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A Note on Dimension and Gaps in Digital Geometry

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Abstract. The notion of *gap* is quite important in combinatorial image analysis and it finds several useful applications in fields as CAD and computer graphics. On the other hand, dimension is a fundamental concept in General Topology and it was recently extended to digital objects. In this paper, we show that the dimension of a 2D digital object equipped with an adjacency relation A_α ($\alpha \in \{0,1\}$) can be determined by the number of its gaps besides some other parameters like the number of its pixel, vertices and edges.

1. Introduction

Digital topology studies the topological properties of computer generated images. First introduced by Rosenfeld in the eighties for binary images [19] (see also [18]), the approach has been reformulated from Kovalevsky [20], Eckhardt and Latecki [13] in the contest of general topology.

Digital topology typically deals with finite sets of basic elements (called *pixels*, *voxels* or in general *n-voxels*) of the *digital space*.

The simplest model of digital space is the *grid point model* $\mathbb{Z}^n$ in which every element is represented by a point of the real space $\mathbb{R}^n$ having integer coordinates.

Although in digital topology it is possible to study gray level and multichannel images like color images or multispectral images obtained by satellite radiometer, throughout this paper we uniquely refer to binary (i.e. black and white) images. The associated digital object will consist of the points of $\mathbb{Z}^n$ corresponding to black pixels.

In order to define a topology in the digital space, we assign an adjacency (i.e. a symmetric and irreflexive) relation A.

For any $p \in \mathbb{Z}^n$, the set $\{q \in \mathbb{Z}^n : p\,A\,q\}$ is called *the adjacency set* (or *adjacency neighborhood*) of p and is indicated by $A(p)$. The set $\{p\} \cup A(p)$ is called the (*incident*) *neighborhood* of p and is denoted by $N(p)$.

In every digital space one or more adjacency relations can be defined. In particular in the *digital plane* $\mathbb{Z}^2$ we can define the adjacency relations A_4 and A_8 by means of the following two adjacency neighborhoods of the generic point $p = (x, y)$:

$$A_4(p) = \{(x+1, y), (x-1, y), (x, y+1), (x, y-1)\}$$

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and

$$A_8(p) = A_4(p) \cup \{(x+i, y+j) : i = \pm 1, j = \pm 1\}.$$

Let p, q be two points of $\mathbb{Z}^2$. We say that p and q are 4-adjacent (resp. 8-adjacent) iff $p \neq q$ and $p \in A_8(q)$ (resp. $p \in A_4(q)$). Moreover, p and q are strictly 8-adjacent if they are 8-adjacent but not 4-adjacent.

Another different model for the digital plane is used, the so-called *grid cell model* (also referred as *cellular model*), originally introduced by Alexandroff and Hopf [2].

We associate to a digital set $D \subseteq \mathbb{Z}^2$ a set of certain objects in the Euclidean plane $\mathbb{R}^2$. The first type of objects are *squares* which are closed respect to the standard topology on $\mathbb{R}^2$ and centered at a grid point.

More precisely if $p = (x, y)$ is a point of $\mathbb{Z}^2$ then:

- the square is defined by the set of real points $\left[x - \frac{1}{2}, x + \frac{1}{2}\right] \times \left[y - \frac{1}{2}, y + \frac{1}{2}\right]$ and is called 2-cell. The set of all these 2-cells is denoted by $\mathbb{C}_2^{(2)}$.
- the edges are defined by the four sets of real points $\left[x - \frac{1}{2}, x + \frac{1}{2}\right] \times \{y \pm \frac{1}{2}\}$ and $\left\{x \pm \frac{1}{2}\right\} \times \left[y - \frac{1}{2}, y + \frac{1}{2}\right]$ and are called 1-cells. The set of all 1-cells is denoted by $\mathbb{C}_2^{(1)}$.
- the points are defined by the four singletons $\left\{x \pm \frac{1}{2}, y \pm \frac{1}{2}\right\}$ and are called 0-cells. The set of all 0-cells is denoted by $\mathbb{C}_2^{(0)}$.

Notation 1.1. In the cellular model the pixels of a digital object D are represented by means of the 2-cells. More precisely, every 2-cell $\left[x - \frac{1}{2}, x + \frac{1}{2}\right] \times \left[y - \frac{1}{2}, y + \frac{1}{2}\right]$ is the representant of the pixel centered at $p = (x, y)$.

The digital plane considered as a cellular model is denoted by $\mathbb{C}_2$. It is the union of all 0-, 1- and 2-cells, that is $\mathbb{C}_2 = \bigcup_{i=0}^{2} C_2^{(i)}$.

The use of the cellular model in order to describe the digital plane has several advantages. For instance, it allow us to simplify both the theoretical treatment and many computational applications (see, for example [21, 22]).

We say that two 2-cells e, e' are 0-adjacent (1-adjacent) if and only if $e \neq e'$ and $e \cap e' \in \mathbb{C}_2^{(0)}$ ($e \cap e' \in \mathbb{C}_2^{(1)}$). The relation of 0-adjacency (resp., 1-adjacency) is denoted by A_0 (resp., A_1). Given a 2-cell e, we denote by $A_0(e)$ and (resp. $A_1(e)$) the A_0 (resp. A_1) *adjacent neighborhoods* of e, that is the sets of all 2-cells which are 0-adjacent (1-adjacent) to e.

