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An Application of Power Increasing Sequences to Infinite Series and Fourier Series

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Abstract. In this paper, we proved a known theorem under more weaker conditions dealing with absolute Riesz summability of infinite series involving a quasi-σ-power increasing sequence. And we applied it to the trigonometric Fourier series.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence c_n and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A positive sequence (X_n) is said to be quasi-σ-power increasing sequence if there exists a constant $K = K(\sigma, X) \geq 1$ such that $Kn^\sigma X_n \geq m^\sigma X_m$ for all $n \geq m \geq 1$. Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [10]). For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [5])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \quad ({t_n}^1 = t_n) \tag{1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad for \quad n > 0. \tag{2}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [7], [9])

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right|^k < \infty. \tag{3}$$

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If we take $\alpha = 1$, then $|C,\alpha|_k$ summability reduces to $|C,1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{4}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{5}$$

defines the sequence (w_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |w_n - w_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N}, p_n|_k$ summability is the same as $|C,1|_k$, (resp. $|\bar{N}, p_n|$) summability.

2. Known Result

The following theorem is known dealing with the $|\bar{N}, p_n|_k$ summability factors of infinite series.

Theorem 2.1 ([11]). Let (X_n) be an almost increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{6}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{7}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty, \tag{8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

$$\sum_{n=1}^{m} \frac{|\ t_n\ |^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $|\ \bar{N}, p_n\ |_k, k \geq 1$.

It should be remarked that Theorem A also implies the known result of Bor dealing with the absolute $|\bar{N}, p_n|_k$ summability factors of infinite series (see [3]).

3. Main Result

The aim of this paper is to prove Theorem 2.1 under more weaker conditions. Now we shall prove the following theorem.

Theorem 3.1 Let (X_n) be a quasi-σ-power increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions (6), (7), (8), and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{{X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n{X_n}^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Remark 3.2 It should be noted that condition (11) is reduced to the condition (9), when k=1. When $k > 1$, condition (11) is weaker than condition (9) but the converse is not true. As in [12] we can show that if (9) is satisfied, then we get that

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{{X_n}^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m).$$

If (11) is satisfied, then for $k > 1$ we obtain that

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = \sum_{n=1}^{m} X_n^{k-1} \frac{p_n}{P_n} \frac{|t_n|^k}{{X_n}^{k-1}} = O(X_m^{k-1}) \sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{{X_n}^{k-1}} = O(X_m^k) \neq O(X_m).$$

The similar argument is also valid for the conditions (12) and (10). Also it should be noted that if we take (X_n) as an almost increasing sequence, then we get some new results.

We need the following lemma for the proof of our theorem.

Lemma 3. 3 ([4]) Under the conditions of Theorem 3.1, we have that

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty, \tag{13}$$

$$n X_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty. \tag{14}$$

4. Proof of Theorem 3.1 Let (T_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v \lambda_v. \tag{15}$$

Then, for $n \geq 1$, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1} \lambda_v}{v} v a_v. \tag{16}$$

Applying Abel's transformation to the right-hand side of (16), we have

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta \left(\frac{P_{v-1} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{p_n \lambda_n}{n P_n} \sum_{r=1}^{n} v a_v \\
&= \frac{(n+1) p_n t_n \lambda_n}{n P_n} - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta \lambda_v t_v \frac{v+1}{v} + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} t_v \frac{1}{v} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,r}\right|^k<\infty, \quad for \quad r=1,2,3,4.$$

Firstly, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,1}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{p_n}{P_n}|t_n|^k=O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}\\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{p_n}{P_n}\frac{|t_n|^k}{X_n{}^{k-1}}\\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m=O(1) \quad as \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.3. Also, as in $T_{n,1}$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|T_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}p_v|t_v|^k|\lambda_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|\frac{p_v}{P_v}\frac{|t_v|^k}{X_v{}^{k-1}}=O(1) \quad as \quad m\to\infty.\end{aligned}$$

Again, by using (8), we get that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left\{\sum_{v=1}^{n-1}P_v|\Delta\lambda_v||t_v|\right\}^k\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}v|\Delta\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}(v|\Delta\lambda_v|)^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}\frac{P_v}{v}(v|\Delta\lambda_v|)^{k-1}v|\Delta\lambda_v|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{|t_v|^k}{vX_v{}^{k-1}}\\ &= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r{}^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{vX_v{}^{k-1}}\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\left(v|\Delta\lambda_v|\right)|X_v+O(1)m|\Delta\lambda_m|X_m\\ &= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and and Lemma 3.3. Finally, by using (8), we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|T_{n,4}\right|^k &\leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP^k_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left(\sum_{v=1}^{n-1}\frac{P_v}{v}|\lambda_{v+1}|^k|t_v|^k\right)\times\left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}\frac{P_v}{v}\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}\frac{P_v}{v}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\ &= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v{}^{k-1}} = O(1) \quad as \quad m\to\infty.\end{aligned}$$

This completes the proof of Theorem 3.1.

5. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Write

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}C_n(x),$$

$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$, and $\phi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)du$, $(\alpha>0)$.
It is well know that if $\phi_1(t)\in\mathcal{BV}(0,\pi)$, then $t_n(x)=O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [6]). Using this fact, we get the following main result dealing with the trigonometric Fourier series.
Theorem 5.1 Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum C_n(x)\lambda_n$ is summable $\left|\bar{N},p_n\right|_k$, $k\geq 1$.

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