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Some Classes of Ideal Convergent Sequences and Generalized Difference Matrix Operator

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Abstract. The aim of paper is to define and study some ideal convergent sequence spaces with the help of generalized difference matrix $B^n_{(m)}$ and Orlicz functions. We also make an effort to study some algebraic and topological properties of these difference sequence spaces.

1. Background and Preliminaries

The concept of statistical convergence is a generalization of the usual notion of convergence that, for real-valued sequences, parallels the usual theory of convergence (see [9]). Kostyrko et al. [15] and Nuray and Ruckle [21] independently studied in detalis about the notion of ideal convergence which is a generalization of statistical convergence and is based on the structure of the admissible ideal I of subsets of natural numbers $\mathbb{N}$. Later on it was further investigated by Tripathy and Hazarika [25, 26], Hazarika and Mohiuddine [10], Hazarika [12] and references therein. Hazarika [11] introduced the notion of generalized difference I-convergence in random 2-normed spaces and proved some interesting results. Çakalli and Hazarika [5] introduced the new concept ideal quasi Cauchy sequences and studied some results in analysis.

Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S iff $\phi \in I$, I is additive and hereditary. An ideal $I \subseteq P(S)$ is said to be non trivial if $S \notin I$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S iff $\phi \notin F$, for each $A, B \in F$ we have $A \cap B \in F$ and for each $A \in F$ and $B \supset A$, implies $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. A non-trivial ideal $I \subseteq P(S)$ is said to be (a) an *admissible ideal* on S if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in S\}$ (b) *maximal*, if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset. Recall that a sequence $\mathbf{x} = (x_k)$ of real numbers is said to be I-convergent to the number ℓ if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$.

We denote w for the set of all real sequences $x = (x_k)$. The difference sequence space was introduced by Kızmaz [14] as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\}, \tag{1.1}$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = \Delta^1 x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, where the standard notations ℓ_∞, c and c_0 are used to denote the set of bounded, convergent and null sequences, respectively. Later this idea was generalized

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by Et and Çolak [8] by considering Δ^n instead of Δ where $(\Delta^n x_k) = \Delta^1(\Delta^{n-1}x_k)$ for $n \geq 2$ (see also Et and Başarir [7]). In case of $n = 0$ we obtain x_k. The author of [24] generalized these spaces by taking Δ_m in (1.1) where the operator Δ_m is defined by $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$. By combining the above two operators Δ^n and Δ_m, Tripathy et al. [27] defined and studied Kızmaz spaces for the operator Δ^n_m and it is given by $\Delta^n_m x = (\Delta^n_m x_k) = (\Delta^{n-1}_m x_k - \Delta^{n-1}_m x_{k+m})$. In [6], Dutta considered $\Delta^n_{(m)} x = (\Delta^n_{(m)} x_k) = (\Delta^{n-1}_{(m)} x_k - \Delta^{n-1}_{(m)} x_{k-m})$ and introduced difference sequences spaces for the sets of bounded, statistically convergent and statistically null sequences, respectively. Başar and Altay [2] introduced the generalized difference matrix $B(r,s) = (b_{nk}(r,s))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$b_{nk}(r,s) = \begin{cases} r, & \text{if } k = n; \\ s, & \text{if } k = n-1; \\ 0, & \text{if } 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N}, r, s \in \mathbb{R} - \{0\}$. Başarir and Kayikci [3] have defined the generalized difference matrix B^n of order n, and the binomial representation of this operator is

$$B^n x_k = \sum_{v=0}^{n} \binom{n}{v} r^{n-v} s^v x_{k-v},$$

where $r, s \in \mathbb{R} - \{0\}$ and $n \in \mathbb{N}$. Another generalization of above difference matrix was given by Başarir et al. [4] as $B^n_{(m)}$ by taking into account operator introduced by Dutta [6], where $B^n_{(m)} x = (B^n_{(m)} x_k) = (rB^{n-1}_{(m)} x_k + sB^{n-1}_{(m)} x_{k-m})$ and $B^0_{(m)} x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B^n_{(m)} x_k = \sum_{v=0}^{n} \binom{n}{v} r^{n-v} s^v x_{k-mv}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(0) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. It is well known if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$. An Orlicz function M is said to be satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$(see Krasnoselskii and Rutitsky [16]).

Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some} \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(t) = |t|^p$ for $1 \leq p < \infty$.

For some recent work related to Orlicz sequence spaces, we refer to Alotaibi et al. [1], Mohiuddine et al. [18, 19], Savaş [23] and references therein.

If X is a linear space and $g : X \to \mathbb{R}$ is such that (i) $g(x) \geq 0$, (ii) $x = 0 \Rightarrow g(x) = 0$, (iii) $g(x + y) \leq g(x) + g(y)$, (iv) $g(-x) = g(x)$ and (v) $g(t_k x_k - tx) \to 0$ as $k \to \infty$ whenever $t_k \to t$ and $x_k \to x$ as $k \to \infty$ for scalars t_k, t and the vectors x_k, x, then g is said to be a *paranorm* on X and the pair (X, g) is called a *paranormed space*. A paranorm g which satifies $g(x) = 0 \Rightarrow x = 0$ is called a *total paranorm*.

A sequence space E is said to be (i) *normal* (or *solid*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, (ii) *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A *K-step space* of E is a sequence space $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$. A canonical preimage of a sequence $\{(x_{k_n})\} \in \lambda_K^E$ is a sequence $\{y_k\} \in w$ defined as

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E. A sequence space E is said to be *monotone* if E contains the cannical pre-image of all its step spaces. Note that every normal space is monotone (see [13], page 53).

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H, D = \max\{1, 2^{H-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \text{ for all } k \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{C}.$$

Also $|a|^{p_k} \leq \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

2. Main Results

We introduce the following new type of ideal convergent sequence spaces using the generalized difference matrix $B_{(m)}^n$ and Orlicz functions. Let M be an Orlicz function, and $p = (p_k)$ be a sequence of positive real numbers and m, n be nonnegative integers. Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{i+1} \leq \lambda_i + 1$, $\lambda_1 = 1$ (such type of sequence also used in [20] to define summability methods). For $\rho > 0$, we define the following new sequence spaces:

$$c_0^I(\lambda, M, B_{(m)}^n, p) = \left\{(u_k) \in w : \left\{i \in \mathbb{N} : \tfrac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\tfrac{|B_{(m)}^n u_k|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I\right\},$$

$$c^I(\lambda, M, B_{(m)}^n, p) = \left\{(u_k) \in w : \left\{i \in \mathbb{N} : \tfrac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\tfrac{|B_{(m)}^n u_k - u_0|}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{ for some } u_0 \in \mathbb{R}\right\},$$

$$\ell_\infty(\lambda, M, B_{(m)}^n, p) = \left\{(u_k) \in w : \sup_i \tfrac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\tfrac{|B_{(m)}^n u_k|}{\rho}\right)\right]^{p_k} < \infty\right\},$$

where $J_i = [i - \lambda_i + 1, i]$. It is easy to see that the inclusions $c_0^I(\lambda, M, B_{(m)}^n, p) \subset c^I(\lambda, M, B_{(m)}^n, p) \subset \ell_\infty(\lambda, M, B_{(m)}^n, p)$ are proper. We can write the following spaces by using the above spaces

$$m^I(\lambda, M, B_{(m)}^n, p) = c^I(\lambda, M, B_{(m)}^n, p) \cap \ell_\infty(\lambda, M, B_{(m)}^n, p)$$

and

$$m_0^I(\lambda, M, B_{(m)}^n, p) = c_0^I(\lambda, M, B_{(m)}^n, p) \cap \ell_\infty(\lambda, M, B_{(m)}^n, p).$$

