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Derivations on FCIN Algebras

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Abstract. Let $\mathcal{L}$ be an algebra generated by the commuting independent nests, $\mathcal{M}$ is an ultra-weakly closed subalgebra of $\mathbf{B(H)}$ which contains $alg\mathcal{L}$ and ϕ is a norm continuous linear mapping from $alg\mathcal{L}$ into $\mathcal{M}$. In this paper we will show that a norm continuous linear derivable mapping at zero point from $Alg\mathcal{L}$ to $\mathcal{M}$ is a derivation.

1. Introduction

Definition 1.1. *Let $\mathcal{A}$ be a subalgebra of $\mathbf{B(H)}$, let ϕ be a linear mapping from $\mathcal{A}$ to $\mathbf{B(H)}$. We say that ϕ is a derivation if $\phi(AB) = \phi(A)B + A\phi(B)$ for any $A, B \in \mathcal{A}$.*

We say that ϕ is a derivable mapping at the zero point if $\phi(AB) = \phi(A)B + A\phi(B)$ for any $A, B \in \mathcal{A}$ with $AB = 0$.

Several authors have studied linear mappings on operator algebras are derivations. In [2] Jing and Liu showed that every derivable mapping ϕ at 0 with $\phi(I) = 0$ on nest algebras is an inner derivation. In [6, 7] Zhu and Xiong proved that every norm continuous generalized derivable mapping at 0 on a finite CSL algebra is a generalized derivation, and every strongly operator topology continuous derivable mapping at the unit operator I in nest algebras is a derivation. It is natural and interesting to ask whether or not a linear mapping is a derivation if it is derivable only at one given point. An and Hou [1] investigated derivable mapping at $0, P$, and I on triangular rings, where P is a fixed non-trivial idempotent. In [5] Zhao and Zhu characterized Jordan derivable mappings at 0 and I on triangular algebras.

Now we will give some required definitions.

Definition 1.2. *Let $\mathcal{L}$ be a lattice on a Hilbert space $\mathbf{H}$. If $\mathcal{L}$ is generated by finitely many commuting independent nests, it will be called by an $alg\mathcal{L}$ FCIN algebra.*

Let $\mathcal{L}$ be a subspace lattice. For each projection $E \in \mathcal{L}$, let

$$E_- = \bigvee \{F : F \in \mathcal{L}, F \ngeq E\} \quad \text{and} \quad E_* = \bigvee \{F_- : F \in \mathcal{L}, F \nleq E\}.$$

Definition 1.3. *A subspace lattice $\mathcal{L}$ is called* **completely distributive if** $E_* = E, \forall E \in \mathcal{L}$.

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Definition 1.4. *If $\mathcal{L}$ is completely distributive and commutative, we will call an $Alg\mathcal{L}$ CDCSL algebra.*

Throughout we consider x, y be vectors in $\mathbf{H}$, we use notation $x \otimes y$ for rank one operators defined by $(x \otimes y)z = (z, x)y$ for all $z \in \mathbf{H}$. Let $\mathcal{R}_L$ be the spanning space of rank one operators in $Alg\mathcal{L}$. Laurie and Longstaff [3] proved the following result.

Theorem 1.5. *A commutative subspace lattice $\mathcal{L}$ is completely distributive if and only if $\mathcal{R}_L$ is ultra-weakly dense in $Alg\mathcal{L}$.*

Definition 1.6. *Let $\mathcal{L}$ be a CSL. Then the von Neumann algebra $(Alg\mathcal{L}) \cap (Alg\mathcal{L})^*$ is called* **diagonal of** $Alg\mathcal{L}$ **and denoted by** $D(\mathcal{L})$**.**

Assume that $\mathcal{L}$ is generated by the commuting independent

$$\{E(Alg\mathcal{L})E^{\perp} : E \in \mathcal{L}\}.$$

It is clear that $\mathcal{R}_L$ is a norm closed ideal of the CSL algebra $Alg\mathcal{L}$.

Assume that $\mathcal{L}$ is generated by the commuting independent nests $\mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_n$, then $\mathcal{M}$ is an ultra-weakly closed subalgebra of $\mathbf{B}(\mathbf{H})$ which contains $alg\mathcal{L}$, and ϕ is a norm continuous linear mapping from $Alg\mathcal{L}$ into $\mathcal{M}$.

2. The Main Result

o prove the main result of this paper we require the following Lemma from [4].

Lemma 2.1. *Let $\mathcal{L}$ be an arbitrary CSL on the complex separable Hilbert space $\mathbf{H}$, and $\mathcal{M}$ be an ultra-weakly closed subalgebra of $\mathbf{B}(\mathbf{H})$ which contains $Alg\mathcal{L}$.*
If $\phi : Alg\mathcal{L} \to \mathcal{M}$ is a norm continuous linear mapping, then $\phi(XAY) = \phi(XA)Y + X\phi(AY) - X\phi(A)Y$ for all A in $Alg\mathcal{L}$ and all X, Y in $\mathcal{D}(\mathcal{L}) + \mathcal{R}(\mathcal{L})$.

