



On the Steady Solutions of Fractional Reaction-Diffusion Equations

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Abstract. In this paper, we study the existence of weak solutions for stationary fractional reaction-diffusion equations with Riemann-Liouville boundary conditions. An appropriate fractional Hilbert space is introduced and a compact embedding theorem demonstrated. Existence results are established using generalized Weierstrass theorem and relatively simple techniques from nonlinear functional analysis.

1. Introduction

Fractional calculus have applications in many areas including fluid flow, electrical networks, probability and statistics, viscoelasticity, chemical physics and signal processing, and so on, see [1, 3–5, 9–11, 17, 18, 22, 24] and references therein. Over the past 30 years, many researchers paid attention to the existence results of solutions of the initial and boundary value problems for fractional differential equations, such as [13, 23, 28].

In this paper, we want to study the existence of weak solutions to the steady fractional reaction-diffusion equation

$${}_x D_1^\alpha ({}_0 D_x^\alpha y(x)) + {}_0 D_x^\beta ({}_x D_1^\beta y(x)) = f(x, y(x)), \quad (1)$$

with Riemann-Liouville boundary conditions

$$\lim_{x \rightarrow 0^+} {}_0 I_x^{1-\alpha} y(x) = \lim_{x \rightarrow 1^-} {}_x I_1^{1-\beta} y(x) = 0, \quad (2)$$

where $0 < \alpha, \beta < 1$, $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function and ${}_0 D_x^\alpha$, ${}_x D_1^\alpha$ denote left and right Riemann-Liouville fractional derivatives of order α , respectively.

Our interest in studying problem (1) comes from the fractional reaction-diffusion equation

$$\frac{\partial y(t, x)}{\partial t} = {}_x D_1^\alpha ({}_0 D_x^\alpha y(t, x)) + {}_0 D_x^\beta ({}_x D_1^\beta y(t, x)) + f(t, x, y(x)),$$

that is a generalization of the classical reaction-diffusion equation in which the second-order derivative is replaced with a fractional derivative of order less than two. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry, depending on the fractional derivative used, see [7, 8, 25] for more details.

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Owing to its wide applications in modeling physical phenomena using equations involving left and right fractional derivatives [19–21], the existence results of the initial and boundary value problem for such equations have been studied by several researchers. We refer for instance to [2, 6, 26]. They all constructed a solution in the space of continuous functions. Indeed, they used the continuous functions on $[0, 1]$ as a solution space so that initial and boundary conditions could make sense. On the other hand, we know the natural conditions at boundary points for problems involving Riemann-Liouville fractional derivatives are Riemann-Liouville conditions as we can see in the literature (see e.g., [13, 18]). Recently, the physical meaning of such conditions to fractional differential equations with Riemann-Liouville derivatives has been discussed [15, 16].

Here, we wish to investigate the existence results for problem (1) subject to the Riemann-Liouville boundary conditions (2). For this, we first construct an appropriate solution space so that boundary conditions make sense. Secondly, we prove some useful properties of our solution space and using generalized Weierstrass theorem among with relatively simple techniques from nonlinear functional analysis, we show that problem (1)-(2) admit at least one weak solution.

The paper is structured as follows. In Section 2, a number of definitions and lemmas concerning the fractional Riemann-Liouville derivatives and integrals are collected. Next, in Section 3, the fractional Hilbert space as a solution space is stated. Finally, in Section 4, the existence of appropriately defined weak solution and almost everywhere solution of main problem is studied.

2. Preliminaries

The purpose of this section is to collect a number of definitions and lemmas concerning the fractional derivatives and integrals.

Definition 2.1. [18]. *The Riemann-Liouville fractional integrals ${}_0I_x^\alpha y$ and ${}_xI_1^\alpha y$ of order $\alpha > 0$ of a function $y : [0, 1] \rightarrow \mathbb{R}$ are defined by*

$${}_0I_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t) dt,$$

and

$${}_xI_1^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_x^1 (t-x)^{\alpha-1} y(t) dt,$$

respectively. Here and in what follows Γ denotes the Gamma function. These integrals are called the left and the right fractional integrals.

Definition 2.2. [18]. *The Riemann-Liouville fractional derivatives ${}_0D_x^\alpha y$ and ${}_xD_1^\alpha y$ of order $0 < \alpha \leq 1$ of a function $y : [0, 1] \rightarrow \mathbb{R}$ are defined by*

$${}_0D_x^\alpha y(x) = \frac{d}{dx} {}_0I_x^{1-\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} y(t) dt,$$

and

$${}_xD_1^\alpha y(x) = -\frac{d}{dx} {}_xI_1^{1-\alpha} y(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^1 (t-x)^{-\alpha} y(t) dt,$$

respectively. These operators are called the left and the right Riemann-Liouville fractional derivatives.

