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On a Regularity of Biharmonic Approximations to a Nonlinear Degenerate Elliptic PDE

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Abstract. Under appropriate assumption on the coefficients, we prove that a sequence of biharmonic regularization to a nonlinear degenerate elliptic equation with possibly rough coefficients preserves certain regularity as the approximation parameter tends to zero. In order to obtain the result, we introduce a generalization of the Chebyshev inequality. We also present numerical example.

1. Introduction

In this paper, we consider a biharmonic regularization to the following nonlinear degenerate elliptic equation

$$Qu = \sum_{i=1}^{d}\left[\partial_{x_i}\Big(\sum_{j=1}^{d} a_{ij}(x,Du)\partial_{x_j}u + b_i(x,u,Du)\Big)+c_i(x,u,Du)\partial_{x_i}u\right]+d(x)u \tag{1}$$
$$= f + \sum_{i=1}^{d}\partial_{x_i}g^i,\ \ x\in\Omega\subset\mathbb{R}^d,\ \ Du=\nabla u=(\partial_{x_1},\dots,\partial_{x_d}),$$

where the coefficients will be specified later. By degenerate ellipticity, we imply that the coefficients a_{ij}, $i,j=1,\dots,d$, satisfy degenerate ellipticity conditions

$$0\le\lambda(x,p)|\xi|^2\le a_{ij}(x,p)\xi_i\xi_j,\ \ x\in\Omega,\ \ p\in\mathbb{R}^d, \tag{2}$$

for all $\xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d\backslash 0$. This means that at some points $x\in\Omega$ and for certain values of Du, the principal part of equation (1) (this is $\partial_{x_i}(a_{ij}(x,Du)\partial_{x_j}u)$) can become zero. If this happens, then we are loosing H^1-estimates of the solution and thus none of the standard $C^{k,\alpha}$-estimates are available (see e.g. [10]). Therefore, existence of solutions to such equations in general are not known.

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However, if we assume that $a_{ij}(x,|Du|) = |Du|^{N-2}$, $N \geq 2$, we reach to the well known N-Laplacian for which the existence of solution is well known (see e.g. [2, 12] and references therein). In this article we aim to make one of the first steps in the direction of generalizing the N-Laplacian to a heterogeneous anisotropic case, i.e. to the case of degenerate elliptic equations whose coefficients depend on both x and $|Du|$, and they are not diagonal or even symmetric. To this end, beside degenerate ellipticity, we needed another property of the N-Laplacian, and that is co-monotonicity of the minimal eigenvalue of the matrix $a(x,\lambda) = (a_{ij}(x,\lambda))_{i,j=1,\dots,d}$ and the function λ^2 (see (6) below).

Indeed, the Poisson equation corresponding to the N-Laplacian has the form

$$\nabla \cdot (|Du|^{N-2}Du) = f \ \text{ on a bounded domain } \Omega \subset \mathbb{R}^d.$$

If we assume that $u \in H_0^1(\Omega)$ then, multiplying the N-Laplacian by u and integrating over Ω, we get:

$$\int_\Omega |Du|^{N-2}|Du|^2 dx \leq \int_\Omega f u dx. \tag{3}$$

Since λ^{N-2} and λ^2 have the same monotonicity for $\lambda > 0$ (consult (6) again), we can use the Chebyshev inequality in the integral form (Theorem 2.1 below) to conclude that $u \in W^{1,N-2}(\Omega) \cap W^{1,2}(\Omega)$. The latter property follows directly from (3), but it still reveals an intrinsic property of the N-Laplacian which enables us to discover regularity properties of (2). We shall specify the assumptions in Section 3.