Adjacencies in grid point and cellular models are strictly connected. In fact, let p_1 and p_2 be the points of $\mathbb{Z}^2$ representing the 2-cells e_1 and e_2, respectively. Then e_1 and e_2 are 1-adjacent (respectively 0-adjacent) if and only if p_1 and p_2 are 4-adjacent (respectively 8-adjacent).

We say that two cells e and e' are *incident* each other if and only if either $e = e'$, or $e \subseteq e'$, or $e' \subseteq e$. If two cells e and e' are incident we write eIe'. The set of all the cells incident to e is denoted by $I(e)$ and called *incidence neighborhood* of the cell e.

We can also consider the digital plane as an *abstract cell complex* $(\mathbb{C}_2, <, dim)$ (see [20, 22, 23]). Here *dim* denotes the function that associates to any cell e its dimension and $<$ is a *bounding relation*, that is antisymmetric, irreflexive, and transitive and such that for every $e, e' \in \mathbb{C}_2$, $e < e'$ if and only if eIe' and $dim(e) < dim(e')$.

Hence $<$ is a partial order on $\mathbb{C}_2$ and the corresponding order topology $\tau(<)$ is called the *grid cell topology*. In this topology the *open sets* are precisely the sets $U \subseteq \mathbb{C}_2$ such that for every $u \in U$ and every $v \in \mathbb{C}_2$ with $u < v$, we have $v \in U$. The grid cell topology is also a T_0-space and an *Alexandroff* space (i.e. the intersection of any family of its open sets of X is an open set). So, for any cell $e \in \mathbb{C}_2$, it is possible to define the *minimal neighborhood* $\eta(e)$ as the intersection of all open sets containing e.

The reader is referred to [4] for more results concerning Alexandroff spaces.

Throughout this paper, we assume that the abstract cell complex $(\mathbb{C}_2, <, dim)$ is equipped with the grid cell topology $\tau(<)$.

2. Dimension in Digital Geometry

Dimension is a fundamental concept in topology. It is a topological invariant [15] and plays an important role in defining and studying properties of basic geometric objects, such as curves and surfaces [16].

In digital topology the notion of dimension has attracted comparatively little attention, unlike some other topological notions (such as connectivity, tunnels, gaps, cavities, genus, and others, see, e.g., [6, 8, 10, 17, 18]).

For a long time, the only available definition of dimension for subsets of discrete spaces was that one given in 1971 by Mylopoulos and Pavlidis [27].

Recently in [9] was noted that such a notion generates some paradoxes which makes it absolutely unsuitable for a general purpose even in the particular case of the digital plane. In the same paper a new consistent definition of dimension was proposed, and some useful properties coherent with the classical ones were proved.

2.1. Review of Mylopoulos-Pavlidis Theory of Dimension

Mylopoulos and Pavlidis [27] proposed a definition of dimension of a subset (finite or infinite) in a generic digital space $\mathbb{C}_n$ with respect to an adjacency relation A_α (see [26] for more details; for its use see also [17]).

Let $\overline{N_\alpha}(e)$ be the union of $N_\alpha(e)$ with all n-cells e' for which there exist $e_1, e_2 \in N_\alpha(e)$ such that a shortest α-path from e_1 to e_2 not passing through e passes through e'. Note that for $n = 2$ we have $\overline{N_1}(e) = \overline{N_0}(e) = N_0(e)$[1)]. We also denote $\overline{A_\alpha}(e) = \overline{N_\alpha}(e) \setminus \{e\}$.

A non-empty set $D \subseteq \mathbb{C}_n$ is called *totally α-disconnected* if, for each n-cell $e \in D$, $A_\alpha(e) \cap D = \emptyset$, that is if no pair of pixels of D is α-adjacent or, equivalently, if every α-connected set of D has cardinality 1.

A subset $D \subseteq \mathbb{C}_n$ is called *linearly α-connected* whenever $|A_\alpha(e) \cap D| \leq 2$ for all $e \in D$ and $|A_\alpha(e) \cap D| > 0$ for at least one $e \in D$.

Definition 2.1. (Mylopoulos-Pavlidis, [27]) Let D be a digital object and A_α an adjacency relation on $\mathbb{C}_n$. The *dimension* $\dim_\alpha(D)$ is defined as follows:

1) $\dim_\alpha(D) = -1$ if and only if $D = \emptyset$,

2) $\dim_\alpha(D) = 0$ if D is a totally α-disconnected nonempty set,

3) $\dim_\alpha(D) = 1$ if D is linearly α-connected,

4) $\dim_\alpha(D) = \max_{e \in D} \dim_\alpha(\overline{A_\alpha}(e) \cap D) + 1$ otherwise.

We call 2×2-*block* a set of four pixels that are pairwise 1-adjacent, 2×1-*block* a set of two 1-adjacent pixels and *L-block* a 2×2-block with one pixel missing. The number of 2×2- (respectively, 2×1, L-) block will be denoted by $\beta_{22}(D)$ (respectively $\beta_{21}(D)$ and $\lambda(D)$) or simply by β_{22}, β_{21} and λ when no confusion about the digital object is possible.