Definition 2.3. *Let $0 < \alpha \leq 1$. A function $y \in L^1[0, 1]$ is said to have a summable fractional derivative ${}_0D_x^\alpha y$ (resp. ${}_xD_1^\alpha y$), if ${}_0I_x^{1-\alpha} y \in AC[0, 1]$ (resp. ${}_xI_1^{1-\alpha} y \in AC[0, 1]$), where $AC[0, 1]$ represents the space of absolutely continuous functions on $[0, 1]$.*

Remark 2.4. Let $0 < \alpha \leq 1$. If ${}_0D_x^\alpha y(x)$ (resp. ${}_xD_1^\alpha y(x)$) exists in the usual sense, i.e. ${}_xI_1^{1-\alpha} y(x)$ (resp. ${}_xI_1^{1-\alpha} y(x)$) is differentiable at every point, then, evidently, $y(x)$ has a derivative in the sense of Definition 2.3.

Proposition 2.5. [18]. Let $p \geq 1$. The fractional integration operators ${}_0I_x^\alpha$ and ${}_xI_1^\alpha$ with $\alpha > 0$ are bounded in $L^p(0, 1)$,

$$\|{}_0I_x^\alpha y\|_p \leq K\|y\|_p, \quad \|{}_xI_1^\alpha y\|_p \leq K\|y\|_p, \quad \left(K = \frac{1}{\Gamma(\alpha + 1)}\right).$$

Proposition 2.6. [12]. Let $0 < \frac{1}{p} < \alpha \leq 1$ and $y(x) \in L^p(0, 1)$, then ${}_0I_x^\alpha y$ is Hölder continuous on $(0, 1]$ with exponent $\alpha - \frac{1}{p}$ and $\lim_{x \rightarrow 0^+} {}_0I_x^\alpha y(x) = 0$. Consequently, ${}_0I_x^\alpha y$ can be continuously extended by 0 in $x = 0$.

Proposition 2.7. [18]. Let $\alpha \geq 1$ and $p \geq 1$. Let $y(x) \in L^p(0, 1)$ and $\tilde{y}(x) \in L^{p'}(0, 1)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\int_0^1 {}_0I_x^\alpha y(x) \cdot \tilde{y}(x) dx = \int_0^1 y(x) \cdot {}_xI_1^\alpha \tilde{y}(x) dx.$$

Proposition 2.8. [18]. Let $\alpha \geq 0$ and $p \geq 1$. If $y(x) \in L^p(0, 1)$, then

$${}_0D_x^\alpha {}_0I_x^\alpha y(x) = y(x), \quad \text{and} \quad {}_1D_x^\alpha {}_xI_1^\alpha y(x) = y(x),$$

almost everywhere on $[0, 1]$.

Proposition 2.9. [22]. Let $0 < \alpha \leq 1$ and $p \geq 1$. If ${}_0D_x^\alpha y(x) \in L^p(0, 1)$ (resp. ${}_xD_1^\alpha y(x) \in L^p(0, 1)$), then

$${}_0I_x^\alpha {}_0D_x^\alpha y(x) = y(x) - \frac{{}_0I_x^{1-\alpha} y(0)}{\Gamma(\alpha)} x^{\alpha-1},$$

and

$${}_xI_1^\alpha {}_xD_1^\alpha y(x) = y(x) - \frac{{}_xI_1^{1-\alpha} y(1)}{\Gamma(\alpha)} (1-x)^{\alpha-1},$$

almost everywhere on $[0, 1]$.

Theorem 2.10. [27].(Generalized Weierstrass Theorem). Let X be a Hilbert space and $\mathcal{L} : X \rightarrow \bar{\mathbb{R}}$ be weakly lower semicontinuous and weakly coercive. Then

- $\inf_{u \in X} \mathcal{L}(u) > -\infty$,
- there is at least one $u_0 \in X$ such that $\mathcal{L}(u_0) = \inf_{u \in X} \mathcal{L}(u)$.

Moreover, if \mathcal{L} is Gâteaux differentiable at u_0 , then $\mathcal{L}'(u_0) = 0$.

3. Fractional Hilbert Space

In this section we introduce the fractional Hilbert space in which boundary conditions are nonlocal.

