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On Fixed Point Theorems for Contractive Mappings of Integral Type with w-distance

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Abstract. Three fixed point theorems for mappings satisfying contractive inequalities of integral type with w-distance in complete metric spaces are proved. Three examples are included. The results presented in this paper extend substantially some known results.

1. Introduction

In 1968, Kannan [8] extended the Banach contraction principle from continuous mappings to noncontinuous mappings and proved the following fixed point theorem.

Theorem 1.1. ([8]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \le c[d(x, Tx) + d(y, Ty)], \quad \forall x, y \in X, \tag{2.1}$$

where $c \in \left(0, \frac{1}{2}\right)$ is a constant. Then T has a unique fixed point in X.

In 1996, Kada et al. [7] used w-distance to generalize Caristi's fixed point theorem, Ekeland's ε-variational principle, Takahashi's nonconvex minimization theorem and to prove a fixed point theorem. In 2002, Branciari [4] introduced the concept of contractive mappings of integral type and obtained the following fixed point theorem, which extends the Banach contraction principle.

Theorem 1.2. ([4]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{2.2}$$

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where $c \in (0,1)$ is a constant and $\varphi : [0,+\infty) \to [0,+\infty)$ is Lebesgue integrable, summable on each compact subset of $[0,+\infty)$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Since then, a lot of fixed and common point theorems dealing with various contractive mappings of integral type in metric spaces, modular spaces and symmetric spaces have been established by many researchers, see, for example, [1]-[3], [5], [11]-[17] and the references cited therein. In particular, Rhoades [14] extended the result of Branciari and got fixed point theorems for more general contractive mappings of integral type in complete metric spaces, Vijayaraju et al. [17] proved a common fixed point theorem for a pair of mappings satisfying a general contractive condition of integral type in complete metric spaces, Altun et al. [2] showed common fixed point theorems of weakly compatible mappings satisfying a general contractive of integral type in complete metric spaces, Beygmohammadi et al. [3] discussed the existence of fixed points for mappings defined in complete modular spaces satisfying contractive inequality of integral type, Aliouche [1] gained a fixed point theorem using a general contractive condition of integral type in symmetric spaces.

Motivated by the results in [1]-[17], in this paper we introduce new mappings satisfying contractive conditions of integral type with w-distance and prove the existence, uniqueness and iterative approximations of fixed points for these mappings in complete metric spaces. Our results extend and improve the results due to Branciari [4] and Kannan [8]. Three nontrivial and illustrative examples are also furnished to support the results in this paper.

2. Preliminaries

Throughout this paper, we assume that $\mathbb{R} = (-\infty,+\infty)$, $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, and

$\Phi = \Big\{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\Big\}$.

Definition 2.1. *([7]) Let (X,d) be a metric space. A function $p : X \times X \to \mathbb{R}^+$ is called a w-distance in X if it satisfies the following*

(w_1) $p(x,z) \le p(x,y) + p(y,z)$, $\forall x,y,z \in X$;

(w_2) for each $x \in X$, a mapping $p(x,\cdot) : X \to \mathbb{R}^+$ is lower semi-continuous, that is, if $\{y_n\}_{n\in\mathbb{N}}$ is a sequence in X with $\lim_{n\to\infty} y_n = y \in X$, then $p(x,y) \le \liminf_{n\to\infty} p(x,y_n)$;

(w_3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

Example 2.2. *Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, k be a positive constant and $p : X \times X \to \mathbb{R}^+$ be defined by*

$$p(x,y) = y^k, \quad \forall x,y \in X.$$

Then p is a w-distance in X.

Proof. Let $x,y,z \in X$. It is clear that (w_2) holds and

$$p(x,z) = z^k \le y^k + z^k = p(x,y) + p(y,z),$$

that is, (w_1) holds. Let $\varepsilon > 0$ and put $\delta = \varepsilon^k$. Suppose that $p(z,x) \le \delta$ and $p(z,y) \le \delta$. It follows that

$$d(x,y) = |x-y| \le \max\{x,y\} \le \max\{\delta^{\frac{1}{k}}, \delta^{\frac{1}{k}}\} = \varepsilon,$$

which implies (w_3). □

Example 2.3. *Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d = |\cdot|$, $k \in \mathbb{R}^+$, m be a positive constant and $p : X \times X \to \mathbb{R}^+$ be defined by*

$$p(x,y) = |x|^k + |y|^m, \quad \forall x,y \in X.$$

Then p is a w-distance in X.