The following characterization of 2-dimensionality in $\mathbb{C}_2$ was given in [17]:

Proposition 2.2. *A digital object $D \subseteq \mathbb{C}_2$ is two dimensional with respect to adjacency relation A_α if and only if:*

- *for $\alpha = 0$, D contains an L-block as a proper subset;*
- *for $\alpha = 1$, D contains a 2×2-block as a proper subset.*

The above proposition suggests us that the 2-dimensionality of a digital object is equivalent to the existence of some L- (resp. 2×2-) block in a digital object. Note, however, that, according to Definition 2.1, an L-block itself, is one-dimensional with respect to A_0.

[1)] For higher dimensions these sets may not coincide; e.g., for $n = 3$, we have $\overline{N_2}(e) = \overline{N_1}(e) = N_1(e)$ and $\overline{N_0}(e) = N_0(e) \neq N_1(e)$, i.e., $\overline{N_2}(e) \neq \overline{N_0}(e)$.

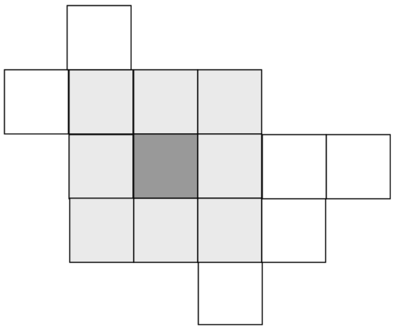

Figure 1: A paradox: an example of a 2D digital object which has dimension 3 according to the Mylopoulos-Pavlidis definition of dimension.

Another "defect" of Definition 2.1, which seems to be even more serious to us, is that a digital object in the 2-dimensional digital space $\mathbb{C}_2$ may have dimension 3! This can be easily seen if we apply Definition 2.1, related to 0-adjacency, to an object that contains a (3×3)-block (see Figure 1).

2.2. *A New Definition for Digital Dimension*

In order to solve the problem of Mylopoulos-Pavlidis definition, in [9] the following new definition of digital dimension was introduced.

Definition 2.3. Let D be a digital object and let the space $\mathbb{Z}_2$ be equipped with an adjacency relation A_α, $\alpha \in \{0, 1\}$. The *dimension* of D *relative to the* α-adjacency is denoted by $\dim_\alpha(D)$ and defined as follows:

1) $\dim_\alpha(D) = -1$ if $D = \emptyset$,

2) $\dim_\alpha(D) = 0$ if D is totally α-disconnected,

3) $\dim_\alpha(D) = 1$ if $\alpha = 0$, D is not totally α-disconnected and does not contain any L-block; or $\alpha = 1$, D is not totally α-disconnected and does not contain any 2×2-block,

4) $\dim_\alpha(D) = 2$ otherwise (more precisely, if $\alpha = 0$ and D contains at least one L-block or $\alpha = 1$ and D contains at least one 2×2-block).

Notation 2.4. Note that in Definition 2.3, points 3) and 4) can be reformulated by using mathematical morphology [12, 29] (Remember that in these cases D is not totally α-disconnected). More precisely, we can define $\dim(D) = 1$ iff $\alpha = 1$ and $\epsilon_B(D) = D \ominus B = \emptyset$ where the structuring element B is a 2×2-block, or $\alpha = 0$ and $\bigcup_{i=1}^{4} \epsilon_{L_i}(D) = \bigcup_{i=1}^{4} D \ominus L_i = \emptyset$ where L_i, $i = 1, \dots, 4$, represents all possible L-blocks. Furthermore, $\dim(D) = 2$ iff $\alpha = 1$ and $\epsilon_B(D) \neq \emptyset$, or $\alpha = 0$ and $\bigcup_{i=1}^{4} \epsilon_{L_i}(D) \neq \emptyset$.

In order to give a sort of "local" characterization of dimension, we now define the dimension of a point of a digital object D.

Definition 2.5. Let p be a 2-cell of a non-empty digital object D. The *local dimension* of p *within* D with respect to A_0 is denoted by $\dim_0(p, D)$ and defined as follows:

1) $\dim_0(p, D) = 0$ if $A_0(p) \cap D = \emptyset$;

2) $\dim_0(p, D) = 1$ if $A_0(p) \cap D$ is totally 0-disconnected;

3) $\dim_0(p, D) = 2$ otherwise (i.e., if $A_0(p) \cap D$ is not totally 0-disconnected).

Definition 2.6. Let D be a nonempty digital object and let $p \in D$. The *local dimension* of p *within* D with respect to A_1 is the nonnegative integer $\dim_1(p, D) = \dim_0(A_0(p) \cap D)$.

The following properties were proved in [9].

Proposition 2.7. *Let D be a nonempty digital object and $p \in D$. Then the following properties hold:*

1) $\dim_0(p, D) = 2$ *(resp.* $\dim_1(p, D) = 2$*) iff p belongs to an L-block (resp.* 2×2*-block) in D.*

2) $\dim_\alpha(D) = \max\{\dim_\alpha(p, D) : p \in D\}$*, where* $\alpha = \{0, 1\}$.

3) If $p \in E \subseteq D$ *then* $\dim_\alpha(p, E) \leq \dim_\alpha(p, D)$*, for* $\alpha = \{0, 1\}$.

4) If $E \subseteq D$ *then* $\dim_\alpha(E) \leq \dim_\alpha(D)$*, where* $\alpha = \{0, 1\}$.

5) If D_1 and D_2 are two disjoint digital objects, then $\dim_\alpha(D_1 \cup D_2) = \max(\dim_\alpha(D_1), \dim_\alpha(D_2))$*, where* $\alpha \in \{0, 1\}$.