Standard way to solve equations of type (1) is to use a fixed points arguments [7, Section 11] or some kind of vanishing viscosity procedures [4]. In order to use the fixed point theory, we need appropriate estimates (e.g. [7, 10]) which we do not have due to lack of the strict ellipticity. As for the vanishing viscosity approach, in our situation, we have the equation in which the nonlinearity acts on the derivative of the unknown function, and thus we need additional assumptions on the nonlinearity (similarly to the viscosity solutions approach from [4]). Therefore, we cannot (in general) apply either of the two procedures. Instead, we shall regularize the problem with the fourth order approximation as follows:

$$\begin{aligned} &\sum_{i=1}^d \left[\partial_{x_i}\left(\sum_{j=1}^d a_{ij}(x,Du_n)\partial_{x_j}u_n + b_i(x,u_n,Du_n)\right) + c_i(x,u_n,Du_n)\partial_{x_i}u_n\right] + d(x)u_n \\ &= f + \sum_{i=1}^d \partial_{x_i} g^i + \frac{1}{n}\Delta^2 u_n, \quad n \in \mathbf{N}. \end{aligned} \tag{4}$$

We augment the latter equation with appropriate boundary conditions:

$$u|_{\partial\Omega} = 0, \ \ \Delta u|_{\partial\Omega} = 0. \tag{5}$$

Existence of the solution to (4), (5) can be obtained by using standard fixed point arguments. Our aim is to inspect behavior of sequence of solutions (u_n) to (4) as $n \to \infty$ assuming that the solution exists. Let us also remark that it is not unusual to consider properties of approximations whose existence is not granted in general. The typical example is the diffusion-dispersion limit for instance (see e.g. [1, 11] and references therein).

Let us note that approximate solution to (4), (5) can be constructed using the numerical methods e.q. fixed point iterations and finite element method.

Dealing with regularity of solutions which are crucial for convergence of sequences of approximate solutions in general is much more complicated for biharmonic equations than for problems that can be treated by well-developed standard methods, such as second-order elliptic problems. First of all, there is no maximum principle for the biharmonic problem so we can not get some estimates of the solution by the methods used to deal with second-order elliptic problems. Secondly, we know little about the properties of the eigenfunctions of the biharmonic operator in $\mathbb{R}^d$.

The novelty in the approach is usage of the Chebyshev inequality (in the integral form; see Theorem 2.1). In order to use the inequality, one needs co-monotonicity of the involved functions. This is in principle quite strict requirement but we succeeded to relax it by introducing so called almost similarly ordered sequences or functions and deriving a Chebyshev type inequality merely from the almost similarly ordered property.

The paper is organized as follows.

In Section 2, we introduce assumptions, auxiliary notions and notations. In particular, we prove a generalization of the Chebyshev inequality. In Section 3, we prove that the sequence of approximations to (1) remains uniformly bounded in $H^1_{loc}(\Omega)$ and in the last section we present numerical examples together with a convergence test.

2. Preliminaries

Let us first introduce a variant of the Chebyshev integral inequality. To this end, recall that the classical Chebyshev integral inequality has been proved in [3]. More precisely, it holds

Theorem 2.1 (Chebyshev integral inequality). *Let $f, g : \Omega \to \mathbb{R}$ be two non-negative integrable functions. If they are monotonic in the same direction (co-monotonic), i.e. if it holds*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0 \text{ for all } x, y \in \Omega, \tag{6}$$

then it holds

$$\int_\Omega f(x)dx \int_\Omega g(x)dx \leq \text{meas}(\Omega) \int_\Omega f(x)g(x)dx.$$

We introduce the following definition.

Definition 2.2. Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be two vectors of real numbers. We say that the vectors a and b are *almost similarly ordered*, if there exists a positive real constant C such that

$$a_i(Cb_i - b_j) + a_j(Cb_j - b_i) \geq 0. \tag{7}$$

If $C = 1$ then (7) is reduced to the standard co-monotonicity property (6) (rewritten in the terminology of vectors).

