aligned}$$

Substituting in the above inequality, using the linearity of $L_{n,m}$ and applying Cauchy-Schwarz inequality we obtain

$$\left|G^*_{n,m}(f;x,y) - f(x,y)\right| = \left|L_{n,m}\left(\Delta_{(x,y)} f[t,s;x,y];x,y\right)\right|$$

$$\leq 3\left\|D_B f\right\|_\infty \sqrt{L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)}$$

$$+\left[L_{n,m}\left(|t-x|\left|s-y\right|;x,y\right) + \delta_1^{-1} L_{n,m}\left((t-x)^2\left|s-y\right|;x,y\right)\right.$$

$$+\delta_2^{-1} L_{n,m}\left(|t-x|(s-y)^2;x,y\right)$$

$$\left.+ +\delta_1^{-1}\delta_2^{-1} L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)\right]\omega_B(D_B f;\delta_1,\delta_2)$$

$$\leq 3\left\|D_B f\right\|_\infty \sqrt{L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)}$$

$$+\left[\sqrt{L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)} + \delta_1^{-1}\sqrt{L_{n,m}\left((t-x)^4(s-y)^2;x,y\right)}\right.$$

$$+\delta_2^{-1}\sqrt{L_{n,m}\left((t-x)^2(s-y)^4;x,y\right)}$$

$$\left.+\delta_1^{-1}\delta_2^{-1} L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)\right]\omega_B(D_B f;\delta_1,\delta_2).$$

Taking into account (6), (7) and using the following equality, for $(x,y), (t,s) \in A_{ab}$ and $i,j \in \{1,2\}$,

$$L_{n,m}\left((t-x)^{2i}(s-y)^{2j};x,y\right) = L_{n,m}\left((t-x)^{2i};x,y\right)L_{n,m}\left((s-y)^{2j};x,y\right)$$

with $\delta_1 = \alpha_n n^{-1}$ and $\delta_2 = \gamma_m^{-1}$, we reach the desired result (8). □

In order to improve measure of smoothness the mixed K-functional is introduced in [6], [15] (see also [1]). For $f \in C_b(I_{ac})$, we define the mixed K-functional by

$$K_{mixed}(f;t_1,t_2) = \inf_{g_1,g_2,h}\left\{\left\|f-g_1-g_2-h\right\|_\infty + t_1\left\|D_B^{2,0}g_1\right\|_\infty + t_2\left\|D_B^{0,2}g_2\right\|_\infty + t_1t_2\left\|D_B^{2,2}h\right\|_\infty\right\} \tag{9}$$

where $g_1 \in C_B^{2,0}, g_2 \in C_B^{0,2}, h \in C_B^{2,2}$ and, for $0 \le i,j \le 1, C_B^{i,j}$ denotes the space of the function $f \in C_b(I_{ac})$ with continuous mixed partial derivatives $D_B^{p,q}f,\ 0 \le p \le i, 0 \le q \le j$. The concept of mixed K-functional will be useful to estimate the order of approximation by Boolean sum associated to positive linear operators.

We give an estimate for the order of approximation of the sequence $\left\{G_{n,m}^*(f)\right\}$ to the function $f \in C_b(I_{ac})$ in terms of the mixed K-functional given by (9).

Theorem 4.4. *Let $G_{n,m}^*$ be GBS operator of $L_{n,m}$ given by (3). Then,*

$$\left|G_{n,m}^*(f;x,y) - f(x,y)\right| \le CK_{mixed}\left(f;\alpha_n n^{-1},\gamma_m^{-1}\right)$$

for each $f \in C_b(I_{a,c})$, with C is a constant independent of n,m.

Proof. From Taylor formula for the function $g_1 \in C_B^{2,0}(I_{ac})$, we get

$$g_1(t,s) = g_1(x,y) + (t-x)D_B^{1,0}g_1(x,s) + \int_x^t (t-u)D_B^{2,0}g_1(u,s)\,du$$

([12], page 67-69). Since the operator $G_{n,m}^*$ reproduces linear functions

$$G_{n,m}^*(g_1;x,y) = g_1(x,y) + G_{n,m}^*\left(\int_x^t (t-u)D_B^{2,0}g_1(u,s)\,du;x,y\right)$$

and by the formula (3)

$$\begin{aligned}\left|G_{n,m}^*(g_1;x,y) - g_1(x,y)\right| &= \left|L_{n,m}\left(\int_x^t (t-u)\left[D_B^{2,0}g_1(u,y) - D_B^{2,0}g_1(u,s)\right]du;x,y\right)\right| \\ &\le L_{n,m}\left(\left|\int_x^t |t-u|\left|D_B^{2,0}g_1(u,y) - D_B^{2,0}g_1(u,s)\right|du;x,y\right|\right) \\ &\le \left\|D_B^{2,0}g_1\right\|_\infty L_{n,m}\left((t-x)^2;x,y\right) \le M_1\left\|D_B^{2,0}g_1\right\|_\infty \alpha_n n^{-1}.\end{aligned}$$