2.3. Relations Between Dimension and Euler Characteristic

In combinatorial topology, Euler characteristic is a fundamental theoretic concept and basic topological invariant. Recall that, given a subset D of the abstract cell complex $(\mathbb{C}_2, <, \dim)$, its Euler characteristic is the number

$$\chi(D) = c_0 - c_1 + c_2, \tag{1}$$

where $c_i = |D \cap \mathbb{C}_2^{(i)}|$ is the number of the i-dimensional cells of D (with $i = 0, 1, 2$), that is the number of vertices, edges, and faces of the digital object D, respectively.

In this section we establish relations between dimension of digital objects and their Euler characteristic. For this purpose we introduce the following definition.

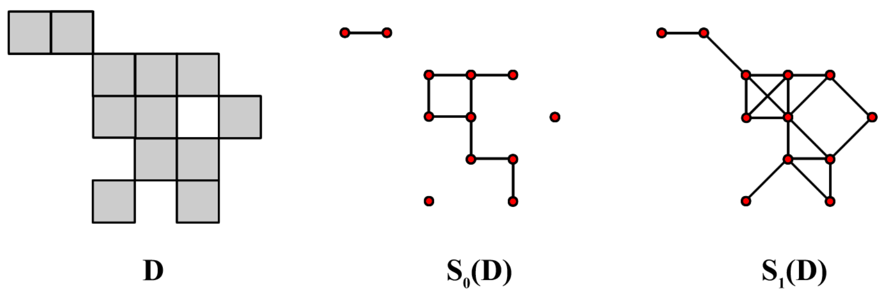

Figure 2: The skeletons $S_0(D)$ and $S_1(D)$ respect to A_0 and A_1 adjacency of a same digital object D.

Definition 2.8. Let D be a non-empty object of the digital space $\mathbb{C}_2$ equipped with an adjacency relation A_α (with $\alpha \in \{0, 1\}$). We call *skeleton* of D respect to A_α the graph $S_\alpha(D) = (\mathcal{V}, \mathcal{E})$ in which the set of vertices is $\mathcal{V} = D \cap \mathbb{C}_2^{(2)}$ and the set of edges is $\mathcal{E} = \{(e, e') : e, e' \in \mathbb{C}_2^{(2)}$ and $eA_\alpha e'\}$, that is the vertices are all the 2-cells of D and a couple of vertices forms an edge iff they are α-adjacent (see Figure 2 for a graphical representation of the skeletons of a digital object).

In what follows, we will characterize dimensionality in $\mathbb{C}_2$ with respect to A_1 adjacency, the characterization with respect to A_0 adjacency being similar. Because of Proposition 2.7(5), it is enough to consider the case of connected digital objects. We have the following theorem.

Theorem 2.9. *Let D be a 1-connected digital object whose skeleton $S_1(D) = (\mathcal{V}, \mathcal{E})$ has $|\mathcal{V}| = c_2$ vertices and $|\mathcal{E}| = m$ edges. Then the following holds:*

1) $\dim_1(D) = -1$, *if* $c_2 = 0$

2) $\dim_1(D) = 0$, *if* $c_2 \neq 0$ *and* $m = 0$

3) $\dim_1(D) = 1$, *if* $c_2 > m > 0$

4) *If* $m = c_2 > 0$, *then*

  (a) $\dim_1(D) = 1$ *if* $\chi(D) = 0$
  (b) $\dim_1(D) = 2$ *if* $\chi(D) > 0$

5) *If* $m > c_2 > 0$,

  (a) $\dim_1(D) = 1$ *if* $\chi(D) < 0$
  (b) $\dim_1(D) = 2$ *if* $\chi(D) \geq 0$.

Notation 2.10. Let us note that in the above theorem the cases "$m < c_2$" and "$c_2 = m$ and $\chi(D) < 0$" seem to be missing. However it can be proved that they are non-admissible. In fact the graph $S_1(D)$ results connected and, because it is well know that a connected graph with c_2 vertices and m edges has a unique cycle of length greater than 3 if and only if $c_2 = m$ (see [5]).

Throughout the rest of the paper, every digital object $D \subseteq \mathbb{Z}^2$ will be also considered as a planar graph $\mathcal{G}$ and we will denote by $V(D)$, $L(D)$, $C(D)$ and $H(D)$ (or simply by V, L, C and H when no confusion arises) the number of its vertices, edges, connected components and holes, respectively.

Such a point of view, allow us to obtain a lighter (and hence more efficient for practical application) and simpler version of Theorem 2.9 which does not involve Euler characteristic.

Corollary 2.11. *Let D be a* 1*-connected digital object whose skeleton $S(D)$ has c_2 vertices and m edges. Then the following implications hold:*

1) *If* $c_2 > m$ *then* $\dim_1(D) = 1$,

2) *If* $c_2 = m$ *and* $H < 1$ *then* $\dim_1(D) = 2$,

3) *If* $c_2 = m$ *and* $H = 1$ *then* $\dim_1(D) = 1$,

4) *If* $m > c_2$ *and* $H \leq 1$ *then* $\dim_1(D) = 2$,

5) *If* $m > c_2$ *and* $H > 1$ *then* $\dim_1(D) = 1$.