Theorem 2.3 (Generalized Chebyshev inequality). *Let $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ be almost similarly ordered non-negative real vectors for some $C > 1$. Then there holds*

$$\frac{1}{n^2} \sum_{k=1}^n a_k \sum_{k=1}^n b_k \leq \frac{C}{n} \sum_{k=1}^n a_k b_k. \tag{8}$$

Proof. It holds

$$Cn \sum_{j=1}^n a_j b_j - \sum_{i=1}^n a_i \sum_{j=1}^n b_j = \sum_{i,j=1}^n \Big(a_i(Cb_i - b_j) + a_j(Cb_j - b_i) \Big) \geq 0.$$

The above inequalities obviously yield (8). □

Assumption (7) is clearly weaker than (6). For instance take $a_1 = 1$, $a_2 = 2$, $b_1 = 2$, $b_2 = 1$, and $C = 2$ in (2.2). Vectors (a_1, a_2) and (b_1, b_2) are not co-monotonic but they satisfy (7) with $C = 2$.

Moreover, under the conditions of Theorem 2.3 and following its proof, we can prove the inequality

$$\sum_{k=1}^n p_k a_k \sum_{k=1}^n p_k b_k \leq C \sum_{k=1}^n p_k a_k b_k,$$

where (p_k) is a vector of positive numbers such that $\sum_{k=1}^n p_k = 1$.

It is natural to extend the previous concept in the continuous setting.

Definition 2.4. Two functions $f, g : \Omega \to \mathbb{R}$ are *almost similarly ordered* on Ω if there exists a constant $C \geq 1$, such that

$$f(x)(Cg(x) - g(y)) + f(y)(Cg(y) - g(x)) \geq 0. \tag{9}$$

The following variant of the result from [3] holds.

Theorem 2.5. *(Generalized Chebyshev integral inequality) Let $f, g : \Omega \to \mathbb{R}$ be almost similarly ordered functions in the sense of* (9) *which are integrable on the bounded domain $\Omega \subset \mathbb{R}^n$ and non-negative out of a set of measure zero. Then it holds*

$$\frac{1}{C\mu(\Omega)} \int_\Omega f(x)\, dx \int_\Omega g(x)\, dx \leq \int_\Omega f(x)g(x)\, dx, \tag{10}$$

where $\mu(\Omega) = \text{meas}(\Omega)$.

Proof. It is enough to prove the theorem under the assumptions that $f, g \in C(\Omega)$. The general case follows by the density arguments (since the continuous functions are dense in the space of locally integrable functions; for details see [3]). This means that we can work with Riemann instead of Lebesgue sums in the sequel.

Having this in mind, assume that we can split Ω on, say N^d, hyper-cubes with the edge-length $1/N$ centered at ξ_r, $r = 1, \dots, N$ (we can assume the latter without loosing on generality since we can approximate the domain Ω by such cubes), and remark that it holds according to Theorem 2.3:

$$C \sum_{r=1}^{N^d} f(\xi_r) g(\xi_r) \frac{1}{\mu(\Omega) N^d} \geq \sum_{r=1}^{N^d} f(\xi_r) \frac{1}{\mu(\Omega) N^d} \sum_{r=1}^{N^d} g(\xi_r) \frac{1}{\mu(\Omega) N^d}.$$

Letting $N \to \infty$ in the above inequality, we find that

$$C\mu(\Omega) \int_\Omega f(x)g(x)\, dx \geq \int_\Omega f(x)\, dx \int_\Omega g(x)\, dx,$$

and from here (10) immediately follows. □

Remark 2.6. If $f, g : \Omega \to \mathbb{R}$ from Theorem 2.5 are almost oppositely ordered on $\Omega \subset \mathbb{R}^n$ functions, that is, if the relation $f(x)(cg(x) - g(y)) + f(y)(cg(y) - g(x)) \leq 0$ for some $c > 0$ is valid, then as in the proof of Theorem 2.5, we can prove that

$$\int_\Omega f(x)\, dx \int_\Omega g(x)\, dx \geq c\mu(\Omega) \int_\Omega f(x)g(x)\, dx.$$

Let us now introduce the assumptions on the coefficients of (1). In order to avoid unsubstantial technicalities, we shall assume that we are dealing with the equation

$$\sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}(x, Du)\partial_{x_j} u) = f(x, u, Du), \;\; x \in \Omega \subset \mathbb{R}^d. \tag{11}$$

We assume

(a) $\max_{(v,p)\in\mathbb{R}^{d+1}} f(\cdot, v, p) \in L^2(\Omega)$;