Proof. Let $x, y, z \in X$. It is clear that (w_2) holds and

$$p(x,z) = |x|^k + |z|^m \le |x|^k + |y|^m + |y|^k + |z|^m = p(x,y) + p(y,z),$$

which yields (w_1). Let $\varepsilon > 0$ and put $\delta = \left(\frac{\varepsilon}{2}\right)^m$. Suppose that $p(z,x) \le \delta$ and $p(z,y) \le \delta$. It follows that

$$d(x,y) = |x-y| \le |x| + |y| \le \delta^{\frac{1}{m}} + \delta^{\frac{1}{m}} = \varepsilon,$$

which implies (w_3). □

Recall that a self mapping T in a metric space (X,d) is called *orbitally continuous at $u \in X$* if $\lim_{n\to\infty} T^n x = u$, $x \in X$, implies that $\lim_{n\to\infty} TT^n x = Tu$. The mapping T is orbitally continuous in X if T is orbitally continuous at each $u \in X$.

The following lemmas play important roles in this paper.

Lemma 2.4. ([7]) *Let X be a metric space with metric d and let p be a w-distance in X. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in X, let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}^+$ converging to 0, and let $x, y, z \in X$, then the following hold:*

(a) If $p(x_n, y) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$;

(b) if $p(x_n, y_n) \le \alpha_n$ and $p(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ converges to z;

(c) if $p(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $n > m$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence;

(d) if $p(x, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Lemma 2.5. ([12]) *Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 2.6. ([12]) *Let $\varphi \in \Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then $\lim_{n\to\infty} \int_0^{r_n} \varphi(t)dt = 0$ if and only if $\lim_{n\to\infty} r_n = 0$.*

3. Main Results

In this section, we establish three fixed point theorems for three classes of contractive mappings of integral type with w-distance in complete metric spaces.

Theorem 3.1. *Let (X,d) be a complete metric space and let p be a w-distance in X. Assume that $T : X \to X$ satisfies that*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \le c \int_0^{p(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{3.1}$$

where $c \in [0,1)$ is a constant and $\varphi \in \Phi$. Then T has a unique fixed point $u \in X$, $p(u,u) = 0$, $\lim_{n\to\infty} p(T^n x_0, u) = 0$ and $\lim_{n\to\infty} T^n x_0 = u$ for each $x_0 \in X$.

Proof. Pick an arbitrary point x_0 in X and define $x_n = T^n x_0$ for each $n \in \mathbb{N}_0$. Now we consider the following two cases:

Case 1. Assume that $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It's easy to see that x_{n_0-1} is a fixed point of T, $x_n = x_{n_0-1}$ for each $n \ge n_0$ and $\lim_{n\to\infty} T^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$\begin{aligned}\int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt &= \int_0^{p(Tx_{n_0-1},Tx_{n_0-1})} \varphi(t)dt \le c \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt \\ &< \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence $p(x_{n_0-1}, x_{n_0-1}) = 0$, which yields that

$$\lim_{n\to\infty} p(x_n, x_{n_0-1}) = p(x_{n_0-1}, x_{n_0-1}) = 0;$$

Case 2. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Suppose that

$$p(x_{n_0-1}, x_{n_0}) = 0 \quad \text{for some} \quad n_0 \in \mathbb{N}. \tag{3.2}$$

In light of (3.1), (3.2) and $\varphi \in \Phi$, we infer that

$$0 \leq \int_0^{p(x_{n_0}, x_{n_0+1})} \varphi(t)dt = \int_0^{p(Tx_{n_0-1}, Tx_{n_0})} \varphi(t)dt \leq c\int_0^{p(x_{n_0-1}, x_{n_0})} \varphi(t)dt = 0,$$

which means that

$$\int_0^{p(x_{n_0}, x_{n_0+1})} \varphi(t)dt = 0,$$

which together with $\varphi \in \Phi$ gives that

$$p(x_{n_0}, x_{n_0+1}) = 0. \tag{3.3}$$

Note that (3.2), (3.3) and (w_1) ensure that

$$0 \leq p(x_{n_0-1}, x_{n_0+1}) \leq p(x_{n_0-1}, x_{n_0}) + p(x_{n_0}, x_{n_0+1}) = 0,$$

that is,

$$p(x_{n_0-1}, x_{n_0+1}) = 0. \tag{3.4}$$

It follows from (3.2), (3.4) and Lemma 2.4 that $x_{n_0} = x_{n_0+1}$, which is absurd and hence

$$p(x_{n-1}, x_n) > 0, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

In view of (3.1), (3.5) and $\varphi \in \Phi$, we deduce that

$$\begin{aligned}\int_0^{p(x_n, x_{n+1})} \varphi(t)dt &= \int_0^{p(Tx_{n-1}, Tx_n)} \varphi(t)dt \leq c\int_0^{p(x_{n-1}, x_n)} \varphi(t)dt \\ &< \int_0^{p(x_{n-1}, x_n)} \varphi(t)dt, \quad \forall n \in \mathbb{N},\end{aligned}$$

which together with (3.5) implies that

$$0 < p(x_n, x_{n+1}) < p(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{3.6}$$