Definition 3.1. For $0 < \alpha, \beta < 1$, we define the following space

$$\mathbb{H}_0^{\alpha, \beta}(0, 1) := \left\{ y \in L^2(0, 1) : {}_0D_x^\alpha y, {}_xD_1^\beta y \in L^2(0, 1), \lim_{x \rightarrow 0^+} {}_0I_x^{1-\alpha} y(x) = \lim_{x \rightarrow 1^-} {}_xI_1^{1-\beta} y(x) = 0 \right\},$$

with the inner product

$$\langle y, \tilde{y} \rangle_{\alpha, \beta} := \int_0^1 {}_0D_x^\alpha y \cdot {}_0D_x^\alpha \tilde{y} dx + \int_0^1 {}_xD_1^\beta y \cdot {}_xD_1^\beta \tilde{y} dx, \tag{3}$$

and corresponding norm

$$\|y\|_{\alpha,\beta} := \left(\int_0^1 ({}_0D_x^\alpha y)^2 dx + \int_0^1 ({}_1D_x^\beta y)^2 dx \right)^{\frac{1}{2}}. \tag{4}$$

Clearly, if $\alpha = \beta = 1$, the space $\mathbb{H}_0^{\alpha,\beta}(0, 1)$ coincides with the Sobolev space $\mathbb{H}_0^1(0, 1)$.

Remark 3.2. In view of Proposition 2.9, for every $y \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$, we have

$${}_0I_x^\alpha {}_0D_x^\alpha y(x) = y(x), \quad \text{and} \quad {}_xI_1^\beta {}_1D_1^\beta y(x) = y(x).$$

Therefore, using the continuity of Riemann-Liouville fractional integral operator from L^p to L^p , we deduce

$$\|y\|_2 \leq K \left(\|{}_0D_x^\alpha y\|_2^2 + \|{}_1D_x^\beta y\|_2^2 \right)^{\frac{1}{2}} = K \|y\|_{\alpha,\beta}. \tag{5}$$

for some $K > 0$. Therefore, the continuous embedding $\mathbb{H}_0^{\alpha,\beta}(0, 1) \hookrightarrow L^2(0, 1)$ holds.

Lemma 3.3. Let $0 < \alpha, \beta < 1$. The space $\mathbb{H}_0^{\alpha,\beta}(0, 1)$ is a Hilbert space.

Proof. Let $\{y_n\}$ be a Cauchy sequence in $\mathbb{H}_0^{\alpha,\beta}(0, 1)$, then $\{y_n\}$, $\{{}_0D_x^\alpha y_n\}$ and $\{{}_xI_1^\beta y_n\}$ are Cauchy sequences in $L^2(0, 1)$. It follows that

$$y_n \xrightarrow{L^2} y, \quad {}_0D_x^\alpha y_n \xrightarrow{L^2} y^{(\alpha)}, \quad {}_xI_1^\beta y_n \xrightarrow{L^2} y^{(\beta)}. \tag{6}$$

A similar argument in Remark 3.2, we deduce

$$y_n \xrightarrow{L^2} {}_0I_x^\alpha y^{(\alpha)}, \quad y_n \xrightarrow{L^2} {}_xI_1^\beta y^{(\beta)}. \tag{7}$$

Therefore, we have

$$y(x) = {}_0I_x^\alpha y^{(\alpha)}(x), \quad y(x) = {}_xI_1^\beta y^{(\beta)}(x) \tag{8}$$

almost everywhere on $[0, 1]$. Finally, using Proposition 2.8, we immediately get that $y^{(\alpha)} = {}_0D_x^\alpha y$, $y^{(\beta)} = {}_xI_1^\beta y$. On the other hand, from (8), we have

$$y(x) = {}_0I_x^\alpha {}_0D_x^\alpha y(x), \quad y(x) = {}_xI_1^\beta {}_1D_1^\beta y(x), \tag{9}$$

almost everywhere on $[0, 1]$. Therefore, using Proposition 2.9, we deduce

$$\lim_{x \rightarrow 0^+} {}_0I_x^{1-\alpha} y(x) = \lim_{x \rightarrow 1^-} {}_xI_1^{1-\beta} y(x) = 0,$$

and the proof is finished. \square

Now we intend to state and prove a compactness result for the space $\mathbb{H}_0^{\alpha,\beta}(0, 1)$. This will be crucial for our analysis in the sequel. For any $h \in \mathbb{R}$ and any $y \in L^2(0, 1)$, we define the translation of y by h , to be the function $\tau_h(y)$ from \mathbb{R} to \mathbb{R} , given by

$$\tau_h(y)(x) = \begin{cases} y(x+h) & x+h \in (0, 1), \\ 0 & x+h \notin (0, 1). \end{cases}$$

Theorem 3.4. Let $0 < \alpha, \beta < 1$. The compact embedding $\mathbb{H}_0^{\alpha,\beta}(0, 1) \hookrightarrow L^2(0, 1)$ holds.