Theorem 2.12. *Let D be a* 0*-connected digital object which skeleton $\mathcal{S}(D)$ has c_2 vertices and m edges. Then the following holds:*

1) *If* $P = c_2 = 0$ *then* $dim_0(D) = -1$;

2) *If* $c_2 \neq 0$ *and* $m = 0$ *then* $dim_0(D) = 0$;

3) *If* $c_2 > m > 0$ *then* $dim_0(D) = 1$;

4) *If* $m = c_2$ *and*

- $L - V < P$ *then* $dim_0(D) = 2$;
- $L - V = P$ *then* $dim_0(D) = 1$.

5) *If* $m > c_2 > 0$ *and* $L - V \leq P$ *then* $dim_0(D) = 2$.

Notation 2.13. In Theorem 2.12, almost all the points of Theorem 2.9 have been reformulated for 0-connected digital objects but item (a) of point 5. In fact there exist some 0-connected objects with $m > c_2 > 0$ and $\chi(D) < 0$ that have not 0-dimension 1 (see for example Fig. 3).

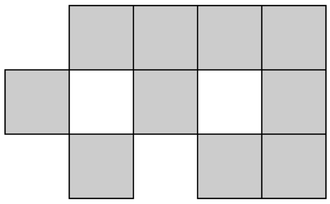

Figure 3: A 0-connected object D with $m > c_2 > 0$, $\chi(D) < 0$ and $dim_0(D) = 2$

In Section 4, we obtain two more general results than Theorem 2.9, 2.12 and Corollary 2.11 which hold both for 0- and 1-adjacency and remove the singularity of the last part of Theorem 2.12. Such properties are obtained by means of the notion of gap of a digital object.

3. Notion of Gap in Digital Geometry

Roughly speaking, a *gap* of a 2-D binary digital object (i.e. a set of pixels) is a location of the object that can be locally penetrated by some discrete path (usually called *ray*). This concept is the discrete equivalent of the topological notion of tunnel, and it is very important in digital geometry. In fact, it finds several useful applications in fields as computer aided design (CAD) and computer graphics where it is relevant to know whether an apparently "solid" surface can have some "unreal" (or "immaterial") holes.

Several definitions of gap are available in the literature (see, e.g., [1, 3]). In what follows, we will refer to the following one, that was introduced in [8] and that fits better our purposes.

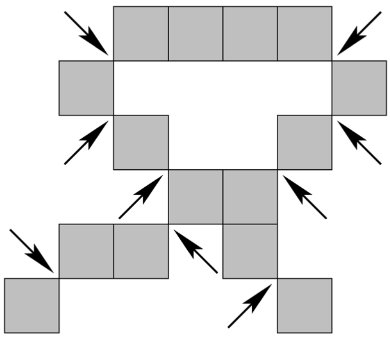

Figure 4: Gaps in a 2-dimensional binary picture.

Definition 3.1. Let v be a 0-cell of a digital object D in $\mathbb{C}^2$. We say that D has a *gap* at v if there are two 2-cells p_1 and p_2, such that:

1) $v \in p_1 \cap p_2$

2) $p_1 \in A_0(p_2) \setminus A_1(p_2)$ (i.e. p_1 and p_2 are strictly 0-adjacent), and

3) $A_1(p_1) \cap A_1(p_2) \cap D = \emptyset$.

We denote by $G(D)$ (or simply by G when no confusion arises) the number of gaps of a digital object D. Figure 4 illustrates the notion of gap.

Notation 3.2. Let us note that in the above definition Condition 1 has mainly the role to locate the gap, but it is essentially contained in Condition 2.

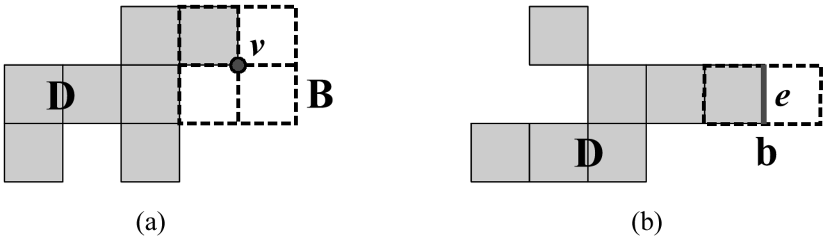

Figure 5: (a) v is a free vertex of the digital object D: the 2×2-block B is not complectly contained in D. (b) l is a free edge of D: the 2×1-block b is not complectly contained in D.

It is also possible to reformulate Definition 3.1 using the bounding relation < defined in Section 1 for the abstract cell complex.

Definition 3.3. Let x be a 0-cell of a digital object D in $(\mathbb{C}^2, I, dim)$. We say that D has a gap in x if there are two 2 cells e' and e'' such that

1) $x < e'$ and $x < e''$ and

2) for any $e \in D$, if $e < e'$ or $e < e''$ than $x \nless e$.

Let D be a digital object with P pixels, V vertices, L edges, C connected components, H holes and β_{22} 2×2-blocks.
In [6], it was proved that the number of gaps of D is given by

$$G = V - 2(P + C - H) + \beta_{22}. \quad (2)$$

In [8], another formula to express the number of gaps for a digital object was found by using the notions of free vertices and free edges. More precisely, let v be a vertex and l an edge of D. We say that v (respectively l) is *free* if the 2×2-block centered in v (respectively 2×1-block centered in l) is not completely contained in D (See Figure 5).
We denote by V^* (resp. L^*) the number of free vertices (resp. edges) and by V' (resp. L') the number of non-free vertices (resp. edges).
We have the following characterization of free vertices and edges.