Note that (3.6) yields that the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}_0}$ is positive and strictly decreasing. Thus there exists a constant $v \geq 0$ with

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = v. \tag{3.7}$$

Suppose that $v > 0$. By means of (3.1), (3.7), $\varphi \in \Phi$ and Lemma 2.5, we conclude that

$$\begin{aligned}\int_0^{v} \varphi(t)dt &= \lim_{n\to\infty}\int_0^{p(x_n, x_{n+1})} \varphi(t)dt = \lim_{n\to\infty}\int_0^{p(Tx_{n-1}, Tx_n)} \varphi(t)dt \\ &\leq \lim_{n\to\infty} c\int_0^{p(x_{n-1}, x_n)} \varphi(t)dt = c\int_0^{v} \varphi(t)dt \\ &< \int_0^{v} \varphi(t)dt,\end{aligned}$$

which is impossible and hence $v = 0$, that is,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.8}$$

Similarly, we get that

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0. \tag{3.9}$$

Now we show that

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{3.10}$$

Otherwise there is a constant $\varepsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$p(x_{n(k)}, x_{m(k)}) > \varepsilon.$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying the above inequality. It follows that

$$p(x_{n(k)}, x_{m(k)}) > \varepsilon \quad \text{and} \quad p(x_{n(k)}, x_{m(k)-1}) \le \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.11}$$

Note that

$$\begin{aligned}
\varepsilon &< p(x_{n(k)}, x_{m(k)}) \\
&\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\
&\le p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) + \varepsilon + p(x_{m(k)-1}, x_{m(k)}), \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.12}$$

Letting $k \to \infty$ in (3.12) and using (3.8), (3.9) and (3.11), we know that

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} p(x_{n(k)-1}, x_{m(k)-1}) = \varepsilon. \tag{3.13}$$

By virtue of (3.1), (3.13), $\varphi \in \Phi$ and Lemma 2.5, we deduce that

$$\begin{aligned}
\int_0^{\varepsilon} \varphi(t)dt &= \lim_{k\to\infty} \int_0^{p(x_{n(k)}, x_{m(k)})} \varphi(t)dt = \lim_{k\to\infty} \int_0^{p(Tx_{n(k)-1}, Tx_{m(k)-1})} \varphi(t)dt \\
&\le \lim_{k\to\infty} c \int_0^{p(x_{n(k)-1}, x_{m(k)-1})} \varphi(t)dt = c \int_0^{\varepsilon} \varphi(t)dt \\
&< \int_0^{\varepsilon} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Thus, (3.10) holds.

Let $\varepsilon > 0$ and δ denote the number in (w_3). It follows from (3.10) that there exists $N \in \mathbb{N}$ satisfying

$$p(x_N, x_n) < \delta \quad \text{and} \quad p(x_N, x_m) < \delta, \quad \forall n, m > N,$$

which together with (w_3) yields that

$$d(x_n, x_m) < \varepsilon, \quad \forall n, m > N,$$

that is, $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Since (X, d) is a complete metric space, it follows that there exists a point $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Observe that (3.10) guarantees that for each $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ satisfying

$$0 \le p(x_n, x_m) < \varepsilon, \quad \forall n, m \ge N_\varepsilon,$$

which together with (w_2) and $\lim_{n\to\infty} x_n = u$ yields that

$$0 \le p(x_n, u) \le \liminf_{m\to\infty} p(x_n, x_m) \le \varepsilon, \quad \forall n \ge N_\varepsilon,$$

which gives that

$$\lim_{n\to\infty} p(x_n, u) = 0. \tag{3.14}$$

Making use of (3.1), (3.14), $\varphi \in \Phi$ and Lemma 2.5, we obtain that

$$0 \le \int_0^{p(Tx_n,Tu)} \varphi(t)dt \le c \int_0^{p(x_n,u)} \varphi(t)dt \to 0 \quad \text{as} \quad n \to \infty,$$

that is,

$$\lim_{n\to\infty} \int_0^{p(Tx_n,Tu)} \varphi(t)dt = 0,$$

which together with Lemma 2.6 means that

$$\lim_{n\to\infty} p(x_{n+1}, Tu) = \lim_{n\to\infty} p(Tx_n, Tu) = 0,$$

which together with (w_1) and (3.8) yields that

$$0 \le p(x_n, Tu) \le p(x_n, x_{n+1}) + p(x_{n+1}, Tu) \to 0 \quad \text{as} \quad n \to \infty,$$

that is,

$$\lim_{n\to\infty} p(x_n, Tu) = 0. \tag{3.15}$$

Combining (3.14) and (3.15) and using Lemma 2.4, we derive at $u = Tu$.