Particular cases: For $n = 0$, the spaces $c^I(\lambda, M, B_{(m)}^n, p), c_0^I(\lambda, M, B_{(m)}^n, p)$, $\ell_\infty(\lambda, M, B_{(m)}^n, p)$, $m^I(\lambda, M, B_{(m)}^n, p)$ and $m_0^I(\lambda, M, B_{(m)}^n, p)$ becomes $c^I(\lambda, M, p)$, $c_0^I(\lambda, M, p)$, $\ell_\infty(\lambda, M, , p)$, $m^I(\lambda, M, p)$ and $m_0^I(\lambda, M, p)$ respectively.

The following is easy to prove.

Theorem 2.1. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. The spaces $c_0^I(\lambda, M, B_{(m)}^n, p), c^I(\lambda, M, B_{(m)}^n, p)$, $\ell_\infty(\lambda, M, B_{(m)}^n, p)$, $m^I(\lambda, M, B_{(m)}^n, p)$ and $m_0^I(\lambda, M, B_{(m)}^n, p)$ are linear.*

Theorem 2.2. *Let $p = (p_k) \in \ell_\infty$. Then $m^I(\lambda, M, B_{(m)}^n, p)$ and $m_0^I(\lambda, M, B_{(m)}^n, p)$ are paranormed spaces with the paranorm $g_{B_{(m)}^n}$ defined by*

$$g_{B_{(m)}^n}(u) = \inf\left\{\rho^{\frac{p_k}{G}} > 0 : \sup_i \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\frac{|B_{(m)}^n u_k|}{\rho}\right)\right] \leq 1, \textit{ for } \rho > 0\right\},$$

where $G = \max\{1, \sup_k p_k\}$.

Proof. Clearly $g_{B^n_{(m)}}(-u) = g_{B^n_{(m)}}(u)$ and $g_{B^n_{(m)}}(0) = 0$. Let $u = (u_k)$ and $v = (v_k)$ be two elements in $m_0^I(\lambda, M, B^n_{(m)}, p)$. Now for $\rho_1, \rho_2 > 0$ we put

$$A_1 = \left\{\rho_1 > 0 : \sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}u_k|}{\rho_1}\right)\right] \leq 1\right\} \text{ and } A_2 = \left\{\rho_2 > 0 : \sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}v_k|}{\rho_2}\right)\right] \leq 1\right\}.$$

Let us take $\rho = \rho_1 + \rho_2$. Then using the convexity of Orlicz function M, we obtain

$$M\left(\frac{|B^n_{(m)}(u_k + v_k)|}{\rho}\right) \leq \frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{|B^n_{(m)}u_k|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} M\left(\frac{|B^n_{(m)}v_k|}{\rho_2}\right)$$

which in turn gives us

$$\sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}(u_k + v_k)|}{\rho}\right)\right]^{p_k} \leq 1$$

and

$$\begin{aligned} g_{B^n_{(m)}}(u+v) &= \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{G}} : \rho_1 \in A_1, \rho_2 \in A_2\right\} \\ &\leq \inf\left\{\rho_1^{\frac{p_k}{G}} : \rho_1 \in A_1\right\} + \inf\left\{\rho_2^{\frac{p_k}{G}} : \rho_2 \in A_2\right\} = g_{B^n_{(m)}}(u) + g_{B^n_{(m)}}(v). \end{aligned}$$

Let $\alpha^i \to \alpha$, where $\alpha^i, \alpha \in \mathbb{R}$ and let $g_{B^n_{(m)}}(u^i - u) \to \infty$ as $i \to \infty$. To prove that $g_{B^n_{(m)}}(\alpha^i u^i - \alpha u) \to \infty$ as $i \to \infty$. We put

$$A_3 = \left\{\rho_m > 0 : \sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}u^i|}{\rho_m}\right)\right]^{p_k} \leq 1\right\} \text{ and } A_4 = \left\{\rho_l > 0 : \sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}(u^i - u)|}{\rho_l}\right)\right]^{p_k} \leq 1\right\}.$$