Similarly, we can write

$$\begin{aligned}\left|G_{n,m}^*(g_2;x,y) - g_2(x,y)\right| &\le \left\|D_B^{0,2}g_2\right\|_\infty L_{n,m}\left((s-y)^2;x,y\right) \\ &\le M_2\left\|D_B^{0,2}g_2\right\|_\infty \gamma_m^{-1}\end{aligned}$$

for $g_2 \in C_B^{0,2}(I_{ac})$. For $h \in C_B^{2,2}(I_{ac})$, considering Taylor formula for B-continuous functions, we have

$$h(t,s) = h(x,y) + (t-x)D_B^{1,0}h(x,y) + (s-y)D_B^{0,1}h(x,y) + (t-x)(s-y)D_B^{1,1}h(x,y)$$

$$+ \int_x^t (t-u)D_B^{2,0}h(u,y)\,du + \int_y^s (s-v)D_B^{0,2}h(x,v)\,dv$$

$$+\int_x^t (s-y)(t-u)\,D_B^{2,1}h(u,y)\,du+\int_y^s (t-x)(s-v)\,D_B^{1,2}h(x,v)\,dv$$
$$+\int_x^t\int_y^s (t-u)(s-v)\,D_B^{2,2}h(u,v)\,dvdu.$$

Taking account into the definition of the operator $G^*_{n,m}$, since $G^*_{n,m}\left((t-x);x,y\right)=0$ and $G^*_{n,m}\left((s-y);x,y\right)=0$, we have

$$\left|G^*_{n,m}(h;x,y)-h(x,y)\right|\le\left|L_{n,m}\left(\int_x^t\int_y^s (t-u)(s-v)\,D_B^{2,2}h(u,v)\,dvdu;x,y\right)\right|$$
$$\le L_{n,m}\left(\left|\int_x^t\int_y^s (t-u)(s-v)\,D_B^{2,2}h(u,v)\,dvdu\right|;x,y\right)$$
$$\le L_{n,m}\left(\int_x^t\int_y^s |t-u||s-v|\left|D_B^{2,2}h(u,v)\right|dvdu;x,y\right)$$
$$\le \frac{1}{4}\left\|D_B^{2,2}h\right\|_\infty L_{n,m}\left((t-x)^2(s-y)^2;x,y\right)$$
$$\le M_3\left\|D_B^{2,2}h\right\|_\infty \alpha_n n^{-1}\gamma_m^{-1}.$$

Therefore, for $f\in C_b(I_{ac})$, we obtain

$$\left|G^*_{n,m}(f;x,y)-f(x,y)\right|\le\left|(f-g_1-g_2-h)(x,y)\right|+\left|\left(g_1-G^*_{n,m}g_1\right)(x,y)\right|$$
$$+\left|\left(g_2-G^*_{n,m}g_2\right)(x,y)\right|+\left|\left(h-G^*_{n,m}h\right)(x,y)\right|+\left|G^*_{n,m}((f-g_1-g_2-h);x,y)\right|$$
$$\le 2\left\|f-g_1-g_2-h\right\|_B+M_1\left\|D_B^{2,0}g_1\right\|_\infty\alpha_n n^{-1}+M_2\left\|D_B^{0,2}g_2\right\|_\infty\gamma_m^{-1}+M_3\left\|D_B^{2,2}h\right\|_\infty\alpha_n n^{-1}\gamma_m^{-1}.$$

Taking the infimum over all $g_1\in C_B^{2,0}, g_2\in C_B^{0,2}, h\in C_B^{2,2}$ with $C=\max\{M_1,M_2,M_3\}$, we reach the result. □

Now, we study the degree of approximation for the operators $G^*_{n,m}(f;x,y)$ by means of the Lipschitz class for B-continuous functions. For $f\in C_b(I_{ac})$, we define the Lipschitz class $Lip_M(\lambda,\mu)$ with $\lambda,\mu\in(0,1]$ as follows

$$Lip_M(\lambda,\mu)=\left\{f\in C_b(A_{ab}):\left|\Delta_{(x,y)}f[t,s;x,y]\right|\le M|t-x|^\lambda\left|s-y\right|^\mu,\text{for }(t,s),(x,y)\in I_{ac},M>0\right\}.$$

Theorem 4.5. *Let $f\in Lip_M(\lambda,\mu)$, then we have*

$$\left|G^*_{n,m}(f;x,y)-f(x,y)\right|\le M\delta_n^{\lambda/2}\delta_m^{\mu/2}$$

where $\delta_n=\left\|{}_xB_n\left((t-x)^2;x\right)\right\|_\infty$, $\delta_m=\left\|{}_yS_m\left((s-y)^2;y\right)\right\|_\infty$ *and* $\lambda,\mu\in(0,1],(x,y)\in I_{ac}$.