Lemma 2.2. *Let $A \in Alg\mathcal{L}$ and $B \in \mathcal{D}(\mathcal{L}) + \mathcal{R}(\mathcal{L})$. If $AB \in \mathcal{D}(\mathcal{L}) + \mathcal{R}(\mathcal{L})$, then $\phi(AB) = \phi(A)B + A\phi(B)$.*

Corollary 2.3. $\phi(XY) = \phi(X)Y + X\phi(Y)$ *for all* $X, Y \in \mathcal{D}(\mathcal{L}) + \mathcal{R}(\mathcal{L})$.

So we are ready to prove the main result of this work.

Theorem 2.4. *Let $\mathcal{L}$ be a commutative subspace lattice generated by finitely many independent nests, and $\mathcal{M}$ be any ultra-weakly closed subalgebra of $\mathbf{B}(\mathbf{H})$ on $\mathbf{H}$, which contains $Alg\mathcal{L}$. Let ϕ be a norm continuous linear derivable mapping at the zero point from $Alg\mathcal{L}$ to $\mathcal{M}$. Then ϕ is a derivation.*

Proof. Let $\phi : Alg\mathcal{L} \to \mathcal{M}$ be a norm continuous derivable linear mapping. Then we just need to prove that ϕ is a derivation. Let $\Omega = \{i : I_- = I \in \mathcal{L}_i\}$. Then we have the following cases.

When $\Omega = \Phi$, for each i=1, 2,....., n, let $Q_i = I_-$ be the projection in $\mathcal{L}_i$ and N=$\prod_{i=1}^{n} Q_i^{\perp}$, then $\mathbf{B}(\mathbf{H})N \subset \mathcal{D}(\mathcal{L}) + \mathcal{R}(\mathcal{L})$. Let $A, B \in alg\mathcal{L}$, it follows from Lemma 2.2 that

$$\phi(ABTN) = \phi(AB)TN + AB\phi(TN)$$

for all $T \in B(H)$. On the other hand, we have from Lemma 2.2 again

$$\begin{aligned} \phi(ABTN) &= \phi(A)BTN + A\phi(BTN) \\ &= \phi(A)BTN + A[\phi(B)TN + B\phi(TN)] \\ &= \phi(A)BTN + A\phi(B)TN + AB\phi(TN). \end{aligned}$$

The last two equations give us that $[\phi(AB) - \phi(A)B - A\phi(B)]TN = 0$ for all $T \in \mathbf{B}(\mathbf{H})$. Thus we have

$$\phi(AB) = \phi(A)B + A\phi(B).$$

When $\Omega \neq \Phi$, for each $i \notin \Omega$, let $Q_i = I_-$ be the projection in $\mathcal{L}_i$, define $M = \Pi_{i\notin\Omega} Q_i^{\perp}$ (If $\Omega = \{1, 2, ..., n\}$, we take $M = I$). For each $i \in \Omega$, there exists an increasing sequence $P_{i,k}$ of projections in $\mathcal{L}_i \setminus \{I\}$ which strongly converges to I. Let $E_k = \Pi_{i\in\Omega} P_{i,k}$ and $F_k = \Pi_{i\in\Omega} P_{i,k}^{\perp}$. Then $\lim_{k\to\infty} E_k = I$. It is clear that $E_k B(H) M F_K \subset \mathcal{R}(\mathcal{L})$ for all $k \in \mathbb{N}$. Let $A, B \in Alg\mathcal{L}$, it follows from Lemma 2.2 that

$$\phi(ABE_kTMF_k) = \phi(AB)E_kTMF_k + AB\phi(E_kTMF_k)$$

for all $T \in \mathbf{B}(\mathbf{H})$ and $k \in \mathbb{N}$. On the other hand, by Lemma 2.2 we have

$$\begin{aligned} \phi(ABE_kTMF_k) &= \phi(A)BE_kTMF_k + A\phi(BE_kTMF_k) \\ &= \phi(A)BE_kTMF_k + A[\phi(B)E_kTMF_k + B\phi(E_kTMF_k)] \\ &= \phi(A)BE_kTMF_k + A\phi(B)E_kTMF_k + AB\phi(E_kTMF_k). \end{aligned}$$

From the last two equations we have $[\phi(AB) - \phi(A)B - A\phi(B)]E_kTMF_k = 0$ for all $T \in \mathbf{B}(\mathbf{H})$ and $k \in \mathbb{N}$. By independence of the nests $\mathcal{L}_i$, $MF_k \neq 0$ for all $k \in \mathbb{N}$. Hence

$$[\phi(AB) - \phi(A)B - A\phi(B)]E_k = 0$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$, we have that $\phi(AB) = \phi(A)B + A\phi(B)$. Hence ϕ is a derivation, namely 0 is a derivable point of $alg\mathcal{L}$ for norm continuous linear mapping. This completes the proof. □

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