By the continuity of M we observe that

$$\begin{aligned} M\left(\frac{|B^n_{(m)}(\alpha^i u^i - \alpha u)|}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l}\right) &\leq M\left(\frac{|B^n_{(m)}(\alpha^i u^i - \alpha u^i)|}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l}\right) + M\left(\frac{|B^n_{(m)}(\alpha u^i - \alpha u)|}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l}\right) \\ &\leq \frac{|\alpha^i - \alpha|\rho_m}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l} M\left(\frac{|B^n_{(m)}u^i|}{\rho_m}\right) + \frac{|\alpha|\rho_l}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l} M\left(\frac{|B^n_{(m)}(u^i - u)|}{\rho_l}\right). \end{aligned}$$

From the last inequality it follows that

$$\sup_i \frac{1}{\lambda_i}\sum_{k\in J_i}\left[M\left(\frac{|B^n_{(m)}(\alpha^i u^i - \alpha u)|}{|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l}\right)\right] \leq 1$$

and consequently

$$\begin{aligned} g_{B^n_{(m)}}(\alpha^i u^i - \alpha u) &= \inf\left\{\left(|\alpha^i - \alpha|\rho_m + |\alpha|\rho_l\right)^{\frac{p_k}{G}} : \rho_m \in A_3, \rho_l \in A_4\right\} \\ &\leq |\alpha^i - \alpha|^{\frac{p_k}{G}} \inf\left\{(\rho_m)^{\frac{p_k}{G}} : \rho_m \in A_3\right\} + |\alpha|^{\frac{p_k}{G}} \inf\left\{(\rho_l)^{\frac{p_k}{G}} : \rho_l \in A_4\right\} \\ &\leq \max\left\{1, |\alpha^i - \alpha|^{\frac{p_k}{G}}\right\} g_{B^n_{(m)}}(u^i) + \max\left\{1, |\alpha|^{\frac{p_k}{G}}\right\} g_{B^n_{(m)}}(u^i - u). \end{aligned} \tag{1}$$

Hence by our assumption the right hand side of (1) tends to 0 as $i \to \infty$. This completes the proof of the theorem. □

Theorem 2.3. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $Z(\lambda, M_2, B^n_{(m)}, p) \subseteq Z(\lambda, M_1 \circ M_2, B^n_{(m)}, p)$,

(ii) $Z(\lambda, M_1, B^n_{(m)}, p) \cap Z(\lambda, M_2, B^n_{(m)}, p) \subseteq Z(\lambda, M_1 + M_2, B^n_{(m)}, p)$,

for $Z = c_0^I, c^I, m_0^I, m^I, \ell_\infty$.

Proof. (i) Let $u = (u_k) \in c^I(\lambda, M_2, B^n_{(m)}, p)$. For $\rho > 0$ we have

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M_2 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \text{ for every } \varepsilon > 0. \tag{2}$$

Let $\varepsilon > 0$ and choose α with $0 < \alpha < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \alpha$. We define

$$v_k = M_2 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right)$$

and consider

$$\lim_{k \in \mathbb{N}; 0 \leq v_k \leq \alpha} [M_1(v_k)]^{p_k} = \lim_{k \in \mathbb{N}; v_k \leq \alpha} [M_1(v_k)]^{p_k} + \lim_{k \in \mathbb{N}; v_k > \alpha} [M_1(v_k)]^{p_k}.$$

We have

$$\lim_{k \in \mathbb{N}; v_k \leq \alpha} [M_1(v_k)]^{p_k} \leq [M_1(2)]^H \lim_{k \in \mathbb{N}; v_k \leq \alpha} [v_k]^{p_k}, H = \sup_k p_k. \tag{3}$$