(b) it holds (the special variant of) (2)

$$0 \leq \lambda(x, |p|)|\xi|^2 \leq a_{ij}(x, p)\xi_i\xi_j, \;\; x \in \Omega, \;\; p \in \mathbb{R}^d. \tag{12}$$

Denote by $\tilde{\Omega}$ set of points $x \in \Omega$ for which there exists a neighborhood $\mathcal{U}(x)$, function $\tilde{\lambda}_x : \mathbb{R} \to [0, \infty)$, and a constant $M > 0$ such that $\tilde{\lambda}_x(q) \leq \lambda(y, q) \leq M\tilde{\lambda}_x(q)$, $y \in \mathcal{U}(x)$. We also asume that the functions

$$\tilde{\lambda}_x(q) \text{ and } q^2, \;\; x \in \tilde{\Omega} \tag{13}$$

are almost similarly ordered for $q \geq 0$ with the constant $C(x) \geq 1$ appearing in (9).

Let us finally recall the notion of the weak solution to

$$\sum_{i,j=1}^{d} \partial_{x_i}(a_{ij}(x, Du_n)\partial_{x_j}u_n) = f + \frac{1}{n}\Delta^2 u_n, \quad x \in \Omega. \tag{14}$$

Definition 2.7. We say that the function $u_n \in H_0^2(\Omega)$ is an approximate solution to (14) with the boundary conditions (5) if for every $\varphi \in H_0^2(\Omega)$, it holds

$$-\int_\Omega \sum_{i,j=1}^{d} a_{ij}(x, Du_n)\partial_{x_i}u_n\partial_{x_j}\varphi\Big)dx = \int_\Omega f\varphi dx + \frac{1}{n}\int_\Omega \Delta u_n \Delta\varphi dx.$$

3. Regularity Estimates for the Fourth Order Approximation

Before we formulate the main theorem, remark that we have no regularity assumptions on λ or $[a_{ij}]$ except the growth rate given in the assumptions on λ from the previous section. However, since we have regularized the equation with the higher order terms, we can expect (relatively mild) L^2-boundedness of the sequence of solutions to (14). We support the above arguing by the numerical examples at the end of the paper. Moreover, if we assume that $\tilde{\Omega} = \Omega$, we can use the Poincare inequality to derive L^2-boundedness from (15) below. Having this in mind, the following theorem holds.

Theorem 3.1. *Assume that the sequence of approximate solutions* (u_n) *of* (14) *satisfying* (5) *is uniformly bounded in* $L^2_{loc}(\Omega)$. *Then, with the notations from the previous section, it holds for every (relatively) compact* $K \subset\subset \tilde{\Omega}$

$$\|Du_n\|_{L^2(K)} \leq C\|\max_{(v,p)\in\mathbb{R}^{d+1}} f(\cdot, v, p)\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)}, \tag{15}$$

for some constant $C > 0$.

Proof. According to the definition of the weak solution given in Definition 2.7, we have

$$\int_\Omega \sum_{i,j=1}^{d} a_{ij}(x, Du_n)\partial_{x_i}u_n\partial_{x_j}u_n dx + \frac{1}{n}\int_\Omega \sum_{i,j=1}^{d} \partial^2_{x_i}u_n\partial^2_{x_j}u_n dx = -\int_\Omega f u_n dx.$$

Using the degenerate ellipticity property (b) of the matrix $[a_{ij}]$, we conclude

$$\int_\Omega \lambda(x, |Du_n|)|Du_n|^2 dx + \frac{1}{n}\int_\Omega \left(\sum_{i=1}^{d} \partial^2_{x_i}u_n\right)^2 dx \leq -\int_\Omega f u_n dx.$$