Proof. By the definition of the operator $G^*_{n,m}$ and by linearity of the operator $L_{n,m}$, we can write

$$\begin{aligned}G^*_{n,m}(f;x,y)&=L_{n,m}(f(x,s)+f(t,y)-f(t,s))\\&=L_{n,m}\left(f(x,y)-\Delta_{(x,y)}f[t,s;x,y];x,y\right)\\&=f(x,y)L_{n,m}(e_{00};x,y)-L_{n,m}\left(\Delta_{(x,y)}f[t,s;x,y];x,y\right).\end{aligned}$$

By the hypothesis, we get

$$\begin{aligned}\left|G^*_{n,m}(f;x,y)-f(x,y)\right| &\leq L_{n,m}\left(\left|\Delta_{(x,y)}f[t,s;x,y]\right|;x,y\right)\\ &\leq ML_{n,m}\left(|t-x|^{\alpha}\left|s-y\right|^{\beta};x,y\right)\\ &= ML_{n,m}\left(|t-x|^{\alpha};x\right)L_{n,m}\left(\left|s-y\right|^{\beta};y\right).\end{aligned}$$

Now, using the Holder's inequality with $p_1=2/\lambda, q_1=2/(2-\lambda)$ and $p_2=2/\mu, q_2=2/(2-\mu)$, we have

$$\begin{aligned}\left|G^*_{n,m}(f;x,y)-f(x,y)\right| &\leq M\,{}_xB_n\left((t-x)^2;x\right)^{\lambda/2}\,{}_xB_n(e_0;x)^{(2-\lambda)/2}\\ &\quad\times{}_yS_m\left((s-y)^2;y\right)^{\mu/2}\,{}_yS_m(e_0;y)^{(2-\mu)/2}.\end{aligned}$$

In [21], from Lemma 3, we have ${}_xB_n\left((t-x)^2;x\right)=O\left(\alpha_n n^{-1}\right)\left(x^2+x\right)$ and ${}_yS_m\left((s-y)^2;y\right)=O\left(\gamma_m^{-1}\right)(y^2+y)$. Taking $\delta_n(x)={}_xB_n\left((t-x)^2;x\right), \delta_m(y)={}_yS_m\left((s-y)^2;y\right)$,we obtain

$$\left|G^*_{n,m}(f;x,y)-f(x,y)\right|\leq M\delta_n^{\lambda/2}\delta_m^{\mu/2}$$

which implies the degree of approximation for $f\in Lip_M(\lambda,\mu),(x,y)\in I_{ac}$. □

Extensions:

As an applications of Theorems1 and 2, similar results can be investigated for the Stancu, Schurer generalizations of the operator $G^*_{n,m}$ and integral modifications of the operators $L_{n,m}$. We give some examples:

1. The Stancu variant of the operator (1) is defined, for all $f\in C(I_{ac})$ and for every $(x,y)\in I_{ac}$, with $0\leq\rho_i\leq\sigma_i, i=1,2$,

$$L_{n,m}(f;x,y)_{\rho,\sigma}=\sum_{k=0}^{n}\sum_{j=0}^{\infty}P\left(\frac{x}{\alpha_n}\right)Q_j(\beta_m y)f\left(\frac{k+\rho_1}{n+\sigma_1}\alpha_n,\frac{j+\rho_2}{\gamma_m+\sigma_2}\right)$$

where $P\left(\frac{x}{\alpha_n}\right)$ and $Q_j(\beta_m y)$ are Bernstein and Szász element functions given as (1).

Let $GL_{n,m}:C_b(I_{ac})\to C_b(I_{ac})$ be the GBS operator of Stancu type defined by

$$\begin{aligned}&GL_{n,m}(f;x,y)_{\rho,\sigma}=\sum_{k=0}^{n}\sum_{j=0}^{\infty}P\left(\frac{x}{\alpha_n}\right)Q_j(\beta_m y)\\ &\times\left\{f\left(x,\frac{j+\rho_2}{\gamma_m+\sigma_2}\right)+f\left(\frac{k+\rho_1}{n+\sigma_1}\alpha_n,y\right)-f\left(\frac{k+\rho_1}{n+\sigma_1}\alpha_n,\frac{j+\rho_2}{\gamma_m+\sigma_2}\right)\right\}\end{aligned}$$

for all $f\in C_b(I_{ac})$ and all $(x,y)\in I_{ac}$. Theorems 1 and 2 can be obtained for the operator $GL_{n,m}(f;x,y)_{\rho,\sigma}$ with $f\in C_b(I_{ac})$ and all $(x,y)\in I_{ac}$.

2. In [24], the Kantorovich type modification of the operator (1) is introduced and studied by İspir and Büyükyazici. We define the Kantorovich variant of the operator $L_{n,m}$ given by (1) as

$$K_{n,m}(f;x,y)=\frac{n}{\alpha_n}\gamma_m\sum_{k=0}^{n}\sum_{j=0}^{\infty}P_{n,k}\left(\frac{x}{\alpha_n}\right)Q_j(\beta_m y)\int\limits_{j/\gamma_m}^{(j+1)/\gamma_m}\int\limits_{k\alpha_n/n}^{(k+1)\alpha_n/n}f(t,s)\,dtds.$$

Hence the GBS operator of Kantorovich type is defined by

$$GK_{n,m}(f;x,y):=K_{n,m}(f;x,y)\left(f(*,y)+f(x,\diamond)-f(*,\diamond);x.y\right)$$

for all $(x,y)\in I_{ac}$. The approximation properties of these type operators will be studied elsewhere.

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