Proof. From Remark 3.2, it is sufficient to prove every bounded sequence in $\mathbb{H}_0^{\alpha,\beta}(0, 1)$ is precompact in $L^2(0, 1)$. On the other hand, from Fréchet-Kolmogorov theorem, it is sufficient to prove that

$$\sup_{n \in \mathbb{N}} \|\tau_h(y_n) - y_n\|_2 \rightarrow 0, \quad \text{as } h \rightarrow 0. \tag{10}$$

From Proposition 2.9, for every $n \in \mathbb{N}$, we have

$$y_n(x) = {}_0I_{x0}^\alpha D_x^\alpha y_n(x).$$

Therefore, for $h > 0$ and $x, x + h \in [0, 1]$, we have

$$\begin{aligned} \|y_n(x+h) - y_n(x)\|_2^2 &= \|{}_0I_{x0}^\alpha D_x^\alpha y_n(x+h) - {}_0I_{x0}^\alpha D_x^\alpha y_n(x)\|_2^2 = \int_0^1 |{}_0I_{x0}^\alpha D_x^\alpha y_n(x+h) - {}_0I_{x0}^\alpha D_x^\alpha y_n(x)|^2 dx \\ &= \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left| \int_0^{x+h} (x+h-t)^{\alpha-1} {}_0D_x^\alpha y_n(t) dt - \int_0^x (x-t)^{\alpha-1} {}_0D_x^\alpha y_n(t) dt \right|^2 dx \\ &= \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left| \int_0^x ((x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}) {}_0D_x^\alpha y_n(t) dt + \int_x^{x+h} (x+h-t)^{\alpha-1} {}_0D_x^\alpha y_n(t) dt \right|^2 dx \\ &\leq \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left| \int_0^x ((x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}) {}_0D_x^\alpha y_n(t) dt \right|^2 dx \\ &\quad + \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left| \int_x^{x+h} (x+h-t)^{\alpha-1} {}_0D_x^\alpha y_n(t) dt \right|^2 dx \\ &\leq \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left(\int_0^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}|^{\frac{1}{2}} |{}_0D_x^\alpha y_n(t)| |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}|^{\frac{1}{2}} dt \right)^2 dx \\ &\quad + \frac{1}{(\Gamma(\alpha))^2} \int_0^1 \left(\int_x^{x+h} |(x+h-t)^{\alpha-1}|^{\frac{1}{2}} |{}_0D_x^\alpha y_n(t)| |(x+h-t)^{\alpha-1}|^{\frac{1}{2}} dt \right)^2 dx. \end{aligned}$$

Using Hölder’s inequality for the inner integrals above, we have

$$\begin{aligned} &\left(\int_0^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}|^{\frac{1}{2}} |{}_0D_x^\alpha y_n(t)| |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}|^{\frac{1}{2}} dt \right)^2 \\ &\leq \left(\int_0^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}| |{}_0D_x^\alpha y_n(t)|^2 dt \right) \left(\int_0^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}| dt \right) \\ &\leq \frac{1}{\alpha} [h^\alpha - ((1+h)^\alpha - 1)] \left(\int_0^x |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}| |{}_0D_x^\alpha y_n(t)|^2 dt \right), \end{aligned}$$

and

$$\begin{aligned} &\left(\int_x^{x+h} |(x+h-t)^{\alpha-1}|^{\frac{1}{2}} |{}_0D_x^\alpha y_n(t)| |(x+h-t)^{\alpha-1}|^{\frac{1}{2}} dt \right)^2 \\ &\leq \left(\int_x^{x+h} |(x+h-t)^{\alpha-1}| |{}_0D_x^\alpha y_n(t)|^2 dt \right) \left(\int_x^{x+h} |(x+h-t)^{\alpha-1}| dt \right) \\ &= h^{2\alpha} \left(\int_0^1 |(1-s)^{\alpha-1}| |{}_0D_x^\alpha y_n(x+sh)|^2 ds \right) \left(\int_0^1 |(1-s)^{\alpha-1}| ds \right). \end{aligned}$$

Therefore, for every $n \in \mathbb{N}$, we have

$$\|y_n(x+h) - y_n(x)\|_2^2 \leq \frac{[h^\alpha - ((1+h)^\alpha - 1)]}{\alpha(\Gamma(\alpha))^2} \int_0^1 \int_t^1 |(x+h-t)^{\alpha-1} - (x-t)^{\alpha-1}| |{}_0D_x^\alpha y_n(t)|^2 dx dt$$

$$\begin{aligned}
 & + \frac{h^{2\alpha}}{\alpha(\Gamma(\alpha))^2} \int_0^1 \int_0^1 |(1-s)^{\alpha-1}| \left| {}_0D_x^\alpha y_n(x+sh) \right|^2 ds dx \\
 \leq & \left(\frac{[h^\alpha - ((1+h)^\alpha - 1)] [h^\alpha - ((1+h)^\alpha - 1)]}{\alpha(\Gamma(\alpha))^2} + \frac{h^{2\alpha}}{\alpha^2(\Gamma(\alpha))^2} \right) \| {}_0D_x^\alpha y_n \|_2^2 \\
 = & \left(\frac{[h^\alpha - ((1+h)^\alpha - 1)]^2}{(\Gamma(\alpha + 1))^2} + \frac{h^{2\alpha}}{(\Gamma(\alpha + 1))^2} \right) \| {}_0D_x^\alpha y_n \|_2^2, \tag{11}
 \end{aligned}$$

whence assertion (10) follows from (11). \square

4. Existence and Uniqueness Results

We will study in this section the existence of appropriately defined weak solutions and almost everywhere solutions of the main problem.