Next we show that $p(u, u) = 0$. Suppose that $p(u, u) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$0 < \int_0^{p(u,u)} \varphi(t)dt = \int_0^{p(Tu,Tu)} \varphi(t)dt \le c \int_0^{p(u,u)} \varphi(t)dt < \int_0^{p(u,u)} \varphi(t)dt, \tag{3.16}$$

which is impossible. That is, $p(u, u) = 0$.

Finally, we show that T possesses a unique fixed point in X. Suppose that α and β are two fixed points of T in X. Similar to the proof of (3.16), we infer easily that $p(\alpha, \alpha) = p(\beta, \beta) = 0$. Suppose that $p(\beta, \alpha) > 0$. It follows from (3.1) and $\varphi \in \Phi$ that

$$0 < \int_0^{p(\beta,\alpha)} \varphi(t)dt = \int_0^{p(T\beta,T\alpha)} \varphi(t)dt \le c \int_0^{p(\beta,\alpha)} \varphi(t)dt < \int_0^{p(\beta,\alpha)} \varphi(t)dt,$$

which is absurd. Consequently $p(\beta, \alpha) = 0$, which together with $p(\beta, \beta) = 0$ and Lemma 2.4 that $\beta = \alpha$. This completes the proof. □

Theorem 3.2. *Let (X, d) be a complete metric space and let p be a w-distance in X. Assume that $T : X \to X$ satisfies that*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt \le a \int_0^{p(Tx,x)} \varphi(t)dt + b \int_0^{p(Ty,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{3.17}$$

where $\varphi \in \Phi$ and

$$a \quad \text{and} \quad b \quad \text{are nonnegative and} \quad a + b < 1. \tag{3.18}$$

Then T has a unique fixed point $u \in X$, $p(u, u) = 0$, $\lim_{n\to\infty} p(T^n x_0, u) = 0$ and $\lim_{n\to\infty} T^n x_0 = u$ for each $x_0 \in X$.

Proof. Let x_0 be an arbitrary point in X and define $x_n = T^n x_0$ for each $n \in \mathbb{N}_0$. Now we consider the following two cases:

Case 1. Assume that $x_{n_0} = x_{n_0-1}$ for some $n_0 \in \mathbb{N}$. It's easy to see that x_{n_0-1} is a fixed point of T, $x_n = x_{n_0-1}$ for each $n \geq n_0$ and $\lim_{n\to\infty} T^n x_0 = x_{n_0-1}$. Suppose that $p(x_{n_0-1}, x_{n_0-1}) > 0$. It follows from (3.17), (3.18) and $\varphi \in \Phi$ that

$$\begin{aligned}
0 < \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt &= \int_0^{p(Tx_{n_0-1},Tx_{n_0-1})} \varphi(t)dt \\
&\leq a \int_0^{p(Tx_{n_0-1},x_{n_0-1})} \varphi(t)dt + b \int_0^{p(Tx_{n_0-1},x_{n_0-1})} \varphi(t)dt \\
&= (a+b) \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt \\
&< \int_0^{p(x_{n_0-1},x_{n_0-1})} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $p(x_{n_0-1}, x_{n_0-1}) = 0$, which yields that

$$\lim_{n\to\infty} p(x_n, x_{n_0-1}) = p(x_{n_0-1}, x_{n_0-1}) = 0;$$

Case 2. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. In terms of (3.17), we obtain that

$$\begin{aligned}
\int_0^{p(x_{n+1},x_n)} \varphi(t)dt &= \int_0^{p(Tx_n,Tx_{n-1})} \varphi(t)dt \\
&\leq a \int_0^{p(Tx_n,x_n)} \varphi(t)dt + b \int_0^{p(Tx_{n-1},x_{n-1})} \varphi(t)dt \\
&= a \int_0^{p(x_{n+1},x_n)} \varphi(t)dt + b \int_0^{p(x_n,x_{n-1})} \varphi(t)dt, \quad \forall n \in \mathbb{N},
\end{aligned}$$

which yields that

$$\int_0^{p(x_{n+1},x_n)} \varphi(t)dt \leq \frac{b}{1-a} \int_0^{p(x_n,x_{n-1})} \varphi(t)dt, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