Proposition 3.4. *A vertex (resp. edge) e of a digital object D is free iff it belong to its boundary $bd(D)$.*

Proof. Let $e \in D$ be free, and let us suppose, by contradiction, that $e \notin bd(D)$. Then $e \in int(D)$. So there is a neighborhood of e, $N_\alpha(e)$ $\alpha = 0, 1$, such that $N_\alpha(e) \subseteq D$. But $\eta(e) \subseteq N_\alpha(e) \subseteq D$, i.e $\eta(e) \subseteq D$. A contradiction.
Conversely, let $e \in bd(D)$. Then for all $N_\alpha(e)$, we have $N_\alpha(e) \cap D \neq \emptyset$ and $N_\alpha(e) \cap (\mathbb{C}_2 \setminus D) \neq \emptyset$. In particular, we have $\eta(e) \cap D \neq \emptyset$. So $\eta(e) \cap \mathbb{C}_2 \setminus D = \emptyset$ and $\eta(e) \nsubseteq D$. □

It is also possible to prove the following topological characterization of free elements.

Proposition 3.5. *Let e be a* 0- *or* 1-*cell of a digital object D. Then, e is free iff $\eta(e) \nsubseteq D$.*

Notation 3.6. Let us note that a vertex v is non-free if and only if it is the unique common vertex of the four pixels in a 2×2-block centered at v. Similarly, an edge l is non-free if and only if it is the common edge of the two pixels in a 2×1-block. Moreover, since it is evident that in every 2×2-block (resp. 2×1-block) there is exactly one non free vertex (resp. edge), we have that $V' = \beta_{22}$ and $L' = \beta_{21}$. It is also clear that $V = V^* + V'$ and $L = L^* + L'$.

More recently, in [24] and [25] two formulas which express, respectively the number of 1-gaps of a generic 3D object of dimension $\alpha = 1, 2$ and the number of $(n-2)$-gaps of a generic digital n-object, by means of a few simple intrinsic parameters of the object itself were found.

Thanks to Proposition 3.4, the boundary $bd(D)$ of a digital object D can be seen as a graph with V^* vertices and L^* edges.

Proposition 3.7. *Let D be a digital object. If D has a gap on a vertices v then* $v \in bd(D)$.

Proof. Let v a vertex of D having a gap. By contradiction, let us suppose that v does not belong to the border $bd(D)$ of D. Then by Proposition 3.4, v belongs to the interior of D. But in this way, v should be a non-free vertex and, by Definition 3.1, D should not have a gap on v. □

Example 3.8. Recall that a closed digital α-curve C ($\alpha = 0$ or 1) is an α-connected set of pixels such that every its pixel has exactly two α-adjacent neighborhoods in C. It is easy to see that all vertices of C are free.

In [8], it was proved (using combinatorial consideration) that the number of gaps of a digital object is also given by the following formula:

$$G = L^* - V^*. \tag{3}$$

Here we give a simpler proof using graph theory.

Proposition 3.9. *Let D be a digital object, and let us denote by* V^* *and* L^* *the number of free vertices and edges, respectively. Then* $G = L^* - V^*$.

Proof. Let $bd(D)$ be the boundary of D. By Proposition 3.4, it forms a graph with V^* vertices and L^* edges. We distinguish two cases depending on D is connected or not.

First, let us suppose that D is connected and that it has no hole. Then $bd(D)$ is an Eulerian graph and every vertex has degree two but the ones in which there is a gap. Such vertices have degree four. Let us denote by Γ the set of all vertices of D having a gap. Obviously, by Property 3.7, $\Gamma \subseteq bd(D)$. Since it is well known that in any graph the sum of the degrees of the vertices is twice the number of the edges, we have that $2L^* = \sum_{v \in bd(D)} d(v) = \sum_{v \in \Gamma} d(v) + \sum_{v \in bd(D) \setminus \Gamma} d(v) = 4|\Gamma| + 2|bd(D) \setminus \Gamma| = 4G + 2(V^* - G) = 2V^* + 2G$ and hence $G = L^* - V^*$.

In the other case, i.e. if D is connected with H holes, the boundary $bd(D)$ is composed by $2H+1$ Eulerian graphs in which the previous case can be applied separately to any graph.

Finally, if D is not connected and has C connected component, the proof easily follows from the previous case by induction over the number of connected components C. □

In the next paragraph we prove the equivalence of the formulas (2) and (3) by directly showing that can be obtained each other. This will make easier and more elegant the proofs given in [6] e [8].

4. The Main Result

Let us recall a well-known generalization of the Euler's formula frequently used in Digital Geometry. Let D be a digital object with P pixels, V vertices, L edges, C connected components and H holes. Considering D as a planar graph $\mathcal{G}$ we can think to apply Euler's formula to D, obtaining:

$$V - L + P = C - H, \tag{4}$$

where P is the number of pixels of the digital object D. We refer to such expression as the *Euler's formula for 2D objects*.

The equivalence between the formulas (2) and (3) is based on two lemmas we are going to prove and that need a special graphical representation called the *pixel language*. In this notation, we refer to basic configurations of 3×3 pixels with a key pixel whose neighborhood is studied. Such a central pixel is denoted by a black square ■ (and sometimes labelled with a letter) and the following graphical rules are used for expressing the existence or not of its adjacent pixels.