Definition 4.1. We say $y \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$ is a weak solution of (1)-(2), if

$$\int_0^1 \left({}_0D_x^\alpha y(x) \cdot {}_0D_x^\alpha v(x) + {}_xD_1^\beta y(x) \cdot {}_xD_1^\beta v(x) \right) dx = \int_0^1 f(x, y(x)) \cdot v(x) dx,$$

for all $v \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$.

We define $F(x, y) = \int_0^y f(x, \xi) d\xi$ and the functional \mathcal{L} on $\mathbb{H}_0^{\alpha,\beta}(0, 1)$ by

$$\mathcal{L}(y) := 2^{-1} \int_0^1 \left({}_0D_x^\alpha y(x) \right)^2 + \left({}_xD_1^\beta y(x) \right)^2 dx - \int_0^1 F(x, y) dx. \tag{12}$$

To prove the main result, we need the following assumptions

- (i) $F(x, s) \geq -\gamma_1|s| - \gamma_2(x)$ for a.e. $x \in [0, 1]$, all $s \in \mathbb{R}$ and some $\gamma_1 > 0, \gamma_2 \in L^1(0, 1)$,
- (ii) $|F(x, s)| \leq \gamma_3(x) + \gamma_4|s|^2$ for a.e. $x \in [0, 1]$, all $s \in \mathbb{R}$ and some $\gamma_3 \in L^1(0, 1)$ and $\gamma_4 > 0$,
- (iii) $|f(x, s)| \leq \gamma_5(x) + \gamma_6|s|$ for a.e. $x \in [0, 1]$, all $s \in \mathbb{R}$ and some $\gamma_5 \in L^2(0, 1)$ and $\gamma_6 > 0$.

Let us prove the following result:

Lemma 4.2. Under the standard conditions above, the functional \mathcal{L} is Gâteaux differentiable at any $y \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$ and

$$\mathcal{L}'(y)(v) = \int_0^1 {}_0D_x^\alpha y(x) \cdot {}_0D_x^\alpha v(x) + {}_xD_1^\beta y(x) \cdot {}_xD_1^\beta v(x) dx - \int_0^1 f(x, y(x)) \cdot v(x) dx,$$

for all $v \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$.

Proof. We set

$$\begin{aligned}
 L(\varepsilon) & := \mathcal{L}(y + \varepsilon v) \\
 & = \underbrace{\int_0^1 \left[2^{-1} \left({}_0D_x^\alpha y + \varepsilon {}_0D_x^\alpha v \right)^2 + 2^{-1} \left({}_xD_1^\beta y + \varepsilon {}_xD_1^\beta v \right)^2 - F(x, y + \varepsilon v) \right] dx}_{\bar{F}}.
 \end{aligned}$$

Since

$$\frac{\partial \bar{F}}{\partial \varepsilon} = \left({}_0D_x^\alpha y + \varepsilon {}_0D_x^\alpha v \right) \cdot {}_0D_x^\alpha v + \left({}_xD_1^\beta y + \varepsilon {}_xD_1^\beta v \right) \cdot {}_xD_1^\beta v - f(x, y + \varepsilon v) \cdot v,$$

and using (iii), we deduce $|\frac{\partial \bar{f}}{\partial \varepsilon}| \in L^1(0, 1)$. Then we obtain

$$L'(\varepsilon) = \int_0^1 \left[({}_0D_x^\alpha y + \varepsilon {}_0D_x^\alpha v) \cdot {}_0D_x^\alpha v + ({}_x D_1^\beta y + \varepsilon {}_x D_1^\beta v) \cdot {}_x D_1^\beta v - f(x, y + \varepsilon v) \cdot v \right] dx.$$

Therefore, we deduce

$$\mathcal{L}'(y)(v) = L'(0) = \int_0^1 {}_0D_x^\alpha y(x) \cdot {}_0D_x^\alpha v(x) + {}_x D_1^\beta y(x) \cdot {}_x D_1^\beta v(x) dx - \int_0^1 f(x, y(x)) \cdot v(x) dx.$$

□

Theorem 4.3. Assume that (i)-(iii) hold. Then the problem (1)-(2) has a weak solution.