Suppose that there exists some $n_0 \in \mathbb{N}$ with

$$p(x_{n_0}, x_{n_0-1}) = 0. \tag{3.20}$$

It follows from (3.19), (3.20) and $\varphi \in \Phi$ that

$$0 \leq \int_0^{p(x_{n_0+1},x_{n_0})} \varphi(t)dt \leq \frac{b}{1-a} \int_0^{p(x_{n_0},x_{n_0-1})} \varphi(t)dt = 0,$$

which together with Lemmas 2.6 yields that

$$p(x_{n_0+1}, x_{n_0}) = 0. \tag{3.21}$$

Linking (3.20), (3.21) and (w_1), we infer that

$$0 \leq p(x_{n_0+1}, x_{n_0-1}) \leq p(x_{n_0+1}, x_{n_0}) + p(x_{n_0}, x_{n_0-1}) = 0,$$

that is,

$$p(x_{n_0+1}, x_{n_0-1}) = 0. \tag{3.22}$$

Using (3.21), (3.22) and Lemma 2.4, we know that $x_{n_0} = x_{n_0-1}$, which is impossible. Consequently, we get that

$$p(x_n, x_{n-1}) > 0, \quad \forall n \in \mathbb{N}. \tag{3.23}$$

In view of (3.18), (3.19), (3.23) and $\varphi \in \Phi$, we deduce that

$$\begin{aligned} 0 < \int_0^{p(x_{n+1},x_n)} \varphi(t)dt &\le \frac{b}{1-a} \int_0^{p(x_n,x_{n-1})} \varphi(t)dt \\ &\le \left(\frac{b}{1-a}\right)^2 \int_0^{p(x_{n-1},x_{n-2})} \varphi(t)dt \le \cdots \\ &\le \left(\frac{b}{1-a}\right)^n \int_0^{p(x_1,x_0)} \varphi(t)dt \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which together with Lemma 2.6 yields that (3.9) holds.

Now we show that

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0. \tag{3.24}$$

Otherwise there is a constant $\varepsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$p(x_{m(k)}, x_{n(k)}) > \varepsilon.$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying the above inequality. It is clear that

$$p(x_{m(k)}, x_{n(k)}) > \varepsilon \quad \text{and} \quad p(x_{m(k)-1}, x_{n(k)}) \le \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.25}$$

Note that (3.9) and (3.25) yield that

$$\begin{aligned} \varepsilon &< p(x_{m(k)}, x_{n(k)}) \\ &\le p(x_{m(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{n(k)}) \\ &\le p(x_{m(k)}, x_{m(k)-1}) + \varepsilon \to \varepsilon \quad \text{as} \quad k \to \infty, \end{aligned}$$

that is,

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{3.26}$$

Making use of (3.9), (3.26), $\varphi \in \Phi$ and Lemma 2.5, we acquire that

$$\begin{aligned} 0 &< \int_0^{\varepsilon} \varphi(t)dt \\ &= \lim_{k\to\infty} \int_0^{p(x_{m(k)},x_{n(k)})} \varphi(t)dt \\ &= \lim_{k\to\infty} \int_0^{p(Tx_{m(k)-1},Tx_{n(k)-1})} \varphi(t)dt \\ &\le \lim_{k\to\infty} \left(a \int_0^{p(Tx_{m(k)-1},x_{m(k)-1})} \varphi(t)dt + b \int_0^{p(Tx_{n(k)-1},x_{n(k)-1})} \varphi(t)dt \right) \\ &= a \lim_{k\to\infty} \int_0^{p(x_{m(k)},x_{m(k)-1})} \varphi(t)dt + b \lim_{k\to\infty} \int_0^{p(x_{n(k)},x_{n(k)-1})} \varphi(t)dt \\ &= 0, \end{aligned}$$

which is a contradiction. Thus (3.24) holds. As in the proof of Theorem 3.1, we conclude that there exists some $u \in X$ satisfying (3.14) and $\lim_{n\to\infty} x_n = u$, which together with (w_2) gives that

$$p(Tu, u) \le \liminf_{n\to\infty} p(Tu, x_n). \tag{3.27}$$

Clearly there exists a subsequent $\{x_{n_i}\}_{i\in\mathbb{N}} \subseteq \{x_n\}_{n\in\mathbb{N}_0}$ satisfying

$$\liminf_{n\to\infty} p(Tu, x_n) = \lim_{i\to\infty} p(Tu, x_{n_i}). \tag{3.28}$$