Now, fix the compact set $K \subset\subset \Omega$. Using property (13) of the function λ and non-negativity of the second term on the right hand side of the last expression, we see that in a neighborhood $\mathcal{U}(x_0) \subset \Omega$ of almost every $x_0 \in K$, it holds

$$\frac{1}{C(x_0)\text{meas}(\mathcal{U}(x_0))}\int_{\mathcal{U}(x_0)} \tilde{\lambda}_{x_0}(|Du_n|)dx \int_{\mathcal{U}(x_0)} |Du_n|^2 dx \leq -\int_\Omega f u_n dx. \tag{16}$$

Since K is compact, we can find finite number of points x_s, $s \in \{1, \dots, N\}$, such that the neighborhoods $\mathcal{U}(x_s)$ cover entire K. Thus, from (16) we conclude

$$\min_{1\le s\le N}\left(\frac{1}{C(x_s)\text{meas}(\mathcal{U}(x_s))}\int_{\mathcal{U}(x_s)}\tilde{\lambda}_{x_s}(|Du_n|)dx\right)\int_K |du|^2dx \le -N\int_\Omega fu_n dx. \tag{17}$$

Using the Cauchy-Schwartz inequality on the right-hand side of (17), and taking (b) into account, we conclude

$$\int_K |Du_n|^2 dx \le \frac{K_1}{\text{meas}(\Omega)\,K_2}\|f\|_{L^2(\Omega)}\|u_n\|_{L^2(\Omega)}, \tag{18}$$

for appropriate constants K_1 and K_2. Since $\|u_n\|_{L^2(\Omega)}$ is bounded, we conclude the proof. □

From the latter theorem, we cannot conclude about global L^p-boundedness of the gradient of u. However, if we assume a growth rate of $\lambda(x,p)$ with respect to p around the critical point x where $\lambda(x,p) = 0$. For instance, if we assume that $0 \in \Omega$ is the critical point, and in some neighborhood of

$$\lambda(|x|,p) \le |x|^\alpha p^{-\beta},$$

where $\alpha, \beta > 0$, then, under the assumptions of Theorem 3.1, we conclude

$$|x|^\alpha |Du_n|^{2-\beta} = F_n \in L^1(\Omega) \ \text{ (uniformly bounded in } L^1(\Omega)).$$

Thus, if $\beta < 2$, using the Hölder inequality, we can find $\gamma > 0$ such that

$$|Du_n|^{(2-\beta)/\gamma} = \left(\frac{F_n}{|x|^\alpha}\right)^{1/\gamma} \in L^1(\Omega)$$

which provides certain regularity of the solution to (4) around the critical points. For instance, in our numerical experiment, we will have function of the form

$$\lambda(|x|, Du_n) = |x||Du_n|,$$

and keeping in mind that Ω is bounded it is easy to see that

$$Du_n \ \text{ is uniformly bounded in } \ L^1(\Omega).$$

Thus, according to the Relich theorem, we conclude that u_n converges strongly in $L^1(\Omega)$. We shall see that numerical experiments show not only the convergence but even much more regularity of the approximate solutions.

4. Numerical Experiment

Let $\Omega = [0,1]^2$. We consider two dimensional boundary value problem

$$-\operatorname{div}\Big(((x_1-0.2)^2 + (x_2-0.3)^2)(10 + |Du|^2)Du\Big) = f - \epsilon\Delta^2 u, \ \ (x_1,x_2) \in \Omega, \tag{19}$$

equipped with (5) where $f(x_1,x_2) = 2\left((x_1+1)^3 + e^{1+x_1x_2}\right)$.

Numerical solution of the regularized problem (19) is obtained by using fixed point iterations and finite element method. Pictures of the solution for different triangularization and ϵ are presented bellow. We also performed convergence test.