- Neighborhoods of the key pixel that MUST exist in the considered configuration are represented by a gray square ■.
- Pixels that MAY OR MAY NOT exist in the configuration will be drawn with a dashed square ⬚. Any subset of such pixels (in particular, no one or all of them) may belong to the configuration or may be missing.
- Grid positions that CANNOT contain any pixels from the configuration will be marked by a cross ╳.

Lemma 4.1. *For every digital object D with P pixels, L edges and β_{21} 2×1-block, we have:*

$$L = 4P - \beta_{21}. \tag{5}$$

Proof. We proceed by induction over P. If $P = 1$, it trivially follows $L = 4$, $\beta_{21} = 0$ and hence the identity holds. Now, suppose that formula (5) holds for any digital object with a fixed number of pixels P and consider a generic digital object D with $P+1$ pixels. Chosen a fixed pixel p of D, $\widetilde{D} = D \setminus \{p\}$ is a digital object with P pixels. Therefore we can use the inductive hypothesis over $\widetilde{D}$ and consider D as a digital object obtained by adding the pixel p to $\widetilde{D}$. Let denote by $\widetilde{V}$, $\widetilde{L}$ and $\widetilde{\beta_{21}}$ the number of vertices, edges and 2×1-blocks of $\widetilde{D}$, respectively. Using the pixel language and up to symmetries, we have only five cases (see Figure 6) according to the insertion of p to $\widetilde{D}$ creates a number $\mu \in \{0, 1, 2, 3, 4\}$ of new 2×1-blocks in D. Since, for each such a case, it results $\beta_{21} = \widetilde{\beta_{21}} + \mu$ and $L = \widetilde{L} + 4 - \mu$, we have $4(P+1) - \beta_{21} = 4P + 4 - (\widetilde{\beta_{21}} + \mu) = 4P - \widetilde{\beta_{21}} + 4 - \mu = L$ which proves the inductive step. □

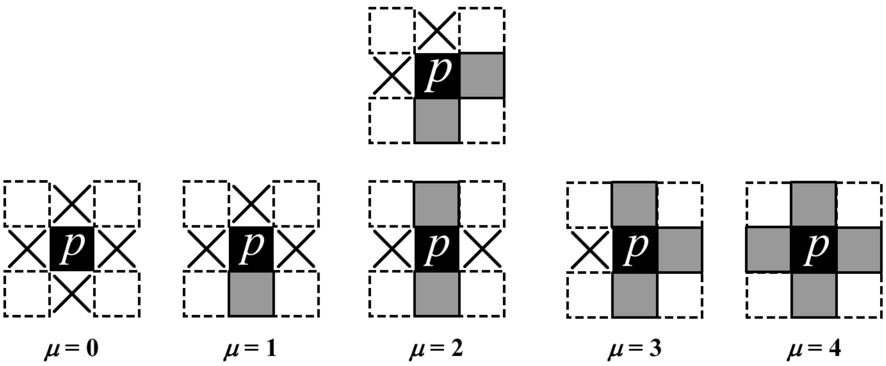

Figure 6: The 5 cases corresponding to 6 families of configurations (expressed in the pixel language) of the object D with $P + 1$ pixels depending on the number μ of new 2×1-blocks in D created inserting the central pixel x in $A_0(p)$.

Lemma 4.2. *For every digital object D with P pixels, V vertices, β_{21} 2×1-block, C connected component, and H holes, we have that*

$$V = 3P + C - H - \beta_{21}. \tag{6}$$

Proof. Suppose, by contradiction, that there exists a digital object D such that $V \neq 3P + C - H - \beta_{21}$. By formula (4) we have $-H = V - L + P - C$ and hence $V \neq 3P + C + V - L + P - C - \beta_{21}$, that is $L \neq 4P - \beta_{21}$ which contradicts Lemma 4.1. □

Now, we are finally able to prove the equivalence between formulas (2) and (3) announced before.

Theorem 4.3. *Let D be a digital object with P pixels, V vertices, G gaps, C connected components, H holes, β_{22} 2×2-blocks and having V^* free vertices and L^* free edges. Then the formula $G = L^* - V^*$ is equivalent to $G = V - 2(P + C - H) + \beta_{22}$.*

Proof. Let $G = L^* - V^*$ hold and suppose – by contradiction – that $G \neq V - 2(P + C - H) + \beta_{22}$. Hence, using formula (6), we obtain $G \neq P - C + H - \beta_{21} + \beta_{22}$. Since, by formula (4) we know that $L - V = P - C + H$, it follows that $G \neq L - V - \beta_{21} + \beta_{22}$, that is $G \neq L^* - V^*$. A contradiction.

Conversely, let us assume that $G = V - 2(P + C - H) + \beta_{22}$ and suppose, by contradiction, that $G \neq L^* - V^*$ or, equivalently,

$$G \neq L - V - \beta_{21} + \beta_{22}. \tag{7}$$

Then, replacing $L - V = P + H - C$, we obtain $G \neq P + H - C - \beta_{21} + \beta_{22}$. By equation (6), we have $\beta_{21} = 3P - V + C - H$ and replacing it in the previous expression, it follows that $G \neq P + H - C + V - 3P - C + H + \beta_{22}$. Hence $G \neq V - 2(P + C - H) + \beta_{22}$ which contradicts our hypothesis and completes our proof. □

Although the notions of gap and dimension of a digital object D seem to be unrelated, as matter of fact a connection exists as the the following theorem proves.