For the second summation (i.e.$v_k > \alpha$), we go through the following procedure. We have

$$v_k < \frac{v_k}{\alpha} < 1 + \frac{v_k}{\alpha}.$$

Since M_1 is non-decreasing and convex, it follows that

$$M_1(v_k) < M_1 \left(1 + \frac{v_k}{\alpha} \right) \leq \frac{1}{2} M_1(2) + \frac{1}{2} M_1 \left(\frac{2 v_k}{\alpha} \right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(v_k) < \frac{1}{2} K \frac{v_k}{\alpha} M_1(2) + \frac{1}{2} K \frac{v_k}{\alpha} M_1(2) = K \frac{v_k}{\alpha} M_1(2).$$

We get the following estimates:

$$\lim_{k \in \mathbb{N}; v_k > \alpha} [M_1(v_k)]^{p_k} \leq \max \left\{ 1, (K \alpha^{-1} M_1(2))^H \right\} \lim_{k \in \mathbb{N}; v_k > \alpha} [v_k]^{p_k}. \tag{4}$$

From (2), (3) and (4), it follows that $(u_k) \in c^I(\lambda, M_1.M_2, B^n_{(m)}, p)$. Hence $c^I(\lambda, M_2, B^n_{(m)}, p) \subseteq c^I(\lambda, M_1 \circ M_2, B^n_{(m)}, p)$.

(ii) Let $(u_k) \in c^I(\lambda, M_1, B^n_{(m)}, p) \cap c^I(\lambda, M_2, B^n_{(m)}, p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M_1 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \text{ and } \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M_2 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

The rest of the proof follows from the following relation:

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[(M_1 + M_2) \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$
$$\subseteq \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M_1 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \bigcup \left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M_2 \left(\frac{|B^n_{(m)} u_k - u_0|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}.$$

$\square$

We remark that if $M_2(x) = x$ and $M_1(x) = M(x)$ for all $x \in [0,\infty)$ in the above theorem then $Z(\lambda, B^n_{(m)}, p) \subseteq Z(\lambda, M, B^n_{(m)}, p)$ for $Z = c_0^I, c^I, m_0^I, m^I, \ell_\infty$, where I is an admissible ideal.

Theorem 2.4. *The spaces $m_0^I(\lambda, M, B^n_{(m)}, p)$ and $m^I(\lambda, M, B^n_{(m)}, p)$ are nowhere dense subsets of $\ell_\infty(\lambda, M, B^n_{(m)}, p)$.*

Proof. From Theorem 3 [25] it follows that $m_0^I(\lambda, M, B^n_{(m)}, p)$ and $m^I(\lambda, M, B^n_{(m)}, p)$ are closed subspaces of $\ell_\infty(\lambda, M, B^n_{(m)}, p)$. Since the inclusion relations $m_0^I(\lambda, M, B^n_{(m)}, p) \subset \ell_\infty(\lambda, M, B^n_{(m)}, p)$ and $m^I(\lambda, M, B^n_{(m)}, p) \subset \ell_\infty(\lambda, M, B^n_{(m)}, p)$ are strict, then the spaces $m_0^I(\lambda, M, B^n_{(m)}, p)$ and $m^I(\lambda, M, B^n_{(m)}, p)$ are nowhere dense subsets of $\ell_\infty(\lambda, M, B^n_{(m)}, p)$. □

Theorem 2.5. *The inclusions $Z(\lambda, M, B^{n-1}_{(m)}, p) \subseteq Z(\lambda, M, B^n_{(m)}, p)$ are strict for $n \geq 1$. In general $Z(\lambda, M, B^i_{(m)}, p) \subseteq Z(\lambda, M, B^n_{(m)}, p)$ for $i = 1, 2, ..., n-1$ and the inclusion is strict, for $Z = c_0^I, c^I, m_0^I, m^I, \ell_\infty$.*