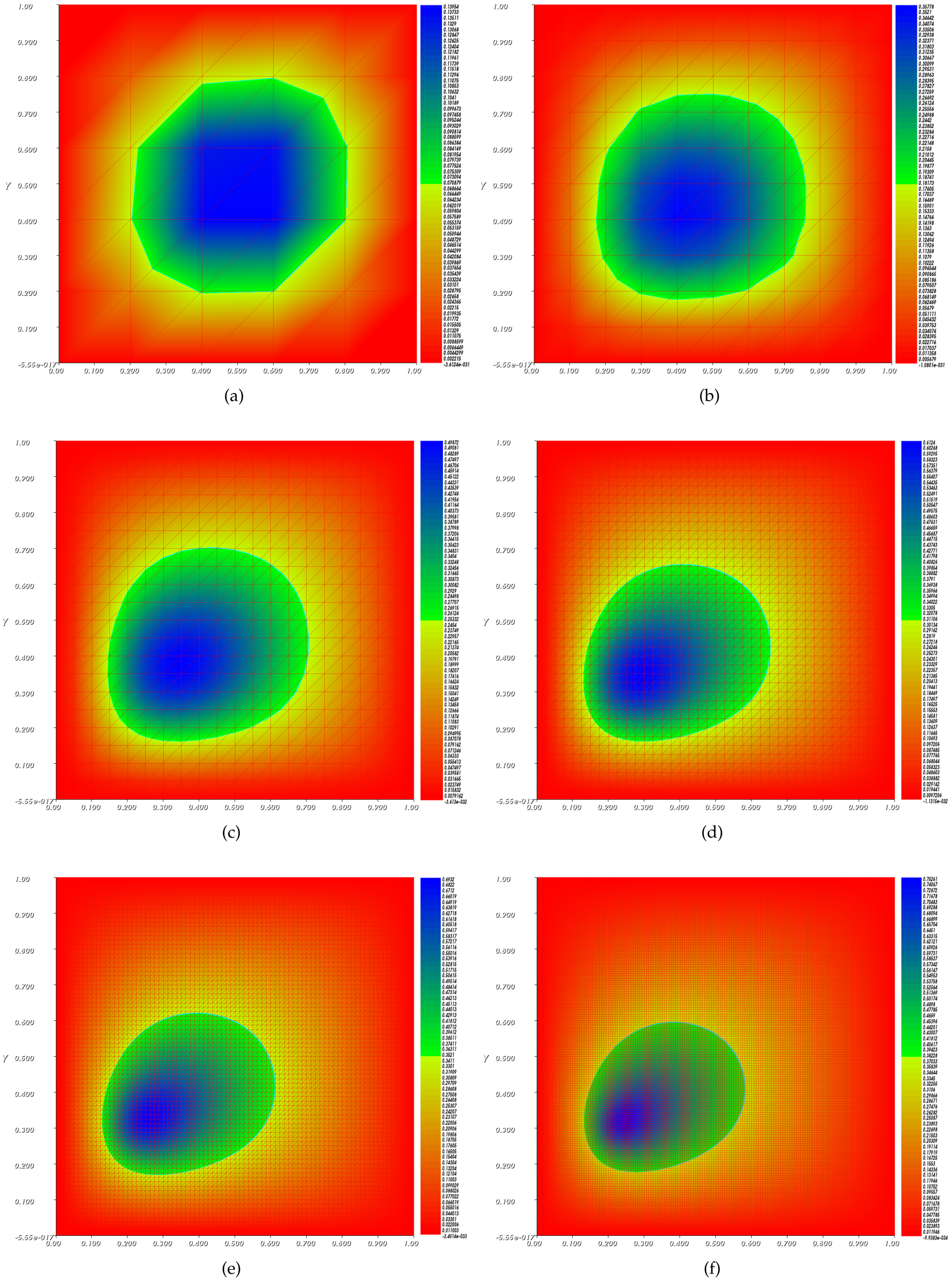

Figure 1: Solution of the equation (19) computed on $N \times N$ mesh with $\epsilon = 1/N^2$ plotted together with triangulation of the domain. a) $N = 5$, b) $N = 10$, c) $N = 20$, d) $N = 40$, e) $N = 80$, f) $N = 160$.

N	$\|\|u_E - u_N\|\|_2/\|\|u_E\|\|_2$
5	0.709188
10	0.454481
20	0.242811
40	0.111621
80	0.0368282
160	0.0099392

Table 1: Convergence test. With u_E we denote solution of (19) computed on $N \times N$ mesh with $N = 320$ and $\epsilon = 1/N^2$.

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