By means of (3.9), (3.17), (3.18), (3.27), (3.28) and Lemma 2.5, we deduce that

$$\begin{aligned}\int_0^{p(Tu,u)}\varphi(t)dt &\le \lim_{i\to\infty}\int_0^{p(Tu,x_{n_i})}\varphi(t)dt\\ &\le \lim_{i\to\infty}\left(a\int_0^{p(Tu,u)}\varphi(t)dt + b\int_0^{p(Tx_{n_i-1},x_{n_i-1})}\varphi(t)dt\right)\\ &= a\int_0^{p(Tu,u)}\varphi(t)dt + b\lim_{i\to\infty}\int_0^{p(x_{n_i},x_{n_i-1})}\varphi(t)dt\\ &= a\int_0^{p(Tu,u)}\varphi(t)dt,\end{aligned}$$

which yields that

$$(1-a)\int_0^{p(Tu,u)}\varphi(t)dt \le 0,$$

which together with (3.18) implies that

$$\int_0^{p(Tu,u)}\varphi(t)dt = 0,$$

that is,

$$p(Tu,u) = 0. \tag{3.29}$$

By virtue of (3.17), (3.18), (3.29) and $\varphi \in \Phi$, we gain that

$$0 \le \int_0^{p(Tu,Tu)}\varphi(t)dt \le a\int_0^{p(Tu,u)}\varphi(t)dt + b\int_0^{p(Tu,u)}\varphi(t)dt = 0,$$

which ensures that

$$p(Tu,Tu) = 0. \tag{3.30}$$

It follows from (3.29), (3.30) and Lemma 2.4 that $u = Tu$ and $p(u,u) = 0$.

Finally, we show that T possesses a unique fixed point in X. Suppose that α and β are two fixed points of T in X. In light of (3.17), (3.18) and $\varphi \in \Phi$, we conclude that

$$\begin{aligned}\int_0^{p(\alpha,\alpha)}\varphi(t)dt &= \int_0^{p(T\alpha,T\alpha)}\varphi(t)dt\\ &\le a\int_0^{p(T\alpha,\alpha)}\varphi(t)dt + b\int_0^{p(T\alpha,\alpha)}\varphi(t)dt\\ &= (a+b)\int_0^{p(\alpha,\alpha)}\varphi(t)dt,\end{aligned}$$

which gives that

$$0 \le (1-a-b)\int_0^{p(\alpha,\alpha)}\varphi(t)dt \le 0,$$

that is,

$$\int_0^{p(\alpha,\alpha)}\varphi(t)dt = 0,$$

which yields that

$$p(\alpha,\alpha) = 0. \tag{3.31}$$

Similarly we infer also that $p(\beta,\beta)=0$. It follows from (3.17), (3.18) and $\varphi\in\Phi$ that

$$\begin{aligned}0 \le \int_0^{p(\alpha,\beta)}\varphi(t)dt &= \int_0^{p(T\alpha,T\beta)}\varphi(t)dt\\ &\le a\int_0^{p(T\alpha,\alpha)}\varphi(t)dt + b\int_0^{p(T\beta,\beta)}\varphi(t)dt\\ &= 0,\end{aligned}$$

which implies that

$$p(\alpha,\beta)=0. \tag{3.32}$$

On account of (3.31), (3.32), $\varphi\in\Phi$ and Lemmas 2.4, we deduce that $\alpha=\beta$. This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete metric space and let p be a w-distance in X. Assume that $T : X\to X$ is an orbitally continuous mapping satisfying*

$$\int_0^{p(Tx,Ty)}\varphi(t)dt \le a\int_0^{p(x,Tx)}\varphi(t)dt + b\int_0^{p(y,Ty)}\varphi(t)dt,\quad \forall x,y\in X, \tag{3.33}$$

where $\varphi\in\Phi$ and (3.18) *holds. Then T has a unique fixed point $u\in X$, $p(u,u)=0$, $\lim_{n\to\infty}p(T^nx_0,u)=0$ and $\lim_{n\to\infty}T^nx_0=u$ for each $x_0\in X$.*

Proof. Let x_0 be an arbitrary point in X and define $x_n=T^nx_0$ for each $n\in\mathbb{N}_0$. Without loss of generality we assume that $x_n\neq x_{n-1}$ for all $n\in\mathbb{N}$. Similar to the proofs of Theorem 3.1 and 3.2, we conclude that (3.10) holds and there exists some $u\in X$ satisfying $\lim_{n\to\infty}x_n=u$ and (3.14). Since T is orbitally continuous, it follows that

$$Tu=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=u.$$

It follows from (3.18), (3.33) and $\varphi\in\Phi$ that

$$\begin{aligned}\int_0^{p(u,u)}\varphi(t)dt &= \int_0^{p(Tu,Tu)}\varphi(t)dt\\ &\le a\int_0^{p(u,Tu)}\varphi(t)dt + b\int_0^{p(u,Tu)}\varphi(t)dt\\ &= (a+b)\int_0^{p(u,u)}\varphi(t)dt,\end{aligned}$$

which implies that

$$0\le(1-a-b)\int_0^{p(u,u)}\varphi(t)dt\le 0,$$

which together with (3.18) means that

$$\int_0^{p(u,u)}\varphi(t)dt=0,$$

that is, $p(u,u)=0$. The rest of the proof is similar to that of Theorem 3.2 and is omitted. This completes the proof. □

Problem 3.4. *If the condition that $T : X\to X$ is an orbitally continuous mapping in Theorem* 3.3 *is removed, and other conditions of Theorem* 3.3 *do not change, the conclusions of Theorem* 3.3 *hold ?*

4. Remarks and Examples

In this section, we construct three nontrivial examples to compare the fixed point theorems obtained in Section 3 with the known results in Section 1.