Proof. In view of Theorem 2.10, the proof consists of two steps:

Step 1. We first check that

$$\mathcal{L}(y) = \mathcal{L}_1(y) + \mathcal{L}_2(y) = 2^{-1} \int_0^1 ({}_0D_x^\alpha y(x))^2 + ({}_x D_1^\beta y(x))^2 dx - \int_0^1 F(x, y(x)) dx,$$

is weakly lower semicontinuous on $\mathbb{H}_0^{\alpha,\beta}(0, 1)$. Note that $\mathcal{L}_1(y) = 2^{-1} \|y\|_{\alpha,\beta}^2$ on $\mathbb{H}_0^{\alpha,\beta}(0, 1)$ which is weakly lower semicontinuous. On the other hand, since the embedding $\mathbb{H}_0^{\alpha,\beta}(0, 1) \hookrightarrow L^2(0,1)$ is compact and $F(x, y)$ is bounded below, from [14, Theorem 1 in §8.2.2], we conclude \mathcal{L}_2 is weakly lower semicontinuous too. Hence \mathcal{L} is weakly lower semicontinuous on $\mathbb{H}_0^{\alpha,\beta}(0, 1)$.

Step 2. \mathcal{L} is weakly coercive on $\mathbb{H}_0^{\alpha,\beta}(0, 1)$. Using (i) and Remark 3.2, we have

$$\begin{aligned} \mathcal{L}(y) &\geq \|y\|_{\alpha,\beta}^2 - \gamma_1 \|y\|_1 - \gamma \\ &\geq \|y\|_{\alpha,\beta}^2 - \gamma_1 \|y\|_2 - \gamma \\ &\geq \|y\|_{\alpha,\beta}^2 - \bar{\gamma} \|y\|_{\alpha,\beta} - \gamma, \end{aligned}$$

and hence $\mathcal{L}(y) \rightarrow \infty$ if $\|y\|_{\alpha,\beta} \rightarrow \infty$. This proves the weak coercivity of \mathcal{L} . Therefore, using Theorem 2.10 and Lemma 4.2, we have

$$\int_0^1 {}_0D_x^\alpha y(x) \cdot {}_0D_x^\alpha v(x) + {}_x D_1^\beta y(x) \cdot {}_x D_1^\beta v(x) dx = \int_0^1 f(x, y(x)) \cdot v(x) dx,$$

as desired. □

Now we derive uniqueness of weak solution to (1)-(2).

Theorem 4.4. Let (i)-(iii) hold and assume the mapping $y \rightarrow -F(x, y)$ is convex for all $x \in [0, 1]$. Then there exists at most one weak solution of the problem (1)-(2).

Proof. Assume that y_1 and y_2 are two weak solutions of (1)-(2). We then divide the proof into two steps:

Step 1. We show that y_1, y_2 are minimizers of the functional

$$\mathcal{L}(y) := 2^{-1} \int_0^1 ({}_0D_x^\alpha y(x))^2 + ({}_x D_1^\beta y(x))^2 dx - \int_0^1 F(x, y) dx, \tag{13}$$

over $\mathbb{H}_0^{\alpha,\beta}(0, 1)$.

Let $L(x, y, {}_0D_x^\alpha y, {}_x D_1^\beta y) = 2^{-1}({}_0D_x^\alpha y)^2 + 2^{-1}({}_x D_1^\beta y)^2 - F(x, y)$. The convexity of the mapping $y \rightarrow -F(x, y)$ implies the mapping $(y, {}_0D_x^\alpha y, {}_x D_1^\beta y) \rightarrow L(x, y, {}_0D_x^\alpha y, {}_x D_1^\beta y)$ is convex for all $x \in [0, 1]$. Therefore, for every $y \in \mathbb{H}_0^{\alpha,\beta}(0, 1)$, we have

$$\begin{aligned} L(x, y, {}_0D_x^\alpha y, {}_x D_1^\beta y) &\geq L(x, y_1, {}_0D_x^\alpha y_1, {}_x D_1^\beta y_1) + \partial_2 L(x, y_1, {}_0D_x^\alpha y_1, {}_x D_1^\beta y_1) \cdot (y - y_1) \\ &\quad + \partial_3 L(x, y_1, {}_0D_x^\alpha y_1, {}_x D_1^\beta y_1) \cdot ({}_0D_x^\alpha y - {}_0D_x^\alpha y_1) + \partial_4 L(x, y_1, {}_0D_x^\alpha y_1, {}_x D_1^\beta y_1) \cdot ({}_x D_1^\beta y - {}_x D_1^\beta y_1), \end{aligned}$$

where ∂_i denotes the partial derivative of L with respect to its i th argument. Therefore, we have

$$L(x, y, {}_0D_x^\alpha y, {}_xD_1^\beta y) \geq L(x, y_1, {}_0D_x^\alpha y_1, {}_xD_1^\beta y_1) - f(x, y_1) \cdot (y - y_1) + {}_0D_x^\alpha y_1 \cdot ({}_0D_x^\alpha y - {}_0D_x^\alpha y_1) + {}_xD_1^\beta y_1 \cdot ({}_xD_1^\beta y - {}_xD_1^\beta y_1).$$