In order to obtain such a result, we will use the notion of *skeleton* (see Definition 2.8).

Theorem 4.4. *Let D be an object of the digital space $\mathbb{C}_2$, equipped with an adjacency relation A_α (with $\alpha \in \{0, 1\}$) but considered as 0-object and having P pixels, V vertices, L edges, and G gaps.*

1) If $P = c_2 = 0$ then $dim_\alpha(D) = -1$;

2) If $c_2 \neq 0$ and $m = 0$ then $dim_\alpha(D) = 0$;

3) If $V - L + m + \alpha G = 0$ then $dim_\alpha(D) = 1$;

4) If $V - L + m + \alpha G > 0$ then $dim_\alpha(D) = 2$

where c_2 and m are respectively the number of vertices and edges of the skeleton $S(D)$ associated to D respect to A_α.

Proof. 1) and 2) are easy to check.
3) Let us suppose firstly that D is equipped with the 1-adjacency and suppose $V - L + m + G = 0$. Hence, $G = L - V - m = L^* + L' - V^* - V' - m$. Because of 1-adjacency, every edge of the skeleton $S_1(D)$ corresponds to a non free edge of the digital object D. So, we have that $m = L'$ and that $G = L^* - V^* - V'$. By Remark 3.6, we have $G = L^* - V^* - \beta_{22}$ and being $G = L^* - V^*$, we obtain that $\beta_{22} = 0$, i.e. $dim_1(D) = 1$.
Now, let D be equipped with the 0-adjacency and suppose that $V - L + m = 0$.
Let us denote by β_{21} the number of 2×1-blocks, by β_{22} the number of 2×2-blocks, by λ the number of L-blocks, and by ξ the number of 0-*tandem* (that is a pair of strictly 0-adjacent pixels) of D. Because of 0-adjacency, every edge of $S_0(D)$ derives either from a 2×1-block or from a 0-tandem. So, we have $m = \beta_{21} + \xi$. In D there are exactly one 0-tandem for every gap, one 0-tandem for every L-block and two 0-tandems for every 2×2-block and all such cases are mutually exclusive. So, it results $\xi = G + \lambda + 2\beta_{22}$ and hence $m = \beta_{21} + G + \lambda + 2\beta_{22}$.

Replacing the latter expression into $V - L + m = 0$ and using relations from Remark 3.6, with some simple computations, we obtain the following system:

$$\begin{cases} \lambda + 3\beta_{22} = 0 \\ \lambda \geq 0 \\ \beta_{22} \geq 0 \end{cases}$$

which admits solutions if and only if $(\lambda, \beta_{22}) = (0, 0)$. This implies that $dim_0(D) = 1$ and completely proves point 3).
4) Let D equipped with the 1-adjacency and suppose $v - L + m + G > 0$. Using the same method of the corresponding part in point 3), we have $\beta_{22} > 0$, i.e. $dim_1(D) = 2$.

Finally, let be D equipped with the 0-adjacency and suppose $V - L + m > 0$. Similarly to the second part of point 3) we obtain the system

$$\begin{cases} L + 3\beta_{22} > 0 \\ L \geq 0 \\ \beta_{22} \geq 0 \end{cases}$$

which admits solutions if and only if $L \neq 0$ or $\beta_{22} \neq 0$. This implies that $dim_0(D) = 2$ and completes our proof. □

Euler's formula for 2-D object (4) allow us to write the statement of Theorem 4.4 in the following way.

Corollary 4.5. *Let D be a digital object equipped with α-adjacency ($\alpha = 0, 1$). The following holds:*

1. *If $P = c_2 = 0$ then $dim_1(D) = -1$;*
2. *If $c_2 \neq 0$ and $m = 0$ then $dim_1(D) = 0$;*
3. *if $C - H + m - P + \alpha G = 0$ then $dim_\alpha(D) = 1$;*
4. *if $C - H + m - P + \alpha G > 0$ then $dim_\alpha(D) = 2$.*

5. Conclusion and Perspectives

The present paper dealt with the study of existing relationship between the notions of gap and that of discrete digital dimension $\dim_\alpha(D)$ of a digital object D in the digital plane $\mathbb{Z}_2$ with the cellular model $\mathbb{C}_2$ and equipped with an adjacency relation A_α ($\alpha \in \{0, 1\}$). More specifically, we proved that the dimension $dim_\alpha(D)$ of such an object is completely determined by the numbers P of its pixels, V of its vertices, L of its edges, and G of its gaps. This result requires some considerations on the number of 2×2-blocks and implies a previous classification of all the possibile configurations of 2×1-blocks that we create in any 0-adjacency neighborhood $A_0(p)$ of the object D every time we add the central pixel p (see Lemma 4.1). However, in dimension 3 a similar classification involves simultaneously at least $2 \times 1 \times 1$-, $2 \times 2 \times 1$- and $2 \times 2 \times 2$-blocks with a larger number of cases that we need to take under consideration and a more complex analysis. Obviously, such approach, which is already difficult to manage for $3D$ objects, is impossible to use for a generic dimension n and, for such a reason, it seems absolutely necessary to adopt some different combinatorial techniques in order to find more general results.

Another paper, with a detailed discussion of the problem above and an attempt to extend the relationship between the number of gaps and the dimension of digital objects in dimension 3 and higher, is currently in preparation and should be available for publication in the near future.

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