Proof. Let $u = (u_k) \in c_0^I(\lambda, M, B^{n-1}_{(m)}, p)$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\frac{|B^{n-1}_{(m)} u_k|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Since M is non-decreasing and convex it follows that

$$\begin{aligned}
\left[M\left(\frac{|B^n_{(m)} u_k|}{2\rho} \right) \right]^{p_k} &\leq D\left[\frac{1}{2} M\left(\frac{|B^{n-1}_{(m)} u_k|}{\rho} \right) \right]^{p_k} + D\left[\frac{1}{2} M\left(\frac{|B^{n-1}_{(m)} u_{k+1}|}{\rho} \right) \right]^{p_k} \\
&\leq DK\left[M\left(\frac{|B^{n-1}_{(m)} u_k|}{\rho} \right) \right]^{p_k} + DK\left[M\left(\frac{|B^{n-1}_{(m)} u_{k+1}|}{\rho} \right) \right]^{p_k},
\end{aligned}$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$. Therefore, we obtain

$$\begin{aligned}
&\left\{ i \in \mathbb{N} : \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\frac{|B^n_{(m)} u_k|}{2\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \\
&\subseteq \left\{ i \in \mathbb{N} : DK \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\frac{|B^{n-1}_{(m)} u_k|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \bigcup \left\{ i \in \mathbb{N} : DK \frac{1}{\lambda_i} \sum_{k \in J_i} \left[M\left(\frac{|B^{n-1}_{(m)} u_{k+1}|}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.
\end{aligned}$$

Hence $(u_k) \in c_0^I(\lambda, M, B^n_{(m)}, p)$. The inclusion is strict follows from the following example.

Example 2.6. *Let $M(x) = x$ for all $x \in [0,\infty)$ and $(\lambda_i) = i$ for all $i \in \mathbb{N}$. Suppose also that $n = 5$, $m = 2$, $r = 1$, $s = -1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Let us define the sequence (u_k) by*

$$u_k = \begin{cases} k^3 + 2k + 1 & , \text{ if } k \text{ is even;} \\ 0 & , \text{ otherwise.} \end{cases}$$

Thus we have

$$B^4_{(2)} u_k = \sum_{\nu=0}^{4} \binom{4}{\nu} r^{4-\nu} s^\nu u_{k-2\nu},$$

which gives $B^4_{(2)} u_k = -64$. So, we have $B^5_{(2)} u_k = 0$. Therefore $(u_k) \in c_0^I(M, B^5_{(2)}, p)$ but $(u_k) \notin c_0^I(M, B^4_{(2)}, p)$.

This completes the proof of the result. □

Theorem 2.7. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(\lambda, M, B_{(m)}^n, p) \subseteq m_0^I(\lambda, M, B_{(m)}^n, q)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k}$ where $K \subseteq \mathbb{N}$ such that $K \notin I$.*

Proof. If we take $(v_k) = M\left(\frac{|B_{(m)}^n u_k|}{\rho}\right)$ for all $k \in \mathbb{N}$. Then the result follows from the Theorem 6, [25]. $\square$

Corollary 2.8. *Let (p_k) and (q_k) be two sequences of positive real numbers. Then $m_0^I(\lambda, M, B_{(m)}^n, p) = m_0^I(\lambda, M, B_{(m)}^n, q)$ if and only if $\liminf_{k\in K} \frac{p_k}{q_k}$ and $\liminf_{k\in K} \frac{q_k}{p_k} > 0$ where $K \subseteq \mathbb{N}$ such that $K \notin I$.*

The proof of the above result follows from the Corollary 7 in [25].

Theorem 2.9. *If I is not a maximal ideal and $I \neq I_f$, then the sequence spaces $c^I(\lambda, M, B_{(m)}^n, p)$ and $m^I(\lambda, M, B_{(m)}^n, p)$ are neither normal nor monotone, where I_f denotes the class of all finite subsets of $\mathbb{N}$.*

We prove this result with the help of following example.