Integrating over $[0, 1]$ and using the fact that $y - y_1 \in \mathbb{H}_0^{\alpha, \beta}(0, 1)$, we have $\mathcal{L}(y_1) \leq \mathcal{L}(y)$. A similar argument shows that, for every $y \in \mathbb{H}_0^{\alpha, \beta}(0, 1)$, $\mathcal{L}(y_2) \leq \mathcal{L}(y)$.

Step 2. We now prove that $y_1 = y_2$.

From the previous step, we know y_1, y_2 are minimizers of the functional $\mathcal{L}(y)$ over $\mathbb{H}_0^{\alpha, \beta}(0, 1)$ and so that $\mathcal{L}(y_1) = \mathcal{L}(y_2) = m$. Denote by $\bar{y} := \frac{y_1 + y_2}{2}$ and observe that $\bar{y} \in \mathbb{H}_0^{\alpha, \beta}(0, 1)$. From the convexity of $(y, {}_0D_x^\alpha y, {}_xD_1^\beta y) \rightarrow L(x, y, {}_0D_x^\alpha y, {}_xD_1^\beta y)$, we can infer that \bar{y} is also a minimizer since

$$m \leq \mathcal{L}(\bar{y}) \leq 2^{-1} \mathcal{L}(y_1) + 2^{-1} \mathcal{L}(y_2) = m,$$

which readily implies that

$$\int_0^1 \left[2^{-1} L(x, y_1, {}_0D_x^\alpha y_1, {}_xD_1^\beta y_1) + 2^{-1} L(x, y_2, {}_0D_x^\alpha y_2, {}_xD_1^\beta y_2) - L(x, \bar{y}, {}_0D_x^\alpha \bar{y}, {}_xD_1^\beta \bar{y}) \right] dx = 0.$$

The convexity of $(y, {}_0D_x^\alpha y, {}_xD_1^\beta y) \rightarrow L(x, y, {}_0D_x^\alpha y, {}_xD_1^\beta y)$ implies that the integrand is non negative, while the integral is zero. This is possible only if

$$2^{-1} L(x, y_1, {}_0D_x^\alpha y_1, {}_xD_1^\beta y_1) + 2^{-1} L(x, y_2, {}_0D_x^\alpha y_2, {}_xD_1^\beta y_2) - L(x, \bar{y}, {}_0D_x^\alpha \bar{y}, {}_xD_1^\beta \bar{y}) = 0, \tag{14}$$

almost everywhere on $[0, 1]$. Now using the convexity of $y \rightarrow -F(x, y)$ and the strict convexity of $\zeta \rightarrow \zeta^2$, we conclude that the mapping $(y, {}_0D_x^\alpha y, {}_xD_1^\beta y) \rightarrow L(x, y, {}_0D_x^\alpha y, {}_xD_1^\beta y)$ is strict convex. Therefore, from equality (14) we obtain $y_1 = y_2$ almost everywhere on $[0, 1]$. \square

Remark 4.5. In a similar way, we can deal with the existence results of solutions for the following problem

$$\begin{cases} {}_xD_1^\alpha ({}_0D_x^\alpha y(x)) = f(x, y(x)), \\ \lim_{x \rightarrow 0^+} {}_0I_x^{1-\alpha} y(x) = 0. \end{cases} \tag{15}$$

Indeed, it is sufficient to construct a space of solutions as follows

$$\mathbb{H}_0^\alpha(0, 1) := \left\{ y \in L^2(0, 1) : {}_0D_x^\alpha y \in L^2(0, 1), \lim_{x \rightarrow 0^+} {}_0I_x^{1-\alpha} y(x) = 0 \right\},$$

and carry out a similar argument to prove existence of solutions.

Theorem 4.6. Let $\alpha < \frac{1}{2}$ and f be continuous. Assume that $y \in \mathbb{H}_0^\alpha(0, 1)$ be a weak solution of (15). Then y is almost everywhere solution of (15) on $[0, 1]$, i.e. y satisfies

$${}_xD_1^\alpha ({}_0D_x^\alpha y(x)) = f(x, y(x)), \tag{16}$$

almost everywhere on $[0, 1]$.