Remark 4.1. *In case $p(x, y) = d(x, y)$ for all $x, y \in X$, then Theorem 3.1 reduces to Theorem 1.2. The following example reveals that Theorem 3.1 extends substantially Theorem 1.2.*

Example 4.2. *Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $T : X \to X$ be defined by*

$$p(x, y) = \sqrt{y}, \quad \forall x, y \in X, \quad \varphi(t) = 4t^3, \quad \forall t \in \mathbb{R}^+$$

and

$$Tx = \begin{cases} 0, & \forall x \in [0, 1] \\ \frac{x}{2}, & \forall x \in (1, +\infty). \end{cases}$$

Put $c = \frac{1}{2}$. It is clear that p is w-distance in X and $\varphi \in \Phi$. In order to verify (3.1)*, we have to consider two possible cases as follows:*

Case 1. Let $x \in X$ and $y \in [0, 1]$. It follows that

$$\int_0^{p(Tx,Ty)} \varphi(t)dt = 0 \le c \int_0^{p(x,y)} \varphi(t)dt;$$

Case 2. Let $x \in X$ and $y \in (1, +\infty)$. Note that

$$\begin{aligned} \int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\sqrt{\frac{y}{2}}} 4t^3 dt = \frac{y^2}{4} \le \frac{y^2}{2} \\ &= \frac{1}{2} \int_0^{\sqrt{y}} 4t^3 dt = c \int_0^{p(x,y)} \varphi(t)dt. \end{aligned}$$

Hence (3.1) *holds. Thus the conditions of Theorem 3.1 are satisfied. It follows from Theorem 3.1 that T has a unique fixed point in X.*

Now we show that the mapping T does not satisfy the conditions of Theorem 1.2. Otherwise, there exist $c \in (0, 1)$ and $\varphi \in \Phi$ satisfying (1.2)*. It follows from* (1.2)*, $\varphi \in \Phi$ and Lemma 2.5 that*

$$\begin{aligned} 0 < \int_0^{\frac{1}{2}} \varphi(t)dt &= \lim_{y \to 1^+} \int_0^{|0-\frac{y}{2}|} \varphi(t)dt = \lim_{y \to 1^+} \int_0^{d(T1,Ty)} \varphi(t)dt \\ &\le c \lim_{y \to 1^+} \int_0^{d(1,y)} \varphi(t)dt = c \lim_{y \to 1^+} \int_0^{|1-y|} \varphi(t)dt \\ &= 0, \end{aligned}$$

which is impossible.

Remark 4.3. *In case $p(x, y) = d(x, y)$ for all $x, y \in X$ and $\varphi(t) = 1$ for all $t \in \mathbb{R}^+$, then Theorem 3.2 reduces to Theorem 1.1. The following example proves that Theorem 3.2 generalizes indeed Theorem 1.1.*

Example 4.4. *Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $T : X \to X$ be defined by*

$$p(x, y) = y^2, \quad \forall x, y \in X, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+$$

and

$$Tx = \begin{cases} 0, & \forall x \in [0, 1) \\ \frac{\sqrt{x}}{2+x^3}, & \forall x \in [1, +\infty). \end{cases}$$

Put $a = \frac{1}{2}$ and $b = \frac{1}{16}$. It is clear that p is w-distance in X, $\varphi \in \Phi$ and (3.18) *holds. In order to verify* (3.17)*, we have to consider two possible cases as follows:*

Case 1. Let $x \in X$ and $y \in [0,1)$. It is clear that

$$\int_0^{p(Tx,Ty)} \varphi(t)dt = 0 \leq a \int_0^{p(Tx,x)} \varphi(t)dt + b \int_0^{p(Ty,y)} \varphi(t)dt;$$

Case 2. Let $x \in X$ and $y \in [1,+\infty)$. Note that

$$\begin{aligned}\int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\left(\frac{\sqrt{y}}{2+y^3}\right)^2} 2tdt = \frac{y^2}{(2+y^3)^4} \leq \frac{y^2}{16} \leq \frac{y^4}{16} \\ &\leq a \int_0^{x^2} 2tdt + b \int_0^{y^2} 2tdt \\ &= a \int_0^{p(Tx,x)} \varphi(t)dt + b \int_0^{p(Ty,y)} \varphi(t)dt.\end{aligned}$$