Example 2.10. *Let $M(x) = x$ for all $x \in [0, \infty)$ and $(\lambda_i) = i$ for all $i \in \mathbb{N}$. Suppose also that $r = 1, s = -1, n = 1, m = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Taking $I = I_\delta$, where $I_\delta = \{A \subset \mathbb{N}$: asymptotic density of A (in symbol, $\delta(A)) = 0\}$ and note that I_δ is an ideal of $\mathbb{N}$. Define the sequence (u_k) by $u_k = k$ for all $k \in \mathbb{N}$. Let*

$$\alpha_k = \begin{cases} -1 & , \text{ if } k \text{ is even;} \\ 1 & , \text{ if } k \text{ is odd.} \end{cases}$$

Then we see that $(u_k) \in Z(\lambda, M, B_{(m)}^n, p)$ for $Z = c^I$. But $(\alpha_k u_k) \notin Z(\lambda, M, B_{(m)}^n, p)$ for $Z = c^I$. Therefore $c^I(\lambda, M, B_{(m)}^n, p)$ is not normal and hence not monotone. Similarly, we can show that $c_0^I(\lambda, M, B_{(m)}^n, p)$ and $m_0^I(\lambda, M, B_{(m)}^n, p)$ are neither normal nor monotone by considering $u_k = 3$ for all $k \in \mathbb{N}$.

Theorem 2.11. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $Z(\lambda, M, B_{(m)}^n, p)$ are not symmetric, where $Z = c_0^I, c^I, m_0^I, m^I$.*

We prove this result only for $c^I(\lambda, M, B_{(m)}^n, p)$ with the help of following example. The rest of the results follow similar way.

Example 2.12. *Let $M(x) = x$ for all $x \in [0, \infty)$ and $(\lambda_i) = i$ for all $i \in \mathbb{N}$. Suppose that $r = 1$, $s = -1$, $n = 1$, $m = 1$. Taking $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. Let us define a sequence (u_k) by*

$$u_k = \begin{cases} -2k + 1 & , \text{ if } k = i^2, i \in \mathbb{N} \\ 0 & , \text{ otherwise.} \end{cases}$$

Thus, we obtain $(u_k) \in c^I(\lambda, M, B_{(m)}^n, p)$. The rearrangement (v_k) of (u_k) defined as

$$v_k = \{u_1, u_4, u_2, u_9, u_3, u_{16}, u_5, u_{25}, u_6, ...\}.$$

This implies that $(v_k) \notin c^I(\lambda, M, B_{(m)}^n, p)$. Hence $c^I(\lambda, M, B_{(m)}^n, p)$ is not symmetric.

Theorem 2.13. *If I is an admissible ideal, then $c_0^I(\lambda, M, B_{(m)}^n, p)$, $c^I(\lambda, M, B_{(m)}^n, p)$ and $\ell_\infty(\lambda, M, B_{(m)}^n, p)$ are convex sets.*

The proof of the above theorem follows directly by using the convexity of Orlicz function.

Theorem 2.14. *If I is an admissible ideal, then the spaces $c_0^I(\lambda, M, B_{(m)}^n, p)$, $c^I(\lambda, M, B_{(m)}^n, p)$ and $\ell_\infty(\lambda, M, B_{(m)}^n, p)$ are topologically isomorphic with the spaces $c_0^I(\lambda, M, p)$, $c^I(\lambda, M, p)$ and $\ell_\infty(\lambda, M, p)$, respectively.*

Proof. Let us consider the mapping $T : Z(\lambda, M, B^n_{(m)}, p) \to Z(\lambda, M, p)$ defined by

$$Tu = v = (B^n_{(m)}u_k) \text{ for every } u = (u_k) \in Z(\lambda, M, B^n_{(m)}, p),$$

where $Z = c^I, c^I_0, \ell_\infty$. Clearly T is linear homeomorphism. □

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