Proof. Since $C_c^\infty(0, 1) \subseteq \mathbb{H}_0^\alpha(0, 1)$ where $C_c^\infty(0, 1)$ is the space of infinitely differentiable functions and compactly supported in $(0, 1)$, we have

$$\int_0^1 [{}_0D_x^\alpha y(x) \cdot {}_0D_x^\alpha v(x)] dx = \int_0^1 f(x, y(x)) \cdot v(x) dx, \tag{17}$$

for all $v \in C_c^\infty(0, 1)$. For any $v \in C_c^\infty(0, 1)$, ${}_0D_x^\alpha v = {}_0I_x^{1-\alpha} v'$. Therefore, Proposition 2.7 gives

$$\int_0^1 [{}_xI_1^{1-\alpha} {}_0D_x^\alpha y(x)] v' dx = \int_0^1 \left[- \int_0^x f(\xi, y(\xi)) d\xi \right] v' dx. \tag{18}$$

So, there exist constant C , such that

$${}_xI_1^{1-\alpha} {}_0D_x^\alpha y(x) = - \int_0^x f(\xi, y(\xi)) d\xi + C. \tag{19}$$

almost everywhere on $[0, 1]$. On the other hand, from Proposition 2.6 we have ${}_xI_1^{1-\alpha} {}_0D_x^\alpha y(x) \in C[0, 1]$. Therefore,

$${}_xI_1^{1-\alpha} {}_0D_x^\alpha y(x) = - \int_0^x f(\xi, y(\xi)) d\xi + C, \tag{20}$$

everywhere on $[0, 1]$. Finally by differentiation, we obtain the desired equality. \square

Remark 4.7. Notice that the condition $\alpha < \frac{1}{2}$ in the previous theorem together with Proposition 2.6 implies that $\lim_{x \rightarrow 1^-} {}_xI_1^{1-\alpha} {}_0D_x^\alpha y(x) = 0$. This is natural because the sequential fractional differential equation (15) involves two fractional derivative operators and therefore, this problem needs two initial or boundary conditions for the existence of the classical solution.

5. Examples

Now we give two examples to illustrate the applicability of the results developed in the previous sections.

Example 5.1. Let us consider the following problem

$${}_x D_1^{\frac{1}{2}} ({}_0 D_x^{\frac{1}{2}} y(x)) + {}_0 D_x^{\frac{1}{3}} ({}_x D_1^{\frac{1}{3}} y(x)) = \mathbf{1}_Q(x)y - \sin y, \tag{21}$$

with Riemann-Liouville boundary conditions

$$\lim_{x \rightarrow 0^+} {}_0 I_x^{\frac{1}{2}} y(x) = \lim_{x \rightarrow 1^-} {}_x I_1^{\frac{2}{3}} y(x) = 0, \tag{22}$$

where

$$\mathbf{1}_Q(x) = \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{Q}^c. \end{cases}$$

In this case, $f(x, y) = \mathbf{1}_Q(x)y - \sin y$ is a Carathéodory function. Indeed, $f(\cdot, y) : [0, 1] \rightarrow \mathbb{R}$ is measurable for every $y \in \mathbb{R}$ and $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous for almost everywhere on $[0, 1]$. On the other hand, it is easy to see that all the conditions (i), (ii) and (iii) hold; it suffices to set $\gamma_i = 1, i = 1, \dots, 6$. Therefore, Theorem 4.3 implies that the above problem admits a weak solution in $\mathbb{H}_0^{\frac{1}{2}, \frac{1}{3}}(0, 1)$.

Example 5.2. Let us consider the following problem

$${}_x D_1^{\frac{1}{3}} ({}_0 D_x^{\frac{1}{3}} y(x)) = \frac{1}{\Gamma(\frac{2}{3})} (1-x)^{\frac{2}{3}} \tag{23}$$

with Riemann-Liouville condition

$$\lim_{x \rightarrow 0^+} {}_0 I_x^{\frac{2}{3}} y(x) = 0. \tag{24}$$

In this case, $f(x, y) = \frac{1}{\Gamma(\frac{2}{3})}(1-x)^{\frac{2}{3}}$. Obviously all the conditions (i), (ii) and (iii) hold true. Therefore, Theorem 4.3 implies that the above problem admits a weak solution in $H_0^{\frac{1}{3}, \frac{1}{3}}(0, 1)$. Furthermore, since $f(x, y)$ does not depend on y , Theorem 4.4 implies that this solution is unique. On the other hand, from Theorem 4.6, this weak solution is almost everywhere solution of (23). Now a simple calculation shows that $y(x) = \frac{1}{\Gamma(\frac{4}{3})}x^{\frac{1}{3}} - \frac{1}{\Gamma(\frac{2}{3})}x^{\frac{4}{3}}$ satisfies the equation (23)-(24).

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