That is, (3.17) *holds. Hence the conditions of Theorem 3.2 are satisfied. It follows from Theorem 3.2 that T has a unique fixed point in X.*

However we cannot use Theorem 1.1 to prove the existence of fixed points of the mapping T in X. Otherwise, there exists $c \in \left(0, \frac{1}{2}\right)$ satisfying (1.1)*. It follows that*

$$\frac{1}{3} = d\left(0, \frac{1}{3}\right) = d(T0, T1) \leq c(d(0, T0) + d(1, T1)) = \frac{2c}{3},$$

which together with $c \in \left(0, \frac{1}{2}\right)$ yields that

$$\frac{1}{2} > c \geq \frac{1}{2},$$

which is impossible.

The following example is an application of Theorem 3.3 and shows that Theorem 3.3 differs from Theorem 1.2.

Example 4.5. *Let $X = [0,2]$ be endowed with the Euclidean metric $d = |\cdot|$, $p : X \times X \to \mathbb{R}^+$, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and $T : X \to X$ be defined by*

$$p(x,y) = x + y, \quad \forall x, y \in X, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+$$

and

$$Tx = \begin{cases} \frac{x}{4}, & \forall x \in [0,1] \\ 0, & \forall x \in (1,2]. \end{cases}$$

Put $a = b = \frac{1}{5}$. It is obvious that p is w-distance in X, $\varphi \in \Phi$ and (3.18) *holds. Observe that*

$$T^n x = \begin{cases} \frac{x}{4^n}, & \forall x \in [0,1],\ n \in \mathbb{N} \\ 0, & \forall x \in (1,2],\ n \in \mathbb{N}, \end{cases}$$

which implies that T is orbitally continuous in X. In order to verify (3.33)*, we have to consider four possible cases as follows:*

Case 1. Let $x, y \in [0,1]$*. It is clear that*

$$\begin{aligned}\int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\frac{x+y}{4}} 2tdt = \frac{(x+y)^2}{16} \le \frac{5(x^2+y^2)}{16} \\ &= a\int_0^{(x+\frac{x}{4})} 2tdt + b\int_0^{(y+\frac{y}{4})} 2tdt \\ &= a\int_0^{p(x,Tx)} \varphi(t)dt + b\int_0^{p(y,Ty)} \varphi(t)dt;\end{aligned}$$

Case 2. Let $x, y \in (1,2]$*. Note that*

$$\int_0^{p(Tx,Ty)} \varphi(t)dt = 0 \le a\int_0^{p(x,Tx)} \varphi(t)dt + b\int_0^{p(y,Ty)} \varphi(t)dt;$$

Case 3. Let $x \in [0,1]$ *and* $y \in [2,3]$*. Note that*

$$\begin{aligned}\int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\frac{x}{4}} 2tdt = \frac{x^2}{16} \le \frac{5x^2}{16} = a\int_0^{p(x,Tx)} \varphi(t)dt \\ &\le a\int_0^{p(x,Tx)} \varphi(t)dt + b\int_0^{p(y,Ty)} \varphi(t)dt;\end{aligned}$$

Case 4. Let $x \in [2,3]$ *and* $y \in [0,1]$*. It is clear that*

$$\begin{aligned}\int_0^{p(Tx,Ty)} \varphi(t)dt &= \int_0^{\frac{y}{4}} 2tdt = \frac{y^2}{16} \le \frac{5y^2}{16} = b\int_0^{p(y,Ty)} \varphi(t)dt \\ &\le a\int_0^{p(x,Tx)} \varphi(t)dt + b\int_0^{p(y,Ty)} \varphi(t)dt,\end{aligned}$$

that is, (3.33) *holds. Thus the conditions of Theorem 3.3 are satisfied. It follows from Theorem 3.3 that T has a unique fixed point in X.*

However, we claim that Theorem 1.2 is unapplicable in proving the existence of fixed points of T in X. Suppose that there exist $c \in (0,1)$ *and* $\varphi \in \Phi$ *satisfying* (1.2)*. It follows from* (1.2)*,* $\varphi \in \Phi$ *and Lemma 2.5 that*

$$\begin{aligned}0 < \int_0^{\frac{1}{4}} \varphi(t)dt &= \lim_{y\to 1^+} \int_0^{d(T1,Ty)} \varphi(t)dt \le \lim_{y\to 1^+} c\int_0^{d(1,y)} \varphi(t)dt \\ &= \lim_{y\to 1^+} c\int_0^{y-1} \varphi(t)dt = 0,\end{aligned}$$

which is a contradiction.

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